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Skewness premium with Lévy processes

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We study the skewness premium (SK) introduced by Bates [*J. Finance*, 1991, **46**(3), 1009–1044] in a general context using Lévy processes. Under a symmetry condition, Fajardo and Mordecki [*Quant. Finance*, 2006, **6**(3), 219–227] obtained that SK is given by Bates' x% rule. In this paper, we study SK in the absence of that symmetry condition. More exactly, we derive sufficient conditions for the excess of SK to be positive or negative, in terms of the characteristic triplet of the Lévy process under a risk-neutral measure.

Keywords: Skewness premium; Lévy processes

JEL Classification: C52, G10

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1. Introduction

The stylized facts of option prices have been studied by many authors. An important fact from option prices is that relative prices of out-of-the-money calls and puts can be used as a measure of the symmetry or skewness of the risk-neutral distribution. Bates (1991) called this diagnosis the 'skewness premium' (henceforth SK). He analysed the behavior of SK using three classes of stochastic processes: constant elasticity of variance (CEV), stochastic volatility and jump-diffusion.

However, as many models in the literature have shown, the behavior of the assets underlying options is very complex and the structure of jumps observed is more complex than Poisson jumps. They have higher intensity (see, for example, Aït-Sahalia (2004)). For that reason, diffusion models cannot consider the discontinuous sudden movements observed in asset prices. In that sense, the use of more general processes such as Lévy processes has shown the ability to provide a better fit to real data, as reported by Eberlein *et al.* (1998) and Carr and Wu (2004). On the other hand, the mathematical tools behind these processes are very well established and known.

When the underlying asset follows a geometric Lévy process, Fajardo and Mordecki (2006) obtained a relationship between calls and puts that they called *put–call duality* and obtained as a particular case the *put–call symmetry*. They also obtained, under a symmetry condition, that SK is given by the Bates' x% rule. Similar symmetry conditions are used in the literature to obtain important applications such as, for example, the construction of semi-static hedges for exotic options, as Carr *et al.* (1998) and Carr and Lee (2009) have shown.

Also, Bates (1997) verified that put–call symmetry does not hold in practice by constructing a hypothesis test that compares the observed relative call–put prices and the ratio given by Bates' rule. The absence of symmetry has also been reported by Carr and Wu (2007). They found asymmetric implied volatility smiles in currency options and, according to a result found by Fajardo and Mordecki (2006) and Carr and Lee (2009), it is known that symmetric markets must present symmetric implied volatilities.

In this paper, we study the SK in the absence of symmetry and obtain sufficient conditions for the excess of SK to be positive or negative. The main idea behind the proofs is to exploit the monotonicity property of option prices with respect to some parameter of the Lévy measure. This monotonicity is not an easy property to obtain. Monotonicity with respect to the intensity parameter of jumps has recently been addressed by Ekström and Tysk (2007), while monotonicity with respect to the triplet characteristics of semimartingales has been partially addressed by Bergenthum and Rüschendorf (2007).

The paper is organized as follows. In section 2 we introduce the Lévy market model and present the duality results. In section 3 we discuss market symmetry and

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present our main results. In section 4 we study the skewness premium. Section 5 discusses monotonicity with respect to the symmetry parameter and section 6 concludes.

2. Lévy processes and duality

Consider a real-valued stochastic process $X=\{X_t\}_{t\geq 0}$, defined on a stochastic basis $\mathcal{B}=(\Omega, \mathcal{F}, \mathbf{F}=(\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, being càdlàg, adapted, satisfying $X_0=0$, such that, for $0\leq s<t$, the random variable X_t-X_s is independent of the σ-field $\mathcal{F}_s$, with a distribution that only depends on the difference $t-s$. Assume also that the stochastic basis $\mathcal{B}$ satisfies the usual conditions (Jacod and Shiryaev 1987). The process X is a Lévy process, and is also called a process with stationary independent increments. For general references on Lévy processes, see Jacod and Shiryaev (1987), Skorokhod (1991), Bertoin (1996) and Sato (1999). For Lévy processes in finance, see Boyarchenko and Levendorskiĭ (2002), Schoutens (2003) and Cont and Tankov (2004).

In order to characterize the law of X under $\mathbb{Q}$, consider for $q\in I\!R$ the Lévy–Khinchine formula, which states

$$\mathbf{E}\mathrm{e}^{iqX_t}=\exp\left\{t\left[iaq-\frac{1}{2}\sigma^2q^2+\int_{I\!R}(\mathrm{e}^{iqy}-1-iqh(y))\Pi(\mathrm{d}y)\right]\right\}, \tag{1}$$

with

$$h(y)=y\mathbf{1}_{\{|y|<1\}}$$

a fixed truncation function, a and $\sigma\geq 0$ real constants, and Π a positive measure on $I\!R\backslash\{0\}$† such that $\int(1\wedge y^2)\Pi(\mathrm{d}y)<+\infty$, called the *Lévy measure*. The triplet (a, σ^2, Π) is the *characteristic triplet* of the process, and completely determines its law.

