SIAM J. APPL. MATH.
Vol. 71, No. 4, pp. 1307–1316
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CUCKER–SMALE FLOCKING UNDER HIERARCHICAL LEADERSHIP AND RANDOM INTERACTIONS*

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Abstract. Consider a flock of birds that fly interacting between them. This flock is a hierarchical system in which each bird, at each time step, adjusts its own velocity according to its past velocity and a linear combination of the relative velocities of its superiors in the hierarchy. This linear combination has nonnegative random coefficients, including the case in which each of the birds can fail to see any of its superiors with certain probability. For this model with random interactions we prove that the flocking results, obtained for similar deterministic models, hold true.

Key words. flocking, Cucker–Smale model, hierarchical leadership, random interactions

AMS subject classifications. 93A14, 60K35, 92D50

DOI. 10.1137/100785910

1. Introduction.

A relevant question in the collective behavior of a system of autonomous interacting agents is to understand under which conditions a certain consensus is reached in the long term. In the present work we consider a flock of birds, the consensus consisting in *flocking*, i.e., the convergence to a common velocity and to a stable configuration in the relative positions of the birds (see Definition 1). Departing from the papers by Cucker and Smale [4] and Shen [9], we introduce *random* interactions between birds to achieve a more realistic modeling of the situation under study.

Consider then a flock of k birds in $\mathbb{R}^3$ that follows a discrete-time dynamic. In this flock we introduce a hierarchy assuming that the birds can be labeled from 1 to k in such a way that 1 is the overall leader of the flock and that j is superior to i implies $j < i$. Let $\mathscr{L}(1) = \varnothing$ and $\mathscr{L}(i) \subseteq \{1, 2, \dots, i-1\}$ be the set of superiors or leader set of bird $i = 2, \dots, k$, assumed to be nonvoid. This defines a flock under hierarchical leadership, for short, an HL-flock.

The velocity of each bird is adjusted, at each time step, according to its own past velocity and a linear combination with random nonnegative coefficients of the relative velocities of its superiors. To specify the dynamic of the flock we denote by $h > 0$ the time step, by $x[t] = (x_1[t], \dots, x_k[t])$ and $v[t] = (v_1[t], \dots, v_k[t])$ the position and velocity of the flock, respectively, at time th ($t = 0, 1, \dots$), where $x_i[t]$ and $v_i[t]$ are the respective individual position and velocity of bird i. The parenthesis-bracket notation follows [9] and applies to other functions of t. The position and velocity of bird i are determined by the equations

$$\begin{cases} x_i[t+1] = x_i[t] + hv_i[t], \\ v_i[t+1] = v_i[t] + h\sum_{j\in\mathscr{L}(i)} a_{ij}(v_j[t] - v_i[t]), \end{cases} \tag{1.1}$$

where a_{ij} quantify the interactions between pairs of birds. These interactions depend, in general, on many different factors. In an attempt to take these factors into account,

*Received by the editors February 16, 2010; accepted for publication (in revised form) May 17, 2011; published electronically July 28, 2011.
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we assume that a_{ij} are random variables defined in a certain probability space. Then $\mathbb{P}(\cdot)$ denotes probability, $\mathbb{E}(\cdot)$ mathematical expectation, $\mathbb{E}_t(\cdot)$ (or $\mathbb{P}_t(\cdot)$) conditional mathematical expectation (or probability) with respect to the past of the system up to time t, i.e., with respect to the vector $(x[s], v[s]\colon s = 0, \dots, t)$.

A relevant consequence of (1.1) is that the first bird moves with constant velocity; i.e., $v_1[t] = v_1[0]$ for all t, and, when flocking, all birds asymptotically reach this same velocity. The model (1.1) with deterministic coefficients a_{ij} was introduced by Shen [9] as a modification of the Cucker–Smale model [4].

Our main result assumes in (1.3) that the conditional expectation of the random coefficients a_{ij} is conveniently bounded from below.

THEOREM 1. *Consider the HL-flock that follows* (1.1) *subject to the conditions*

$$h \le \frac{1}{k-1} \text{ and } 0 \le a_{ij} \le 1. \tag{1.2}$$

Assume that for some $\varepsilon > 0$ *the random coefficients satisfy*

$$\mathbb{E}_t\big(a_{ij}[t+1]\big) \ge \frac{\varepsilon}{(1 + \|x_i[t] - x_j[t]\|)^{\alpha}}. \tag{1.3}$$

Then, if $\alpha < 1$, *or* $\alpha = 1$ *and* $\|v_i[0] - v_1[0]\| < p/(2(k-1))$, $i = 2, \dots, k$, *we have flocking; i.e., for each bird*

$$v_i[t] \to v_1[0] \text{ and } x_i[t] - x_1[t] \to \widehat{x}_i$$

when $t \to \infty$, *where* $\widehat{x}_2, \dots, \widehat{x}_k$ *are vectors in* $\mathbb{R}^3$.

