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Hierarchical Cucker-Smale Model Subject to Random Failure

Federico Dalmao and Ernesto Mordecki

***Abstract*—Consider a system of autonomous interacting agents in $\mathbb{R}^3$, where each agent adjusts its velocity according to its own past velocity and a weighted mean of the relative velocities of its superiors in a certain prescribed hierarchy. Furthermore, each agent, at each time step, can fail to see any of its superiors with a certain probability, the failure rate probability. We prove that for agent interactions with linear or sub-linear distance dependent rate of decay, the flocking results valid for the non-failure deterministic model introduced by Shen [8] also hold true, independently of the failure rate probability.**

***Index Terms*— Cucker–Smale flocking.**

I. Introduction and Main Result

The mathematical modeling of collective behavior in a group of autonomous interacting agents arises in many different fields, a central question being the asymptotic behavior of the system. A relevant contribution in this field was recently made by Shen in the article "Cucker-Smale flocking under hierarchical leadership" [8], where he introduces a variant of the seminal Cucker-Smale model [3], considering hierarchical rules of connectivity, instead of the "all against all" connectivity scheme of [3], preserving the interaction strength between pairs of agents as a power decaying function of its relative distance. The relevance of Cucker and Smale proposal relies on the fact that, maintaining essential characteristics of the modeled situation (a flock of birds in their work), and letting aside others (as the volume of the agents or its difference in weights), gives the possibility of a mathematical treatment of the behavior of the system.

Observe that Cucker and Smale propose a *symmetric* interactive model, with all agents playing an identical rôle, whereas Shen ranks the agents in a leadership scheme, covering different applications, as he discuses in [8]. Furthermore, the symmetric model has been largely considered (see [1], [4]–[6] and the references therein), while the hierarchical case has been less studied, to our knowledge the only reference being [2].

In this technical note, departing from the Hierarchical Cucker-Smale model proposed by Shen in [8], we analyze the situation in which each interaction is subject to random failure. Our main result states that the convergence type result obtained in [8] holds in our framework, with probability one, in the case of agent interactions with linear or sub-linear distance dependent rate of decay ($0 \leq \alpha \leq 1$, see Theorem 1).

Consider then a system of k agents with positions and velocities denoted respectively by $X = (X_1, \ldots, X_k)$ and $V = (V_1, \ldots, V_k)$. All individual positions and velocities are vectors in $\mathbb{R}^3$, and we refer to X and V as the position and velocity of the system respectively.

Assume that the system evolves under hierarchical leadership, following the discrete-time dynamic:

$$\begin{cases} X_i(t+h) = X_i(t) + hV_i(t) \\ V_i(t+h) = V_i(t) + h\sum_{j\in\mathcal{L}(i)} a_{ij}\,(V_j(t) - V_i(t)) \end{cases} \tag{1}$$

where $i = 1, \ldots, k$, $h > 0$ is the time step, a_{ij} are the weighting coefficients, $a_{ij} = 0$ if $j \geq i$, and $\mathcal{L}(i) = \{j : a_{ij} > 0\}$ is the leader set of the agent i, assumed to be fixed in time and non empty for $i \geq 2$. Model (1) was introduced by Shen in [8] under the denomination of *Cucker-Smale model under Hierarchical Leadership*.

In this framework we introduce the possibility that each agent i, at each time step, can fail to see any of its superiors in the hierarchy, in the set $\mathcal{L}(i)$. These failures are assumed to be random, independent with a fixed failure rate probability $\lambda \in (0, 1)$. More precisely, the weighting coefficients in (1), for each pair of agents $j < i$ with $j \in \mathcal{L}(i)$, are given by

$$a_{ij}(t) = \zeta_t^{ij} \frac{1}{(1 + ||X_i(t) - X_j(t)||)^\alpha} \tag{2}$$

where $\zeta_t^{ij} (t \geq 0, j \in \mathcal{L}(i))$ are independent and identically distributed Bernoulli random variables with success probability $1 - \lambda$, i.e.

$$\mathbb{P}\left(\zeta_t^{ij} = 1\right) = 1 - \lambda, \qquad \mathbb{P}\left(\zeta_t^{ij} = 0\right) = \lambda.$$

Here $\mathbb{P}$ and $\mathbb{E}$ denote probability and expectation respectively, and the (random) event $\zeta_t^{ij} = 0$ (resp. $= 1$) means that agent i fails to (resp. does) see agent j at time step t.