Consider the set

$$\mathbb{C}_0=\left\{z=p+iq\in\mathbb{C}\colon \int_{\{|y|>1\}}\mathrm{e}^{py}\Pi(\mathrm{d}y)<\infty\right\}. \tag{2}$$

The set $\mathbb{C}_0$ is a vertical strip in the complex plane, contains the line $z=iq$ $(q\in I\!R)$, and consists of all complex numbers $z=p+iq$ such that $\mathbf{E}\mathrm{e}^{pX_t}<\infty$ for some $t>0$. Furthermore, if $z\in\mathbb{C}_0$, we can define the *characteristic exponent* of the process X by

$$\psi(z)=az+\frac{1}{2}\sigma^2z^2+\int_{I\!R}(\mathrm{e}^{zy}-1-zh(y))\Pi(\mathrm{d}y). \tag{3}$$

This function ψ is also called the *cumulant* of X, having $\mathbf{E}|\mathrm{e}^{zX_t}|<\infty$ for all $t\geq 0$, and $\mathbf{E}\mathrm{e}^{zX_t}=\mathrm{e}^{t\psi(z)}$. The finiteness of this expectation follows from Sato (1999, theorem 25.3). For $t=1$, formula (3) reduces to the exponent of equation (1) with $\mathrm{Re}(z)=0$.

2.1. Lévy market

By a *Lévy market* we mean a model of a financial market with two assets: a deterministic savings account $B=\{B_t\}_{t\geq 0}$, with

$$B_t=\mathrm{e}^{rt},\quad r\geq 0,$$

where we take $B_0=1$ for simplicity, and a stock $S=\{S_t\}_{t\geq 0}$, with random evolution modeled by

$$S_t=S_0\mathrm{e}^{X_t},\quad S_0=\mathrm{e}^x>0, \tag{4}$$

where $X=\{X_t\}_{t\geq 0}$ is a Lévy process.

In this model we assume that the stock pays dividends, with constant rate $\delta\geq 0$, and that the given probability measure $\mathbb{Q}$ is the chosen equivalent martingale measure. In other words, prices are computed as expectations with respect to $\mathbb{Q}$, and the discounted and reinvested process $\{\mathrm{e}^{-(r-\delta)t}\,S_t\}$ is a $\mathbb{Q}$-martingale.

In terms of the characteristic exponent of the process, this means that

$$\psi(1)=r-\delta, \tag{5}$$

based on the fact that $\mathbf{E}\mathrm{e}^{-(r-\delta)t+X_t}=\mathrm{e}^{-t(r-\delta-\psi(1))}=1$, and condition (5) can also be formulated in terms of the characteristic triplet of the process X as

$$a=r-\delta-\sigma^2/2-\int_{I\!R}(\mathrm{e}^y-1-h(y))\Pi(\mathrm{d}y). \tag{6}$$

In the market model considered, we introduce some derivative assets. More precisely, we consider call and put options, of both European and American type. Denote by $\mathcal{M}_T$ the class of stopping times up to a fixed constant time T, i.e.

$$\mathcal{M}_T=\{\tau: 0\leq\tau\leq T,\ \tau \text{ stopping time w.r.t. } \mathbf{F}\}.$$

Then, for each stopping time $\tau\in\mathcal{M}_T$, we introduce

$$c(S_0, K, r, \delta, \tau, \psi)=\mathbf{E}\mathrm{e}^{-r\tau}(S_\tau-K)^+, \tag{7}$$

$$p(S_0, K, r, \delta, \tau, \psi)=\mathbf{E}\mathrm{e}^{-r\tau}(K-S_\tau)^+. \tag{8}$$

In our analysis, equations (7) and (8) are auxiliary quantities, but they are interesting in themselves as random maturity options, as considered, for instance, by Schroder (1999) and Detemple (2001). If $\tau=T$, formulas (7) and (8) give the price of the European call and put option, respectively. For the American finite case, prices and optimal stopping rules τ_{c}^* and τ_{p}^* are defined, respectively, by

$$\begin{aligned}C(S_0, K, r, \delta, T, \psi)&=\sup_{\tau\in\mathcal{M}_T}\mathbf{E}\mathrm{e}^{-r\tau}(S_\tau-K)^+\\&=\mathbf{E}\mathrm{e}^{-r\tau_{\mathrm{c}}^*}(S_{\tau_{\mathrm{c}}^*}-K)^+,\end{aligned} \tag{9}$$

$$\begin{aligned}P(S_0, K, r, \delta, T, \psi)&=\sup_{\tau\in\mathcal{M}_T}\mathbf{E}\mathrm{e}^{-r\tau}(K-S_\tau)^+\\&=\mathbf{E}\mathrm{e}^{-r\tau_{\mathrm{p}}^*}(K-S_{\tau_{\mathrm{p}}^*})^+.\end{aligned} \tag{10}$$

† $\Pi(\{0\})$ could be defined as 0. Here we follow Cont and Tankov (2004).