We end the introduction with a short review of the evolution of the model departing from the proposal of Vicsek et al. in [10]. In this paper, each bird can see only its neighbors within a certain fixed distance. Several simulation studies were carried out (see, for instance, [1]) but the rigorous proofs of convergence results in this case happens to be complex, as can be seen in the work by Jadbabaie, Lin, and Morse [7].

A fundamental advance in the modeling of the flocking phenomena was the introduction by Cucker and Smale of distance-dependent power decaying coefficients that, after minor changes, evolved to

$$a_{ij} = \frac{1}{(1 + \|x_i[t] - x_j[t]\|)^{\alpha}} \tag{1.4}$$

(see [4] for the original formulation), which enabled flocking results to be obtained under conditions on the initial state of the system. A modification in the interaction rules between birds was proposed by Shen [9] who introduced the HL-flock that we consider in this paper.

It is clear that the interactions in Cucker–Smale [4] and Shen [9] models are stronger than the ones in the model proposed by Vicsek et al. [10]. As follows from [4], a key factor in obtaining flocking results is to bound from below the interaction coefficients a_{ij} that vanish in the Vicsek et al. model in [10] when the birds are far enough.

Naturally the question arises whether it is possible to weaken the interactions in order to have more realistic modeling, preserving the flocking property. Our work moves in this direction, incorporating random interactions. In particular, our coefficients can vanish with certain probabilities, including the possibility of having a disconnected flock at some steps. We assume that not the interaction coefficient but its mean remains bounded from below. The argument we present seems to work only in the linear and sublinear cases ($\alpha \le 1$).

Other papers that include randomness in the flock dynamic are [3], where a random external perturbation is included at each step, and [6], where the connectivity at each step and for each pair of birds has a constant failure rate (see the "on-off" case in Example 1).

Finally, let us mention that this family of models can be seen as control laws. For instance, in [8] the Cucker–Smale model is used to keep the cohesion of the spacecrafts composing the Darwin mission. The presented result may be helpful since it deals with the issue of fault tolerance and reliability for these models.

2. Examples. In the first example we incorporate random independent failures in Shen coefficients [9]. As a consequence of the failure, the resulting strength of the interaction is a random part of the original one. This model includes the on-off case when the interactions alternate between the original ones and zero, according to a certain fixed probability. In the second example we assume that the probability that a pair of birds is connected decreases with the distance between them. In this way distant birds are rarely connected.

Example 1 (connectivity under random failure). Departing from Shen model, consider that the (deterministic) connectivity strength between each pair of birds (1.4) can be realized only in a random proportion $X_{t,ij} \in [0,1]$, chosen independently for each time and pair.

Hence, the coefficients are given by

$$a_{ij}[t+1] = \frac{X_{t,ij}}{(1+\|x_i[t]-x_j[t]\|)^{\alpha}}. \tag{2.1}$$

Assuming that $\inf_{t,ij} \mathbb{E}\,(X_{t,ij}) \geq \varepsilon > 0$ for some ε (that is, the expected proportion is not arbitrarily small) we have

$$\mathbb{E}_t\big(a_{ij}[t+1]\big) = \frac{\mathbb{E}\big(X_{t,ij}\big)}{(1+\|x_i[t]-x_j[t]\|)^{\alpha}} \geq \frac{\varepsilon}{(1+\|x_i[t]-x_j[t]\|)^{\alpha}},$$

and condition (1.3) holds.

Our main particular example is the on-off case with Bernoulli random variables, where we have either complete interaction or complete failure; i.e.,

$$\mathbb{P}(X_{t,ij}=1) = 1-\mathbb{P}(X_{t,ij}=0) = p$$

for some $0 < p \leq 1$. In this case we have $\inf_{t,ij} \mathbb{E}X_{t,ij} = p$. Observe that the case $p=1$ corresponds to the Shen model, as $X_{t,ij}=1$ for all t,i,j.