The factor $(1 + ||X_i(t) - X_j(t)||)^{-\alpha}$ in (2), with $0 \leq \alpha \leq 1$ and $||X_i(t) - X_j(t)||$ the Euclidean distance between agents i and j, is the Cucker-Smale coefficient introduced in [3].

The purpose of our work is to find conditions under which, for large times, all agent velocities become equal, with fixed relative positions. Following Cucker and Smale [3] and Shen [8], this asymptotic behavior is called *flocking*. Observe that, as a consequence of the presence of random failures, we can not expect to have flocking in all possible realizations, as in the case when all the connections fail all the time. This is a natural situation when dealing with convergence of random vari-

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ables (the random positions and velocities of the flock in our case). There exist several modes of convergence of random variables, one of the usual and strongest is the almost sure convergence (which is the mode of convergence involved in the Strong Law of Large Numbers). We take this approach (see Definition 1), and in particular we prove in Theorem 1 that the set of realizations where flocking does not occur has zero probability. (For the sake of brevity sometimes we omit the probabilistic "almost sure" abbreviation "a.s." in the corresponding sentence).

In order to simplify the treatment and following [2] and [8] we introduce relative coordinates. As $\mathcal{L}(1) = \emptyset$, it follows that $V_1(t) = V_1(0)$ for all t, so, if we have flocking, all limiting velocities become equal to $V_1(0)$. As a consequence of this fact we introduce a system of coordinates centered in $X_1(t)$ that travels with velocity $V_1(0)$. Denoting the positions of the agents and its velocities in this new system respectively by $x = (x_1, \ldots, x_k)$ and $v = (v_1, \ldots, v_k)$, for each agent i we have

$$x_i(t) = X_i(t) - X_1(t), \qquad v_i(t) = V_i(t) - V_1(0).$$

A relevant observation is that in the new system of coordinates the form of the equations in (1) is preserved. In fact, the new equations are

$$\begin{cases} x_i[t+1] = x_i[t] + h v_i[t], \\ v_i[t+1] = v_i[t] + h \sum_{j \in \mathcal{L}(i)} a_{ij} \left(v_j[t] - v_i[t] \right) \end{cases} \tag{3}$$

where we write $x[t]$ and $v[t]$ instead of $x(th)$ and $v(th)$ respectively. We proceed consistently with all other functions of t, and when writing "for all t" we mean "for all th with $t = 0, 1, \ldots$".

An alternative way to write the velocity equation in (3) is

$$v_i[t+1] = \left(1 - h \sum_{j \in \mathcal{L}(i)} a_{ij} \right) v_i[t] + h \sum_{j \in \mathcal{L}(i)} a_{ij} v_j[t] \tag{4}$$

that, under the natural condition

$$h \leq \frac{1}{k-1} \tag{5}$$

(that makes the first coefficient in the r.h.s. of (4) always positive), implies that the velocity at $t+1$ is a linear convex combination of the system velocities at time t. As a consequence of this fact, we obtain that

$$\max_{1 \leq i \leq k} \|v_i[t+1]\| \leq \max_{1 \leq i \leq k} \|v_i[t]\| \leq \max_{1 \leq i \leq k} \|v_i[0]\| \tag{6}$$

as was noticed in [2]. Consequently, in what follows we assume condition (5), under which (6) holds, and also introduce the following norms for the position x and velocity v of the system

$$\|x[t]\| = \max_{1 \leq i \leq k} \|x_i[t]\|, \qquad \|v[t]\| = \max_{1 \leq i \leq k} \|v_i[t]\|.$$

Due to the fact that the coordinates are relative to the first agent, the norm $\|x\|$ is a measure of the disperssion of the system, whereas $\|v\|$ measures its departure from flocking. These facts motivate the following definition.

Definition 1: In a dynamical system governed by the system of (1), with weighting coefficients given in (2), we have *flocking* when the following two conditions are (almost surely) fulfilled:

$$\|v[t]\| \to 0 \quad and \quad x[t] \to \widehat{x}, \quad when \quad t \to \infty$$

for some $\widehat{x} \in \mathbb{R}^{3k}$, i.e. the relative velocities asymptotically vanish and the relative positions converge.