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2.2. Put–call duality and dual markets

Proposition 2.1 (Fajardo and Mordecki 2006)**:** *Consider a Lévy market with driving process X with characteristic exponent $\psi(z)$, defined in* (3)*, on the set $\mathbb{C}_0$ in* (2)*. Then, for the expectations introduced in* (7) *and* (8) *we have*

$$c(S_0, K, r, \delta, \tau, \psi) = p(K, S_0, \delta, r, \tau, \tilde{\psi}), \tag{11}$$

where

$$\tilde{\psi}(z) = \tilde{a}z + \frac{1}{2}\tilde{\sigma}^2 z^2 + \int_{IR} (\mathrm{e}^{zy} - 1 - zh(y))\tilde{\Pi}(\mathrm{d}y) \tag{12}$$

is the characteristic exponent (*of a certain Lévy process*) *that satisfies*

$$\tilde{\psi}(z) = \psi(1 - z) - \psi(1), \quad \text{for } 1 - z \in \mathbb{C}_0,$$

and, consequently,

$$\begin{aligned} \tilde{a} &= \delta - r - \sigma^2/2 - \int_{IR} (\mathrm{e}^{y} - 1 - h(y))\tilde{\Pi}(\mathrm{d}y), \\ \tilde{\sigma} &= \sigma, \\ \tilde{\Pi}(\mathrm{d}y) &= \mathrm{e}^{-y}\Pi(-\mathrm{d}y). \end{aligned} \tag{13}$$

Now, given a Lévy market with driving process characterized by ψ in (3), consider a market model with two assets, a deterministic savings account $\tilde{B} = \{\tilde{B}_t\}_{t\geq 0}$, given by

$$\tilde{B}_t = \mathrm{e}^{\delta t}, \quad \delta \geq 0,$$

and a stock $\tilde{S} = \{\tilde{S}_t\}_{t\geq 0}$, modeled by

$$\tilde{S}_t = K\mathrm{e}^{\tilde{X}_t}, \quad \tilde{S}_0 = K > 0,$$

where $\tilde{X} = \{\tilde{X}_t\}_{t\geq 0}$, called the *dual process*, is a Lévy process and is given by $\tilde{X}_t = -X_t$ $(t \geq 0)$. The process $\tilde{S}_t$ represents the price of KS_0 dollars measured in units of stock S.

The characteristic exponent of $\tilde{X}_t$, under $\tilde{\mathbb{Q}}$, is given by $\tilde{\psi}$ in (12), where the *dual martingale measure* $\tilde{\mathbb{Q}}$ is given by its restrictions $\tilde{\mathbb{Q}}_t$ to $\mathcal{F}_t$ by

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}_t}{\mathrm{d}\mathbb{Q}_t} = \mathrm{e}^{X_t - (r-\delta)t}, \quad t \geq 0,$$

and where $\mathbb{Q}_t$ (respectively $\tilde{\mathbb{Q}}_t$) is the restriction of $\mathbb{Q}$ (respectively $\tilde{\mathbb{Q}}$) to $\mathcal{F}_t$. Then, it follows from Jacod and Shiryaev (2003, proposition 3.8, p. 163) that, if e^{X_t} is a $\mathbb{Q}$-martingale, then e^{-X_t} is a $\tilde{\mathbb{Q}}$-martingale.

3. Market symmetry

Here we use the symmetry concept introduced by Fajardo and Mordecki (2006). We define a Lévy market to be *symmetric* when the following relation holds:

$$\mathcal{L}(\mathrm{e}^{-(r-\delta)t + X_t} \mid \mathbb{Q}) = \mathcal{L}(\mathrm{e}^{-(\delta - r)t - X_t} \mid \tilde{\mathbb{Q}}), \tag{14}$$

meaning equality in law. Otherwise, we call the Lévy market *asymmetric*. As Fajardo and Mordecki (2006) pointed out, a necessary and sufficient condition for (14) to hold is

$$\Pi(\mathrm{d}y) = \mathrm{e}^{-y}\Pi(-\mathrm{d}y). \tag{15}$$

This ensures that $\tilde{\Pi} = \Pi$, and it follows that $a - (r - \delta) = \tilde{a} - (\delta - r)$, giving (14), as we always have $\tilde{\sigma} = \sigma$. Moreover, in Lévy markets with a jump measure of the form

$$\Pi(\mathrm{d}y) = \mathrm{e}^{\beta y}\Pi_0(\mathrm{d}y), \tag{16}$$

where $\Pi_0(\mathrm{d}y)$ is a symmetric measure, i.e. $\Pi_0(\mathrm{d}y) = \Pi_0(-\mathrm{d}y)$, and β is a parameter that describe the asymmetry of the jumps, everything is with respect to the risk-neutral measure $\mathbb{Q}$. Fajardo and Mordecki (2006) found that the market is symmetric if and only if $\beta = -1/2$.