The Bernoulli case was treated by other methods in [6], where the particular form of the random variables gives a more accurate convergence rate. More general models of noisy connectivity can also be considered under the weaker condition $\inf_{t,ij} \mathbb{E}_t\,(X_{t,ij}) \geq \varepsilon > 0$ that applies, for instance, when $(X_{t,ij})$ is a Markov chain in stationary regime with state space $\{0,1\}$.

Example 2 (connectivity in random environment). Assume that in an HL-flock each possible connection between birds is active or not with a certain probability that decreases with the distance between the birds. Furthermore, in case of interaction, the strength is one. More precisely, the coefficients can take the values 0 and 1 with conditional probabilities

$$\mathbb{P}_t(a_{ij}[t+1]=1) = 1-\mathbb{P}_t(a_{ij}[t+1]=0) = \frac{1}{(1+\|x_i[t]-x_j[t]\|)^{\alpha}}.$$

In this case, we have

$$\mathbb{E}_t\big(a_{ij}[t+1]\big) = \frac{1}{(1+\|x_i[t]-x_j[t]\|)^{\alpha}},$$

and condition (1.3) holds. This example models a situation in which the failure of connectivity may be produced by some space-dependent external cause.

3. Additional results and proofs. We start this section with a corollary and remarks about Theorem 1.

The following result gives, for the case $\alpha < 1$, a precise bound on the discrepancy between the velocity of each bird and the velocity of the leader of the flock. It is a straightforward consequence of the proof of Theorem 1.

COROLLARY 1. *Under the conditions of Theorem* 1, *for* $\alpha < 1$ *and any deterministic sequence* $\{\delta_t\}$ *such that*

$$\sum_{t=1}^{\infty} \delta_t^{-1} t^{k-2} e^{-C(ht)^{1-\alpha}} \tag{3.1}$$

with $C = \varepsilon(1-\alpha)^{-1}(2\max_i \|v_i[0]-v_1[0]\|)^{-\alpha}$, *there exists a random time* τ_0 *such that, for* $t \geq \tau_0$, *we have*

$$\max_{2\leq i\leq k} \|v_i[t]-v_1[0]\| \leq \delta_t.$$

The next remark concerns the situation in which, for $\alpha = 1$, the whole flock may fail to reach a consensus.

Remark 1 (partial flocking). For $\alpha = 1$, in the proof of Theorem 1 below it is actually proved that if

$$\gamma_\ell := \sum_{j\in\mathcal{L}(\ell)} \frac{\varepsilon}{\|v_\ell[0]-v_j[0]\|}, \quad \ell = 2,\dots,k,$$

(where $1/0 = \infty$) satisfy the conditions

$$\gamma_\ell > k-\ell+2 \text{ for } \ell = 3,\dots,k \text{ and } \gamma_2 > k-1, \tag{3.2}$$

we have flocking. In particular, if (3.2) holds only for $\ell = 2,\dots,\ell_0 < k$, we have flocking for the subflock $\{1,\dots,\ell_0\}$; nevertheless, this could be not the case for the full flock.

The last remark is about the statement of the conditions (1.2).

Remark 2. Both conditions $h \leq 1/(k-1)$ and $0 \leq a_{ij} \leq 1$ in (1.2) in Theorem 1 may be replaced by the condition

$$h \sum_{j\in\mathscr{L}(i)} a_{ij} \leq 1 \quad \text{for all} \quad i = 1,\dots,k.$$

3.1. Preliminaries. In order to prove Theorem 1 we adapt methods from [2, 9]. We need to address some issues first.

As $v_1[t] = v_1[0]$ for all t, it is convenient to consider relative positions and velocities with respect to a system of coordinates in $\mathbb{R}^3$ that travels with constant velocity $v_1[0]$ centered in $x_1[t]$. Accordingly, the relative coordinates for bird i, denoted by ξ_i and ν_i, in this new system are

$$\xi_i[t+1] = x_i[t]-x_1[t] \text{ and } \nu_i[t+1] = v_i[t]-v_1[0].$$

It is easy to see that the relative velocities ν_i and positions ξ_i follow the same dynamic as the absolute velocities v_i and positions x_i. For example, for the velocities, from (1.1) we have

$$\begin{aligned} v_i[t+1] - v_1[0] &= v_i[t] - v_1[0] + h \sum_{j \in \mathscr{L}(i)} a_{ij} \left(v_j[t] - v_i[t]\right) \\ &= \left(v_i[t] - v_1[0]\right) + h \sum_{j \in \mathscr{L}(i)} a_{ij} \big(\left(v_j[t] - v_1[0]\right) - \left(v_i[t] - v_1[0]\right) \big). \end{aligned}$$