Remark 1: It is clear that the condition $\sum_t \|v[t]\| < \infty$ a.s. ensures flocking, as does the condition $\|v[t]\| \leq N e^{-\nu t}$ a.s. for all t, for some random variables N and $\nu > 0$ independent of t.

We turn now to the main result of this note.

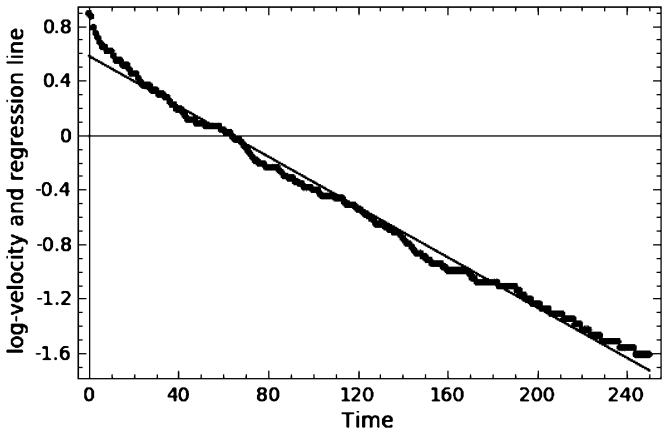


Fig. 1. Logarithm of the system velocity and regression line. The slope of the line is the rate ν.

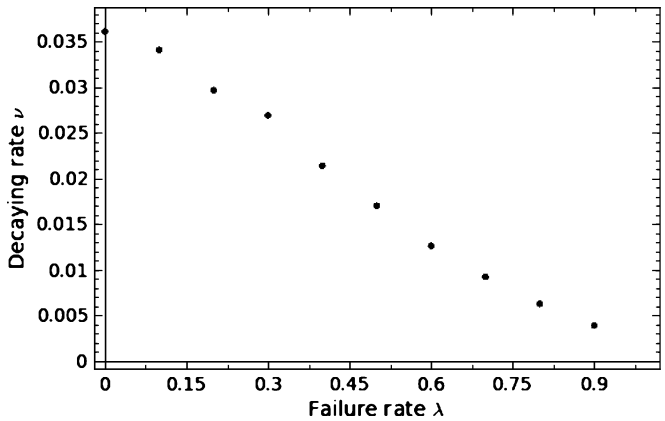


Fig. 2. Decaying rate ν as a function of the failure probability λ.

Theorem 1: Consider a system following the dynamic described in (1) with interaction coefficients subject to random failure given in (2). Assume (5). Then, under either

1) $\alpha < 1$;
2) $\alpha = 1$ and $\|v[0]\| < (1 - \lambda)/4$;

we have flocking. More precisely, there exist random variables N and $\nu > 0$, which do not depend on t, such that we have

$$\|v[t]\| \leq N e^{-\nu t} \quad \text{for all} \quad t. \tag{7}$$

Relevant applications of flocking results can be found in different areas of science and technology. For instance, Cucker-Smale model can be seen as a control law to keep the cohesion of a set of flying space crafts, as in the application to the Darwin mission in [7]. Other possible applications in Robotics, Control, Biology, Ecology, Economy and Language are discussed by Shen [8]. The flocking results presented in this note, where each connection is subject to random failure, can be considered as a robustness result on the control law, that meets natural application in practically all the situations described above. Further research involve the consideration of the case $\alpha > 1$, that does not seem to be covered by our approach.

II. Numerical Simulations

Consider a system of $k = 10$ agents, time step $h = 1/(2(k-1)) = 1/18$, $\alpha = 1/2$ and $t = 1, \ldots, 250$. Each coordinate of the initial position and velocity is sampled as an independent standard normal variable. In Fig. 1, for one simulated trajectory of the system with $\lambda = 0.5$, we plot $\log \|v[t]\|$ to see how the velocity of the system behaves in time. Performing a linear regression we estimate the decaying rate ν as the slope of the regression line. In Fig. 2 and Table I