In this setting, Fajardo and Mordecki (2006) obtained Bates' $x\%$ rule for the case $r = \delta$, i.e. no carrying costs. In the following corollary we extend this result to the case with carrying costs.

Corollary 3.1 (Bates' $x\%$ rule)**:** *Assume that* (15) *holds* (*or in the particular case* (16)*, assume $\beta = -1/2$*)*, then we have*

$$c(F, K_c, r, \delta, \tau, \psi) = (1 + x)\, p(F, K_p, \delta, r, \tau, \psi), \tag{17}$$

where $K_c = (1 + x)F$ and $K_p = F/(1 + x)$, with $x > 0$, i.e. call options $x\%$ out-of-the-money (OTM) are priced exactly $x\%$ higher than the corresponding OTM put.

Proof: Under (15) we know that $\psi = \tilde{\psi}$. Then, taking $K = (1 + x)F\mathrm{e}^{-(r-\delta)T}$ and $S_0 = F\mathrm{e}^{-(r-\delta)T}$ in proposition 2.1, we obtain

$$\begin{aligned} &c(F\mathrm{e}^{-(r-\delta)T}, (1 + x)F\mathrm{e}^{-(r-\delta)T}, r, \delta, \tau, \psi) \\ &\quad = p((1 + x)F\mathrm{e}^{-(r-\delta)T}, F\mathrm{e}^{-(r-\delta)T}, \delta, r, \tau, \psi). \end{aligned}$$

Now, a linear operation performed on a linearly homogeneous function preserves its property of linear homogeneity. Then since call and put prices are linearly homogenous in stock and strike price, we have

$$\begin{aligned} &\mathrm{e}^{-(r-\delta)T} c(F, (1 + x)F, r, \delta, \tau, \psi) \\ &\quad = (1 + x)\mathrm{e}^{-(r-\delta)T} p(F, F/(1 + x), \delta, r, \tau, \psi). \end{aligned}$$

Finally,

$$c(F, (1 + x)F, r, \delta, \tau, \psi) = (1 + x)\, p(F, F/(1 + x), \delta, r, \tau, \psi). \tag{18}$$

Observe that the carrying cost $b = r - \delta$ on the l.h.s. is $b^* = \delta - r$ on the r.h.s., as Bates's $x\%$ rule requires. □

Remark 1 : Put–call relationships for diffusion with jumps have also been obtained by Schroder (1999, example 8) and Carr and Lee (2009, corollary 4.5).

3.1. More evidence of the absence of symmetry

Fajardo and Mordecki (2006) review several concrete models proposed in the literature. Although from the theoretical point of view, assumption (16) is a real restriction, most models in practice share this property,

and, furthermore, they have a jump measure that has a Radon–Nikodym density. In this case, we have

$$\Pi(\mathrm{d}y) = \mathrm{e}^{\beta y} p(y)\mathrm{d}y, \tag{19}$$

where $p(y) = p(-y)$, i.e. the function $p(y)$ is even. More precisely, all parametric models that we found in the literature regarding Lévy markets, including diffusions with jumps, can be re-parametrized in the form (19): the generalized hyperbolic model proposed by Eberlein and Prause (2002), the Meixner model proposed by Schoutens (2001) and the CGMY model proposed by Carr, Geman, Madan and Yor (2002). Recently, Fajardo and Mordecki (2010) showed that, under some conditions, the independently time-changed Brownian motion with drift is also included in this class. They then showed that the resulting processes will satisfy the above symmetry if and only if the drift equals $-1/2$.

Using the risk-neutral market measure and the Esscher transform measure as equivalent martingale measures, Fajardo and Mordecki (2006, 2010) obtained evidence that empirical risk-neutral markets are not symmetric. Therefore, the question naturally arises of how to obtain a put–call relationship in the absence of symmetry. In what follows we answer this question for the class of Lévy processes satisfying (16). It is worth noting that Carr and Lee (2009) obtained a modified put–call symmetry for certain classes of asymmetric dynamics. However, here we are interested in deviations from the original relationship.

Henceforth we use the following assumption.

Assumption 3.2: *Call option prices are monotonically increasing with respect to the symmetry parameter β.*

Our main result is stated as follows.

Theorem 3.3 : *Consider Lévy measures given by* (16). *Under assumption* 3.2, *if* $\beta \gtrless -1/2$, *then*

$$c(F, K_c, r, \delta, \tau, \psi) \gtrless (1+x)\, p(F, K_p, \delta, r, \tau, \psi), \tag{20}$$

where $K_c = (1+x)F$ *and* $K_p = F/(1+x)$, *with* $x > 0$.