Therefore, from now on we are concerned with the dynamic

$$\begin{cases} \xi_i[t+1] = \xi_i[t] + h\nu_i[t], \\ \nu_i[t+1] = \nu_i[t] + h \sum_{j \in \mathscr{L}(i)} a_{ij} \left(\nu_j[t] - \nu_i[t]\right). \end{cases} \tag{3.3}$$

Next, we present some basic facts about the general model under consideration; in particular, in an HL-flock, the norm of the velocities, with an adequate norm, is nonincreasing in time. For $y = (y_1, \ldots, y_k)$ in $\mathbb{R}^{3k}$ let

$$\|y\|_\infty = \max_{1 \le i \le k} \|y_i\|,$$

and denote $\xi_0 = \|\xi[0]\|_\infty$ and $\nu_0 = \|\nu[0]\|_\infty$.

LEMMA 1. *Consider an HL-flock with the dynamic given in* (1.1) *under the constraints* (1.2). *Then:*

(a) *The absolute velocity of the flock is decreasing in norm; i.e., for all* $t \ge 0$, *we have*

$$\|v[t+1]\|_\infty \le \|v[t]\|_\infty.$$

(b) *The relative velocity is decreasing in norm; i.e., for all* $t \ge 0$, *we have*

$$\|\nu[t+1]\|_\infty \le \|\nu[t]\|_\infty.$$

(c) *The relative positions satisfy*

$$\|\xi[t]\|_\infty \le \xi_0 + h\nu_0 t.$$

(d) *The differences of the relative positions verify*

$$\max_{1 \le i,j \le k} \|\xi_i[t] - \xi_j[t]\|_\infty \le 2\xi_0 + 2h\nu_0 t.$$

The proofs of (a) and (b) in this lemma can be easily adapted from [2]. See also Dalmao [5]. The proofs of (c) and (d) are a consequence of $\|\nu[t]\|_\infty \le \|\nu[0]\|_\infty$ and the second equation in (1.1).

Finally, we define flocking in terms of the relative positions and velocities of the birds, and we state a necessary condition for flocking to hold.

DEFINITION 1. *In a dynamical system governed by* (3.3), *we have* flocking *when the following two conditions are fulfilled:*

(a) *The relative velocities asymptotically vanish; i.e.,*

$$\|\nu[t]\|_\infty \to 0 \text{ when } t \to \infty.$$

(b) *The relative positions converge; i.e., there exists* $\widehat{\xi} \in \mathbb{R}^{3k}$ *such that*

$$\xi[t] \to \widehat{\xi} \text{ when } t \to \infty.$$

The norms $\|\xi\|_\infty$ and $\|\nu\|_\infty$ can be considered as a measure of the dispersion of the flock.

Remark 3. It is easy to see that a sufficient condition to have flocking is

$$\sum_{t=0}^{\infty} \mathbb{E}\|\nu[t]\|_\infty < \infty, \tag{3.4}$$

as this condition implies that $\sum_{t=0}^{\infty} \|\nu[t]\|_\infty < \infty$ with probability one, and the latter condition implies that relative positions converge.

3.2. Proof of the main result. We begin by an auxiliary result that describes the contractive nature of the dynamic. Denote

$$A_0 = 1 + 2\xi_0, \qquad B_0 = 2h\nu_0, \qquad w_{0\ell} = \min\{\|\nu_\ell[0] - \nu_j[0]\| : j \in \mathscr{L}(\ell)\},$$

and assume that $w_{0\ell} > 0$ for all $\ell = 2, \dots, d$ (see Remark 4 below).

LEMMA 2. *Under the conditions of Theorem* 1, *if* $\alpha < 1$, *we have*

$$\mathbb{E}_\tau \left[\prod_{\sigma=\tau+1}^{t+1} \left(1 - h \sum_{j \in \mathscr{L}(\ell)} a_{\ell j}[\sigma] \right) \right] \leq \exp\left[-\frac{h\varepsilon}{(1-\alpha)B_0} \left((A_0 + B_0 t)^{1-\alpha} - (A_0 + B_0 \tau)^{1-\alpha} \right) \right],$$

and in case $\alpha = 1$, *we have*

$$\mathbb{E}_\tau \left[\prod_{\sigma=\tau+1}^{t+1} \left(1 - h \sum_{j \in \mathscr{L}(\ell)} a_{\ell j}[\sigma] \right) \right] \leq \left(\frac{A_0 + h w_{0\ell} \tau}{A_0 + h w_{0\ell}(t+1)} \right)^{\gamma_\ell}.$$