TABLE I
DECAYING RATE ν AS A FUNCTION OF THE FAILURE PROBABILITY λ

λ	$\nu_{\min}$	$\overline{\nu}$	$\nu_{\max}$	σ_ν
0.0	3.61E-2	**3.61E-2**	3.61E-2	0.E+0
0.1	2.67E-2	**3.41E-2**	3.10E-2	9.89E-4
0.2	2.24E-2	**2.96E-2**	2.62E-2	1.24E-3
0.3	1.70E-2	**2.69E-2**	2.16E-2	1.33E-3
0.4	1.34E-2	**2.14E-2**	1.73E-2	1.32E-3
0.5	8.93E-3	**1.70E-2**	1.34E-2	1.16E-3
0.6	7.13E-3	**1.26E-2**	9.87E-3	1.03E-3
0.7	4.69E-3	**9.20E-3**	6.92E-3	8.13E-4
0.8	2.46E-3	**6.26E-3**	4.28E-3	6.26E-4
0.9	6.79E-4	**3.88E-3**	2.08E-3	4.72E-4

we investigate the variation of the decaying rate ν of Theorem 1 when $\lambda = 0, 1/10, \ldots, 9/10$, performing 500 simulations for each case.

III. PROOF

In order to prove Theorem 1 we adapt the arguments used in the non-failure case by Shen in [8] and afterwards by Cucker and Dong in [2]. The key idea is to use mathematical induction in the sub-systems $\{1, \ldots, \ell\}$ $(2 \leq \ell \leq k)$.

Roughly speaking, the induction relies on the fact that for each ℓ the sub-system $\{1, \ldots, \ell - 1\}$ is also a system under hierarchical leadership. Therefore, when the latter sub-system has almost reached a common velocity, the system $\{1, \ldots, \ell\}$ can be considered as a two agents system replacing agents $1, \ldots, \ell - 1$ by a single "mean" agent, at the cost of adding a noisy term in the equations. Proposition 1 deals with this noisy situation.

Furthermore, Proposition 1 shows, for a two-agent flock, that in spite of the fact that at some time steps agents may fail to interact, in the long term, the proportion of failures will be small enough as to prevent flocking to occur. Theorem 1 shows that this fact carries on to the full flock.

Proposition 1: Assume that the relative position $x[t]$ and velocity $v[t]$ in $\mathbb{R}^3$ of a two agents system verify the equations

$$\begin{cases} x[t+1] = x[t] + hv[t], \\ v[t+1] = (1 - ha[t])\, v[t] + b[t] \end{cases} \tag{8}$$

subject to the following conditions:

(P1) The random variables $\zeta_t := \mathbf{1}_{\{a[t] \neq 0\}}$ (where $\mathbf{1}_A$ is the characteristic function of the set A) are mutually independent with common Bernoulli distribution and parameter $p \in (0, 1)$.

(P2) There exist a constant $\alpha \in [0, 1]$ and a positive random variable ξ such that, in case $a[t] \neq 0$, we have

$$\frac{1}{(1 + \xi + \|x[t]\|)^\alpha} \leq a[t] \leq h^{-1}$$

for all t.

(P3) The velocities remain bounded: there exists v_0 such that $\|v[t]\| \leq v_0$ for all t.

(P4) There exist random variables B and $\beta > 0$, independent of t, such that $\|b[t]\| \leq Be^{-\beta t}$ for all t.

Then, if any of the following conditions hold:

$$1)\ 0 \leq \alpha < 1, \qquad 2)\ \alpha = 1 \quad \text{and} \quad v_0 < \frac{p}{2}$$

there exist random variables N and $\nu > 0$, independent of t, such that

$$\|v[t]\| \leq Ne^{-\nu t} \quad \textit{for all} \quad t.$$

The first step in the proof of Proposition 1 is presented in the following Lemma.

Lemma 1: Under the conditions of Proposition 1, given $0 < t_0 < t$, there exist random variables ρ_t and α_t such that $0 < \rho_t < 1$, α_t has binomial distribution with parameters $(t - t_0, p)$, and

$$\|v[t]\| \leq \rho_t^{\alpha_t} v_0 + \frac{Be^{-\beta t_0}}{1 - e^{-\beta}}. \tag{9}$$

Furthermore, denoting $\|x_*\| = \|x[t_*]\| := \max_{0 \leq \tau \leq t} \|x[\tau]\|$, and given $0 < \gamma < 1$, there exists a positive random variable τ_0 independent of t, such that

$$\|x_*\| \leq \|x[0]\| + hv_0\tau_0 + \frac{2hB}{(1 - e^{-\beta})^2} + \frac{2v_0}{p(1-\gamma)} (1 + \xi + \|x_*\|)^\alpha. \tag{10}$$

Proof of Lemma 1: Let $\rho_t = 1 - (h/((1 + \xi + \|x_*\|)^\alpha))$. Clearly, we have $0 < \rho_t < 1$ for all t.