Proof: We have that

$$\beta \gtrless -1/2 \Longleftrightarrow \beta \gtrless \widetilde{\beta} := -\beta - 1.$$

Then, $\Pi(\mathrm{d}y) = \mathrm{e}^{\beta y}\, \Pi_0(\mathrm{d}y)$ has $\beta \gtrless \widetilde{\beta}$ of $\widetilde{\Pi} = \mathrm{e}^{-(1+\beta)y} \Pi_0(\mathrm{d}y)$. Now since ψ's component a is determined by equation (6), for convenience we use the notation $\psi = (\sigma, \beta, \Pi_0)$. Then, by monotonicity,

$$\begin{aligned} c(F, K_\mathrm{c}, r, \delta, \tau, \sigma, \beta, \Pi_0) &\gtrless c(F, K_\mathrm{c}, \delta, r, \tau, \sigma, \widetilde{\beta}, \Pi_0) \\ &= (1+x)\, p(F, K_\mathrm{p}, \delta, r, \tau, \sigma, \beta, \Pi_0), \end{aligned}$$

where the last equality is obtained from the put–call duality relationship and the fact that $\widetilde{\Pi} = \Pi$. □

To formulate the result for American options, we observe that the optimal stopping rules for the American call and put options respectively have the form

$$\tau_\mathrm{c}^* = \inf\{t \geq 0 \colon S_t \geq B_\mathrm{c}(t)\} \wedge T,$$
$$\tau_\mathrm{p}^* = \inf\{t \geq 0 \colon S_t \leq B_\mathrm{p}(t)\} \wedge T,$$

where the curves B_c and B_p are the boundaries of the continuation region.†

Corollary 3.4 (American options)**:** *Consider Lévy measures given by* (16). *Under assumption* 3.2, *if* $\beta \gtrless -1/2$, *then for the value functions in* (9) *and* (10) *we have*

$$C(F, K_c, r, \delta, T, \psi) \gtrless (1+x) P(F, K_p, \delta, r, T, \psi). \tag{21}$$

Furthermore, when $\delta > 0$, *for the optimal stopping boundaries, we obtain that*

$$B_c(t) B_p(t) = S_0 K. \tag{22}$$

If $\delta = 0$, *we have* $\tau_c^* = \tau_p^* = T$.

Proof: The proof is analogous to that of the European case. We only need to take the supremum over stopping times and then use the duality result for American options obtained by Fajardo and Mordecki (2006, corollary 2). □

In the next section we address the skewness premium.

4. Skewness premium

We now analyse the following intra-day data for 27 American call option and 25 American put option prices on the S&P500 Futures on 08/31/2006 that mature on 09/15/2006 with future price $F = 1303.82$. To see if Bates' rule holds, we need to interpolate some unobserved option prices. To this end we use a cubic spline, as can be seen in figure 1.

The x% skewness premium is defined as the percentage deviation of x% out-of-the-money (OTM) call prices from x% OTM put prices. The interpolating call and put prices for the unobserved strikes are presented in tables 1 and 2. It can be seen from both tables that this rule does not hold for this sample. Of course, a more rigorous statistical test with more sample data must be constructed to arrive at stronger conclusions, as did Bates (1997). Moreover, for OTM options, usually $x_{\mathrm{obs}} < x$, which implies $(c/p) - 1 < x$ and, for in-the-money (ITM) options, $x_{\mathrm{obs}} > x$, implying $(c/p) - 1 > x$.

Now we define the skewness premium, following Bates (1991).

Definition 4.1:

$$SK(x) = \frac{c(F, K_\mathrm{c}, r, \delta, T, \psi)}{p(F, K_\mathrm{p}, r, \delta, T, \psi)} - 1, \quad \text{for European options,}$$
$$SK(x) = \frac{C(F, K_\mathrm{c}, r, \delta, T, \psi)}{P(F, K_\mathrm{p}, r, \delta, T, \psi)} - 1, \quad \text{for American options,} \tag{23}$$

where $K_\mathrm{p} = F/(1+x)$ and $K_\mathrm{c} = F(1+x)$, $x > 0$.

†See Cont and Tankov (2004, 12.1.3) or Boyarchenko and Levendorskiĭ (2002, theorem 6.1).