Proof. Based on properties of the conditional expectation, we have

$$\begin{aligned} &\mathbb{E}_\tau \left[\prod_{\sigma=\tau+1}^{t+1} \left(1 - h \sum_{j \in \mathscr{L}(\ell)} a_{\ell j}[\sigma] \right) \right] \\ &\quad = \mathbb{E}_\tau \left[\mathbb{E}_t \left[\left(1 - h \sum_{j \in \mathscr{L}(\ell)} a_{\ell j}[t+1] \right) \prod_{\sigma=\tau+1}^{t} \left(1 - h \sum_{j \in \mathscr{L}(\ell)} a_{\ell j}[\sigma] \right) \right] \right] \\ &\quad = \mathbb{E}_\tau \left[\mathbb{E}_t \left(1 - h \sum_{j \in \mathscr{L}(\ell)} a_{\ell j}[t+1] \right) \prod_{\sigma=\tau+1}^{t} \left(1 - h \sum_{j \in \mathscr{L}(\ell)} a_{\ell j}[\sigma] \right) \right]. \end{aligned} \tag{3.5}$$

Assume that $\alpha < 1$. Since $\mathscr{L}(\ell) \neq \varnothing$ and

$$\mathbb{E}_t(a_{\ell j}[t+1]) \geq \frac{\varepsilon}{(1 + \|\xi_j[t] - \xi_\ell[t]\|)^\alpha} \geq \frac{\varepsilon}{(A_0 + B_0 t)^\alpha},$$

we have

$$\mathbb{E}_t\left(1-h\sum_{j\in\mathscr{L}(\ell)}a_{\ell j}[t+1]\right)\le 1-\frac{h\varepsilon}{(A_0+B_0t)^\alpha},$$

which is a deterministic bound. Therefore,

$$\begin{aligned}(3.6)\qquad &\mathbb{E}_\tau\left[\prod_{\sigma=\tau+1}^{t+1}\left(1-h\sum_{j\in\mathscr{L}(\ell)}a_{\ell j}[\sigma]\right)\right]\\ &\le\left(1-\frac{h\varepsilon}{(A_0+B_0t)^\alpha}\right)\mathbb{E}_\tau\left[\prod_{\sigma=\tau+1}^{t}\left(1-h\sum_{j\in\mathscr{L}(\ell)}a_{\ell j}[\sigma]\right)\right]\\ &\le\prod_{\sigma=\tau}^{t}\left(1-\frac{h\varepsilon}{(A_0+B_0\sigma)^\alpha}\right)=\exp\left(\sum_{\sigma=\tau}^{t}\log\left(1-\frac{h\varepsilon}{(A_0+B_0\sigma)^\alpha}\right)\right)\\ &\le\exp\left(-h\varepsilon\sum_{\sigma=\tau}^{t}\frac{1}{(A_0+B_0\sigma)^\alpha}\right).\end{aligned}$$

Bounding the sum in the last line of (3.6) by an integral, we obtain the desired bound.

Assume now that $\alpha=1$. Analogously to the case $\alpha<1$, we have

$$(3.7)\qquad \mathbb{E}_t\left[a_{\ell j}[t+1]\right]\ge\frac{\varepsilon}{A_0+h\|\nu_\ell[0]-\nu_j[0]\|t};$$

therefore,

$$\begin{aligned}(3.8)\quad &\mathbb{E}_\tau\left[\prod_{\sigma=\tau+1}^{t+1}\left(1-h\sum_{j\in\mathscr{L}(\ell)}a_{\ell j}[\sigma]\right)\right]\\ &\le\exp\left[-h\sum_{j\in\mathscr{L}(\ell)}\sum_{\sigma=\tau}^{t}\frac{\varepsilon}{A_0+h\|\nu_\ell[0]-\nu_j[0]\|\sigma}\right]\\ &\le\exp\left[-\varepsilon\sum_{j\in\mathscr{L}(\ell)}\frac{1}{\|\nu_\ell[0]-\nu_j[0]\|}\log\left(\frac{A_0+h\|\nu_\ell[0]-\nu_j[0]\|(t+1)}{A_0+h\|\nu_\ell[0]-\nu_j[0]\|\tau}\right)\right]\\ &\le\left(\frac{A_0+hw_{0\ell}\tau}{A_0+hw_{0\ell}(t+1)}\right)^{\gamma_\ell}.\end{aligned}$$

This concludes the proof of the lemma. □

Proof of Theorem 1 (case $\alpha<1$). We verify condition (3.4) and proceed by induction over $\ell=2,\dots,k$, considering the subflocks $\{1,2,\dots,\ell\}$.