We begin with the proof of the statement (9) about the velocity. By the second equation of (8), using (P2), we have

$$\begin{aligned} \|v[t]\| &\leq (1 - ha[t-1]) \|v[t-1]\| + \|b[t-1]\| \\ &\leq \rho_t^{\zeta_t - 1} \|v[t-1]\| + \|b[t-1]\| \end{aligned}$$

since $\zeta_{t-1} = 0$ if and only if the factor $1 - ha[t-1]$ is equal to one. Iterating this argument, applying (P3), and bounding the sum that appears we obtain

$$\|v[t]\| \leq \rho_t^{\zeta_{t-1} + \cdots + \zeta_{t_0}} v_0 + \sum_{\tau = t_0}^{t-1} \|b[\tau]\| \leq v_0 \rho_t^{\alpha_t} + \frac{Be^{-\beta t_0}}{1 - e^{-\beta}}$$

that is condition (9), where $\alpha_t = \zeta_{t-1} + \ldots + \zeta_{t_0}$.

We turn now to (10). Applying the bound for the velocity (9) with $t = \tau$ and $t_0 = [\tau/2]$, it follows that:

$$\begin{aligned} \|x_*\| &\leq \|x[0]\| + \sum_{\tau=0}^{t-1} \|x[\tau+1] - x[\tau]\| \leq \|x[0]\| + h \sum_{\tau=0}^{t-1} \|v[\tau]\| \\ &\leq \|x[0]\| + hv_0 \sum_{\tau=0}^{t-1} \rho_t^{\alpha_\tau} + \frac{2hB}{(1 - e^{-\beta})^2}. \end{aligned} \tag{11}$$

In what respects the sum in (11), one expects α_t to behave asymptotically as $pt/2$. We claim that given $0 < \gamma < 1$ there exists a random time τ_0 such that

$$\frac{\alpha_t}{t} \geq (1 - \gamma)\frac{p}{2} \quad \text{for all} \quad t \geq \tau_0. \tag{12}$$

Indeed, (12) follows by a direct application of Borel-Cantelli Lemma (see IISection 10 in [9]), since for $0 < \gamma < 1$ we have

$$\begin{aligned} \mathbb{P}\left(\alpha_t < (1 - \gamma)\frac{pt}{2}\right) &\leq \mathbb{P}\left(|\alpha_t - \mathbb{E}\alpha_t| \geq p\left(\frac{\gamma t}{2} - 1\right)\right) \\ &\leq \frac{1}{\left(p\left(\gamma t/2 - 1\right)\right)^4} \mathbb{E}\left(\alpha_t - \mathbb{E}\alpha_t\right)^4 \\ &\leq \frac{Ct^2}{\left(p\left(\gamma t/2 - 1\right)\right)^4} \end{aligned} \tag{13}$$

and the r.h.s in (13) is the term of a convergent series.

We may now finish the proof of Lemma 1. From inequality (12) it follows that $\rho_t^{\alpha_\tau} < \rho_t^{p(1-\gamma)\tau/2}$ for $\tau \geq \tau_0$. Then

$$\begin{aligned} \sum_{\tau=1}^{\infty} \rho_t^{\alpha_\tau} &\leq \tau_0 + \sum_{\tau=1}^{\infty} \left(\rho_t^{\frac{p(1-\gamma)}{2}}\right)^\tau \leq \tau_0 + \left(1 - \rho_t^{\frac{p(1-\gamma)}{2}}\right)^{-1} \\ &\leq \tau_0 + \frac{2\left(1 + \xi + \|x_*\|\right)^\alpha}{p(1-\gamma)h} \end{aligned} \tag{14}$$

where we used that

$$\left(1-\frac{h}{(1+\xi+\|x_*\|)^\alpha}\right)^{\frac{p(1-\gamma)}{2}} \le 1-\frac{p(1-\gamma)h}{2(1+\xi+\|x_*\|)^\alpha}.$$