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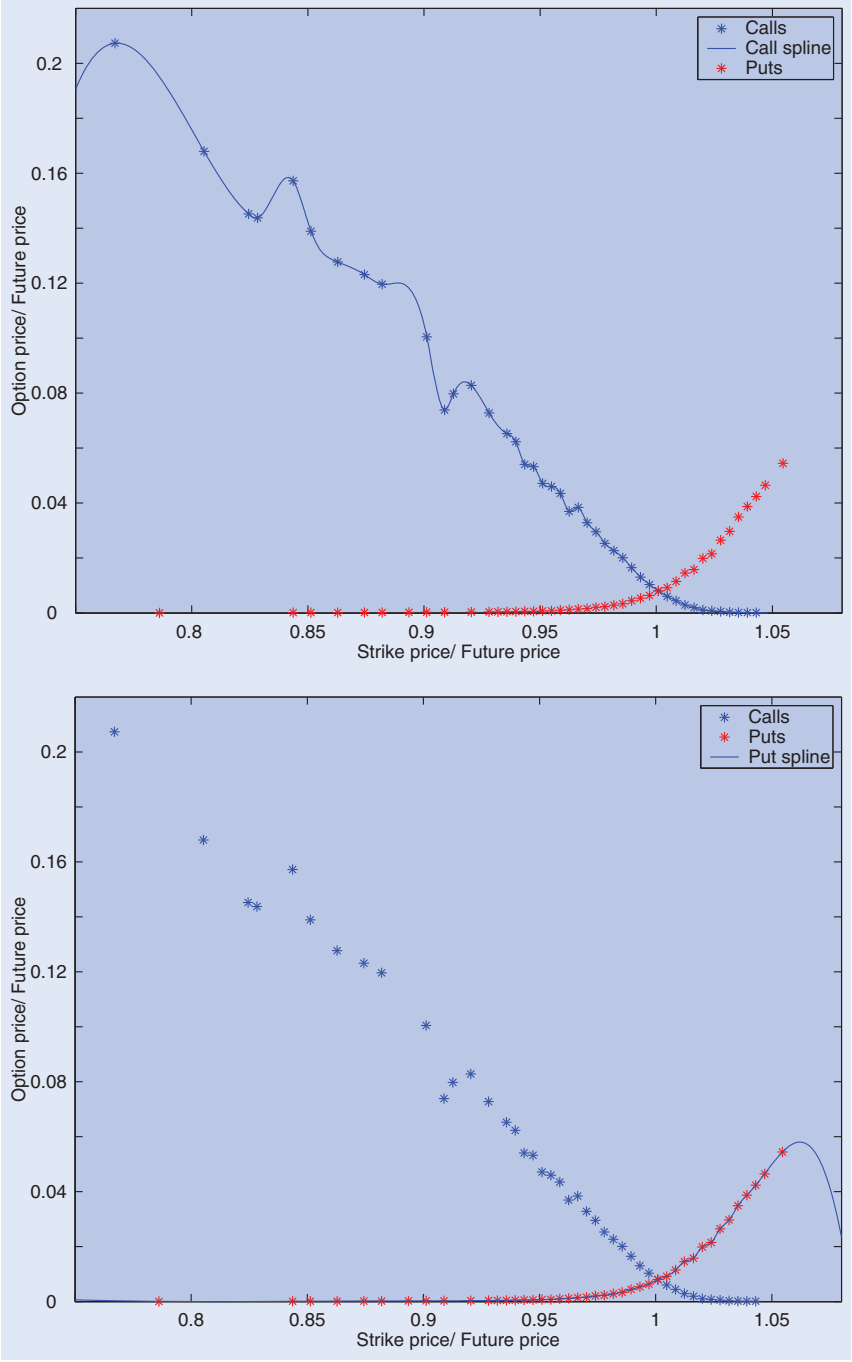


Figure 1. Observed call and put prices on the S&P500 on 08/31/2006.

Under symmetry ($\beta = -0.5$), the skewness premium is given by Bates' x% rule, that is $SK(x) = x$. Under the absence of symmetry, $SK(x)$ can be less or greater than x, so we define the excess of SK as the difference $SK(x) - x$.

The excess of SK was addressed by Bates (1997) for the following process classes: constant elasticity of variance (CEV), stochastic volatility (the benchmark models being those for which volatility evolves independently of the asset price), and jump-diffusion (the benchmark model being that of Merton (1976)). For these classes, he obtained the following result.

Proposition 4.2: (Bates 1997)**:** *For European options in general and for American options on futures, the SK has the following properties for the above distributions*:

(i) $SK(x) \lesseqgtr x$ *for CEV processes with* $\rho \lesseqgtr 1$;
(ii) $SK(x) \lesseqgtr x$ *for jump-diffusions with log-normal jumps depending on whether* $2\mu + \delta^2 \lesseqgtr 0$; *and*
(iii) $SK(x) \lesseqgtr x$ *for stochastic volatility processes depending on whether* $\rho_{S\sigma} \lesseqgtr 0$.

Bates (1997) presents proofs for (i), but only numerical arguments for (ii) and (iii). Below we give a proof for (ii). Unfortunately, we do not have a proof for (iii), but many authors conjecture that it must be true (Cont and Tankov 2004, section 15.1.1).

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Table 1. Interpolating prices of put options.