Plugging (14) into (11) we obtain (10), concluding the proof of Lemma 1. ∎

Proof of Proposition 1: Denote $z=1+\xi+\|x_*\|$, $c_1=2v_0/(p(1-\gamma))$, and $c_2=1+\xi+\|x[0]\|+hv_0\tau_0+(2hB/((1-e^{-\beta})^2))$. (Observe that c_1 is non random.) Equation (10) can then be written as

$$z \le c_2 + c_1 z^\alpha. \tag{15}$$

Consider first $0<\alpha<1$. By Lemma 2 of [3], applied with $s=1, q=\alpha$, we have $1+\xi+\|x_*\| \le U_0$, where $U_0=\max\{(2c_1)^{1/(1-\alpha)}, 2c_2\}$. Since U_0 is independent of t, it follows that $\|x[t]\| \le U_0-\xi-1$ for all t, obtaining that the position is uniformly bounded from above in time. In case $\alpha=0$, from (15) we have the same result with $U_0=c_1+c_2$.

We turn our attention to the velocity. Taking into account that $\rho_t \le 1-(h/U_0^\alpha)$ for all t, from (9) with $t_0=[t/2]$, we obtain that

$$\|v[t]\| \le v_0\rho_t^{\alpha_t} + \frac{Be^{-\beta t/2}}{e^\beta-1} \le v_0\left(1-\frac{h}{U_0^\alpha}\right)^{\frac{p(1-\gamma)t}{2}} + \frac{Be^{-\beta t/2}}{e^\beta-1}$$

for $t \ge \tau_0$, obtaining the existence of positive random variables N and $\nu>0$, independent of t, such that

$$\|v[t]\| \le Ne^{-\nu t} \quad \text{for} \quad t \ge \tau_0. \tag{16}$$

Case $\alpha=1$ is treated similarly to the previous one. As γ can be taken arbitrarily close to one, under the condition $v_0 < (p/2)$ we select γ such that $c_1<1$. From (15) we then obtain that $z \le c_2/(1-c_1)$ and the rest of the proof follows as in the case $\alpha<1$. ∎

We are able now to prove Theorem 1.

Proof of Theorem 1: We proceed by induction in the sub-systems $\{1,\dots,\ell\}$ with $2 \le \ell \le k$. The proof is valid for all cases $0 \le \alpha \le 1$.

Consider then the sub-system {1,2}. By definition we have that the leader sets are $\mathcal{L}(1)=\emptyset$ and $\mathcal{L}(2)=\{1\}$. Let $x[t]=x_2[t]-x_1[t]$ and $v[t]=v_2[t]-v_1[t]=v_2[t]$. In relative coordinates (see (3)) the dynamic is

$$\begin{cases} x[t+1]=x[t]+hv[t], \\ v[t+1]=(1-ha_{21}[t])\,v[t] \end{cases} \tag{17}$$

where $a_{21}[t]=1/((1+\|x[t]\|)^\alpha)$. Hence, under (17) we are in the situation of Proposition 1 with $b[t]=0$ for all t. Condition (P1) is fulfilled with $p=1-\lambda$. The first inequality in (P2) is fulfilled with $\xi=0$, the second because $a_{21}[t] \le 1 \le h^{-1}$. To check condition (P3) we observe that this system corresponds to a system under hierarchical leadership, it follows from (6) that

$$\|v[t]\| \le \|v[0]\| := v_0 \quad \text{for all} \quad t.$$

Finally, condition (P4) is fulfilled with $B=0$. In conclusion, by Proposition 1, we deduce the existence of random variables N_2 and $\nu_2>0$ such that $\|v[t]\| \le N_2e^{-\nu_2 t}$ for all t.

We come now to the induction step. Assume that the result holds for the sub-system $\{1,\dots,\ell-1\}$ with $2 \le \ell \le k$, more precisely, assume the existence of random variables $N_{\ell-1}$ and $\nu_{\ell-1}>0$, independent of t. such that

$$\max_{1\le i\le \ell-1}\|v_i[t]\| \le N_{\ell-1}e^{-\nu_{\ell-1}t} \quad \text{for all} \quad t. \tag{18}$$

Our aim is to state similar bounds for the sub-system $\{1,\dots,\ell\}$. Following the idea of Cucker and Dong [2] we consider mean velocity and position of the leaders of agent ℓ:

$$\overline{x}_\ell[t]=\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}x_i[t], \qquad \overline{v}_\ell[t]=\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}v_i[t]$$

where $d_\ell=\#\mathcal{L}(\ell) \le \ell-1$, and define

$$x[t]=x_\ell[t]-\overline{x}_\ell[t], \qquad v[t]=v_\ell[t]-\overline{v}_\ell[t]. \tag{19}$$

We have

$$\begin{aligned} v[t+1] &= v_\ell[t+1]-\overline{v}_\ell[t+1] \\ &= v_\ell[t]+\sum_{i\in\mathcal{L}(\ell)}ha_{\ell i}\,(v_i[t]-v_\ell[t])-\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}v_i[t+1], \\ &= v_\ell[t]+\sum_{i\in\mathcal{L}(\ell)}ha_{\ell i}\,(\overline{v}_\ell[t]-v_\ell[t]) \\ &\quad +\sum_{i\in\mathcal{L}(\ell)}ha_{\ell i}\,(v_i[t]-\overline{v}_\ell[t]) \\ &\quad -\overline{v}_\ell[t]-\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\sum_{j\in\mathcal{L}(i)}ha_{ij}\,(v_j[t]-v_i[t]) \\ &= \left(1-\sum_{j\in\mathcal{L}(\ell)}ha_{\ell j}\right)v[t] \\ &\quad +\left[\sum_{i\in\mathcal{L}(\ell)}ha_{\ell i}\,(v_i[t]-\overline{v}_\ell[t])\right. \\ &\quad \left.-\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\sum_{j\in\mathcal{L}(i)}ha_{ij}\,(v_j[t]-v_i[t])\right]. \end{aligned}$$

Denoting

$$\begin{aligned} a[t] &= \sum_{j\in\mathcal{L}(\ell)}a_{\ell j}[t], \\ b[t] &= \sum_{j\in\mathcal{L}(\ell)}ha_{\ell j}\,(v_j[t]-\overline{v}_\ell[t]) \\ &\quad -\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\sum_{j\in\mathcal{L}(i)}ha_{ij}\,(v_j[t]-v_i[t]) \end{aligned} \tag{20}$$

we can rewrite (3) as

$$\begin{cases} x[t+1]=x[t]+hv[t], \\ v[t+1]=(1-ha[t])\,v[t]+b[t]. \end{cases}$$

Hence, we have a system of the form (8). Let us check that conditions (P1) to (P4) hold for this system. In view of (20), we have

$$\mathbb{P}(a[t]=0)=\mathbb{P}(a_{\ell j}[t]=0 \quad \text{for all} \quad j\in\mathcal{L}(\ell))=\lambda^{d_\ell}$$

then (P1) holds with $p=1-\lambda^{d_\ell}$.

We turn into (P2), the estimation of the coefficient $a[t]$. From (20), since $a_{ij} \le 1$ we have $a[t] \le d_\ell \le k-1 \le h^{-1}$ for all t. For the lower bound, assuming $a[t] \ne 0$, we have

$$\begin{aligned} a[t] &= \sum_{j\in\mathcal{L}(\ell)}X_t^{\ell j}\frac{1}{(1+\|x_\ell[t]-x_j[t]\|)^\alpha} \\ &\ge \sum_{j\in\mathcal{L}(\ell)}X_t^{\ell j}\frac{1}{\left(1+\max\limits_{1\le j\le \ell-1}\|x_j[t]-x_\ell[t]\|\right)^\alpha} \\ &\ge \frac{1}{\left(1+\max\limits_{1\le j\le \ell-1}\|x_j[t]-x_\ell[t]\|\right)^\alpha}. \end{aligned} \tag{21}$$

Now, we write

$$\begin{aligned}\|x_j[t]-x_\ell[t]\| &\le \|x_\ell[t]-\overline{x}_\ell[t]\|+\|x_j[t]-\overline{x}_\ell[t]\| \\ &\le \|x[t]\|+\|x_j[t]-\overline{x}_\ell[t]\|. \end{aligned} \tag{22}$$

Besides

$$\begin{aligned}\|x_j[t]-\overline{x}_\ell[t]\| &\le \|x_j[0]-\overline{x}_\ell[0]\| \\ &\quad +\sum_{\tau=0}^{t-1}\|(x_j[\tau+1]-\overline{x}_\ell[\tau+1])-(x_j[\tau]-\overline{x}_\ell[\tau])\| \\ &= \|x_j[0]-\overline{x}_\ell[0]\|+h\sum_{\tau=0}^{t-1}\|v_j[\tau]-\overline{v}_\ell[\tau]\|.\end{aligned}$$