K_c	$K_p = F^2/K_c$	$x = K_c/F - 1$	$x_{obs} = c_{obs}/p_{int} - 1$	$x - x_{obs}$
1230	1382.07	−0.05662	0.050681	−0.1073
1235	1376.475	−0.05278	0.13642	−0.1892
1240	1370.925	−0.04895	0.115006	−0.16395
1245	1365.419	−0.04511	0.197696	−0.24281
1250	1359.957	−0.04128	0.277944	−0.31922
1255	1354.539	−0.03744	0.280729	−0.31817
1260	1349.164	−0.03361	0.536286	−0.5699
1265	1343.831	−0.02977	0.574983	−0.60476
1270	1338.541	−0.02594	0.606719	−0.63266
1275	1333.291	−0.0221	0.675372	−0.69748
1280	1328.083	−0.01827	0.691325	−0.70959
1285	1322.916	−0.01443	0.966306	−0.98074
1290	1317.788	−0.0106	0.904839	−0.91544
1295	1312.7	−0.00676	0.794059	−0.80082
1300	1307.651	−0.00293	0.78018	−0.78311
1305	1302.641	0.000905	0.614561	−0.61366
1310	1297.669	0.00474	0.532798	−0.52806
1315	1292.735	0.008575	0.427299	−0.41872
1320	1287.838	0.01241	0.108911	−0.0965
1325	1282.979	0.016245	−0.11658	0.132826
1330	1278.155	0.020079	−0.45097	0.471053
1335	1273.368	0.023914	−0.50378	0.527697
1340	1268.617	0.027749	−0.61306	0.640807
1345	1263.901	0.031584	−0.73872	0.770305
1350	1259.22	0.035419	−0.81448	0.849896
1355	1254.573	0.039254	−0.80297	0.842224
1360	1249.961	0.043089	−0.82437	0.867454

Table 2. Interpolating prices of call options.

K_p	$K_c = F^2/K_p$	$x = F/K_p - 1$	$x_{obs} = c_{int}/p_{obs} - 1$	$x - x_{obs}$
1250	1359.957	0.043056	−0.88837	0.931421
1255	1354.539	0.0389	−0.86897	0.907873
1260	1349.164	0.034778	−0.85655	0.891331
1265	1343.831	0.030688	−0.78107	0.81176
1270	1338.541	0.02663	−0.70531	0.731941
1275	1333.291	0.022604	−0.63926	0.661869
1280	1328.083	0.018609	−0.51726	0.535865
1285	1322.916	0.014646	−0.31216	0.326801
1290	1317.788	0.010713	−0.20329	0.214005
1295	1312.7	0.006811	−0.03659	0.043397
1300	1307.651	0.002938	0.090739	−0.0878
1305	1302.641	−0.0009	0.130843	−0.13175
1310	1297.669	−0.00472	0.252541	−0.25726
1315	1292.735	−0.0085	0.261905	−0.27041
1320	1287.838	−0.01226	0.242817	−0.25507
1325	1282.979	−0.01598	0.346419	−0.3624
1330	1278.155	−0.01968	0.183207	−0.20289
1335	1273.368	−0.02336	0.237999	−0.26135
1340	1268.617	−0.027	0.145858	−0.17286
1345	1263.901	−0.03062	0.152637	−0.18325
1350	1259.22	−0.03421	0.101211	−0.13542
1355	1254.573	−0.03777	−0.03964	0.001869
1360	1249.961	−0.04131	0.028337	−0.06965
1365	1245.382	−0.04482	−0.0101	−0.03472
1375	1236.325	−0.05177	−0.0451	−0.00667

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Now based on the results presented in the last section, we can extend Bates' results to a more general class of Lévy processes.

Corollary 4.3: *If* $\beta \gtrless -1/2$ *and the Lévy measure is given by* (16), *under assumption* 3.2 *we have*

$$SK(x) \gtrless x, \quad x > 0. \tag{24}$$

Proof: The proof follows from theorem 3.3. □

Corollary 4.4: *Under assumption* 3.2, *if* $2\mu + \delta^2 \leqslant 0$, *then* $SK(x) \leqslant x$ *for jump-diffusions with log-normal jumps.*

Proof: Consider the jump-diffusion model proposed by Merton (1976). The driving Lévy process in this model has the Lévy measure given by

$$\Pi(\mathrm{d}y) = \lambda \frac{1}{\delta\sqrt{2\pi}} \mathrm{e}^{-(y-\mu)^2/(2\delta^2)} \mathrm{d}y,$$

since, in this model, the Lévy measure can be expressed as (16), with $\beta = \mu/\delta^2$. Then, applying theorem 3.3, the result follows. □

For a more general distribution in the jump-diffusion model, i.e. $\Pi(\mathrm{d}y) = \lambda F(\mathrm{d}y)$, we have the following proposition.

Proposition 4.5: *Under the covariance inequality*,† *if* F *is such that* (*i*) F *is symmetric around zero and* (*ii*) $\int \mathrm{e}^y F(\mathrm{d}y) > 1$, *then* $SK(x) > x$.