By the definition of $\overline{v}_\ell[t]$ and induction hypothesis (18) we have

$$\|v_j[\tau]-\overline{v}_\ell[\tau]\| \le \frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\|v_j[t]-v_i[t]\| \le 2N_{\ell-1}e^{-\nu_{\ell-1}t}$$

We then define ξ by

$$\begin{aligned}\|x_j[t]-\overline{x}_\ell[t]\| &\le \|x_j[0]-\overline{x}_\ell[0]\|+h\sum_{\tau=0}^{t-1}2N_{\ell-1}e^{-\nu_{\ell-1}\tau} \\ &\le \max\{\|x_j[0]-\overline{x}_\ell[0]\|\}+\frac{2hN_{\ell-1}}{1-e^{-\nu_{\ell-1}}}=:\xi. \end{aligned} \tag{23}$$

Hence, substituting (23) into (22) and the result into (21), for $a[t]\neq 0$, we have

$$a[t] \ge \frac{1}{(1+\xi+\|x[t]\|)^\alpha}$$

and $a[t]$ verifies (P2) with ξ defined in (23).

To check (P3) we observe that $\|v_k[t]\|$ and $\|\overline{v}_k[t]\|$ are bounded by $\|v[0]\|$ by (6), so by (19) we obtain $\|v[t]\| \le 2\|v[0]\| =: v_0$.

Let us finally verify (P4), the bound on $b[t]$. We have

$$\begin{aligned}\|b[t]\| &= \Bigg\| \sum_{j\in\mathcal{L}(\ell)} h a_{\ell j}\,(v_j[t]-\overline{v}_\ell[t]) \\ &\qquad -\frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\sum_{j\in\mathcal{L}(i)} h a_{ij}\,(v_j[t]-v_i[t])\Bigg\| \\ &\le \frac{1}{d_\ell}\sum_{i\in\mathcal{L}(\ell)}\Bigg\|\sum_{j\in\mathcal{L}(i)} h(a_{\ell i}-a_{ij})\,(v_i[t]-v_j[t]) \\ &\qquad + \sum_{j\in\{\mathcal{L}(\ell)\setminus\mathcal{L}(i)\}} h a_{\ell i}\,(v_i[t]-v_j[t])\Bigg\| \\ &\le 2N_{\ell-1}e^{-\nu_{\ell-1}t}\end{aligned}$$

for all t, in view of (18), as $|a_{\ell i}-a_{ij}|\le 1$.

Hence, conditions (P1) to (P4) are fulfilled. By Proposition 1 there exist random variables $N',\nu'>0$ independent of t such that $\|v[t]\| \le N'e^{-\nu' t}$ for all t. Therefore

$$\|v_\ell[t]\| \le \|v[t]\|+\|\overline{v}_\ell[t]\| \le N'e^{-\nu' t}+N_{\ell-1}e^{-\nu_{\ell-1}t}.$$

This inequality, toghether with (18), shows that $\max_{1\le i\le\ell}\|v_i[t]\| \le Ne^{-\nu t}$ with $N=N'+N_{\ell-1}$ and $\nu=\max\{\nu',\nu_{\ell-1}\}$. Theorem 1 follows. ∎

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Steady-State Analysis and Regulation of Discrete-Time Nonlinear Systems

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***Abstract*—This paper presents results on steady-state analysis and regulation for nonlinear discrete-time systems subject to time-varying excitations. In the analysis part of the paper, for convergent nonlinear systems (which have uniquely defined steady-state responses to excitations) we provide a complete characterization of the steady-state responses to excitations generated by an exosystem. As a corollary, we obtain a nonlinear frequency response function which extends the well-known FRF defined for linear systems to the class of nonlinear convergent systems. In the control part of the paper, we present a characterization of all controllers solving the global output regulation problem. All these results are obtained using the machinery of convergent systems, extended to the discrete-time setting. For piecewise affine systems, general results are supplied with a constructive design procedure.**

***Index Terms*—Convergent systems, discrete-time systems, frequency response functions, incremental stability, nonlinear systems, output regulation, PWA systems.**

I. INTRODUCTION

In many theoretical and practical problems where a dynamical control system (either continuous- or discrete-time) is excited by time-

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