Proof: Since $\widetilde{\Pi} = \mathrm{e}^{-y}\Pi(-\mathrm{d}y) = \lambda \mathrm{e}^{-y} F(-\mathrm{d}y)$, define $\widetilde{\lambda} = \lambda \int \mathrm{e}^{-y} F(-\mathrm{d}y)$ and $\widetilde{F}(\mathrm{d}y) = \mathrm{e}^{-y} F(-\mathrm{d}y) / \int \mathrm{e}^{-y} F(-\mathrm{d}y)$. Then $\widetilde{\Pi}(\mathrm{d}y) = \widetilde{\lambda}\widetilde{F}(\mathrm{d}y)$. Now take a convex function g such that the integrals given below exist. Applying the covariance inequality, we obtain

$$\begin{aligned} \int g(y)\lambda F(-\mathrm{d}y) &= \int g(y)\mathrm{e}^{-y}\mathrm{e}^{y}\lambda F(-\mathrm{d}y) \\ &\geq \int g(y)\mathrm{e}^{-y}\lambda F(-\mathrm{d}y) \int \mathrm{e}^{y} F(-\mathrm{d}y) \\ &= \int g(y)\widetilde{\lambda}\widetilde{F}(\mathrm{d}y) \int \mathrm{e}^{y} F(-\mathrm{d}y). \end{aligned}$$

Using the properties of F, we have

$$\begin{aligned} \int g(y)\lambda F(\mathrm{d}y) &\geq \int g(y)\widetilde{\lambda}\widetilde{F}(\mathrm{d}y) \int \mathrm{e}^{y} F(\mathrm{d}y) \\ &> \int g(y)\widetilde{\lambda}\widetilde{F}(\mathrm{d}y). \end{aligned}$$

Then by theorem 2.2 of Bergenthum and Rüschendorf (2007), we obtain an order in option prices. Using the put–call duality the result follows. □

Remark 2 :

- Observe that, in Merton's model, as $\beta = \mu/\delta^2$:

$$\int \mathrm{e}^{-y} F(-\mathrm{d}y) = \mathrm{e}^{\mu + \delta^2/2} \gtrless 1 \Longleftrightarrow \beta \gtrless -1/2.$$

†The covariance inequality states that, given a random vector X and two functions f and g, under certain conditions we have $\mathrm{Cov}(f(X), g(X)) \geq 0$. More details on conditions for the covariance inequality to be satisfied can be found in Houdré and Pérez-Abreu (1995) and Houdré *et al.* (1998).

- For more general F: if F is symmetric around zero, then $\Pi(\mathrm{d}y) = \lambda F(\mathrm{d}y)$ belongs to class (19), with $\beta = 0$. Under assumption 3.2, by corollary 4.3, we have $SK(x) > x$.
- Moreover, for $\mu < 0$ fixed, when β increases, δ^2 necessarily also increases, resulting in an increase of the option price, validating our assumption 3.2.

In the next section we present a discussion of assumption 3.2.

5. Monotonicity and symmetry parameter

As seen in the last section, we need the monotonicity of option prices with respect to the symmetry parameter to obtain our main result. The literature contains extensive studies on the monotonicity properties of option prices. The main idea is to exploit the *convexity-preserving property*,† to obtain the monotonicity of option prices with respect to certain parameters of the model (Bergman *et al*. 1996, El Karoui *et al*. 1998, Ekström and Tysk 2007).

However, we are interested in the possible misspecifications in the models when using a fixed equivalent martingale measure. That is, if we change the parameter β in the Lévy measure described by (16), what happens to the option price? Unfortunately, we do not have a result that guarantees the validity of assumption 3.2. In that sense, the results obtained by Bergenthum and Rüschendorf (2007) and the use of the covariance inequality can give some insight, as shown in proposition 4.5.

Also, for stochastic volatility models we only have numerical evidence of the monotonicity properties of option prices with respect to the correlation parameter. We conclude that this is not an easy task and is left as an open question.

6. Conclusions

Under a given risk-neutral probability measure, we use the measure of *symmetry* of a Lévy market model introduced by Fajardo and Mordecki (2006) to address the skewness premium in the absence of symmetry. In that case we derive sufficient conditions for the excess of SK to be positive in terms of the structure of the Lévy measure, in particular the symmetry parameter. We have also presented a result using the covariance inequality instead of the monotonicity assumption. Clearly, there is a connection between these assumptions, as Merton's jump-diffusion model showed. We leave the study of monotonicity of option prices with respect to the symmetry parameter and its relation with covariance inequality for future research.

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†We say that a model is *convexity preserving* if, for any convex contract function, the corresponding price is convex as a function of the price of the underlying asset at all times prior to maturity. Many models do not satisfy this property, for example general stochastic volatility models.

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