Let us start with $\ell=2$. Recall that the first bird is the overall leader of the flock; in particular, its (relative) velocity $\nu_1[t]=0$ for all t. For the second bird we have $\mathscr{L}(2)=\{1\}$; then

$$\begin{aligned}\nu_2[t+1]&=\nu_2[t]+ha_{21}[t+1](\nu_1[t]-\nu_2[t])\\ &=(1-ha_{2\,1}[t+1])\nu_2[t]\\ &=\prod_{\tau=1}^{t+1}(1-ha_{21}[\tau])\nu_2[0].\end{aligned}\qquad(3.9)$$

By Lemma 2, we have

$$\mathbb{E}\|\nu_2[t+1]\| \leq K \exp\left[-\frac{h\varepsilon}{(1-\alpha)B_0}\left(A_0+B_0t\right)^{1-\alpha}\right],$$

where $K = \|\nu_2[0]\| \exp\left[-\frac{h\varepsilon}{(1-\alpha)B_0}A_0^{1-\alpha}\right]$; this gives flocking by condition (3.4), concluding the proof for $\ell = 2$.

We turn now to the general case. Assume that for $j < \ell$ there exist positive constants C_j such that for all t we have

$$\mathbb{E}\|\nu_j[t]\| \leq C_j t^{j-2} e^{-\kappa(A_0+B_0t)^{1-\alpha}}, \tag{3.10}$$

where $\kappa = -\frac{h\varepsilon}{(1-\alpha)B_0}$. From (3.3), it follows that

$$\begin{aligned}\nu_\ell[t+1] &= \nu_\ell[t] + h\sum_{j\in\mathscr{L}(\ell)} a_{\ell j}[t+1](\nu_j[t]-\nu_\ell[t]) \\ &= \left(1-h\sum_{j\in\mathscr{L}(\ell)} a_{\ell j}[t+1]\right)\nu_\ell[t] + h\sum_{j\in\mathscr{L}(\ell)} a_{\ell j}[t+1]\nu_j[t] \\ &= (1-a[t+1])\nu_\ell[t] + b[t+1], \qquad (3.11)\end{aligned}$$

where we denote

$$a[t+1] = \sum_{j\in\mathscr{L}(\ell)} a_{\ell j}[t+1], \qquad b[t+1] = h\sum_{j\in\mathscr{L}(\ell)} a_{\ell j}[t+1]\nu_j[t]. \tag{3.12}$$

Iterating (3.11) and taking norms, it follows that

$$\|\nu_\ell[t+1]\| \leq \prod_{\tau=1}^{t+1}(1-ha[\tau])\|\nu_\ell[0]\| + \sum_{\tau=1}^{t+1}\|b[\tau]\| \prod_{\sigma=\tau+1}^{t+1}(1-ha[\sigma]), \tag{3.13}$$

where we agree that $\prod_{\sigma=t+2}^{t+1}(1-ha[\sigma]) = 1$. By Lemma 2, there exists a constant C such that

$$\mathbb{E}\prod_{\tau=1}^{t+1}(1-ha[\tau]) \leq Ce^{-\kappa(A_0+B_0t)^{1-\alpha}}.$$

We have

$$\mathbb{E}\|b[\tau]\| \leq h\sum_{j\in\mathscr{L}(\ell)} \mathbb{E}\left(a_{\ell j}[\tau]\|\nu_j[\tau-1]\|\right) \leq h\sum_{j\in\mathscr{L}(\ell)} \mathbb{E}\|\nu_j[\tau-1]\|,$$

and, by the induction hypothesis (3.10), for all t we have

$$\begin{aligned}\sum_{\tau=1}^{t+1}\mathbb{E}\|b[\tau]\| &\leq h\sum_{j\in\mathscr{L}(\ell)}\sum_{\tau=1}^{t+1}\mathbb{E}\|\nu_j[\tau-1]\| \leq h\sum_{j\in\mathscr{L}(\ell)}\sum_{\tau=1}^{t+1} C_j\tau^{j-2}e^{-\kappa(A_0+B_0(\tau-1))^{1-\alpha}} \\ &\leq C'(t+1)^{\ell-3}\sum_{\tau=1}^{t+1} e^{-\kappa(A_0+B_0(\tau-1))^{1-\alpha}},\end{aligned}$$

where $C' = \max_{j\in\mathcal{L}(\ell)} C_j$. Using this information and Lemma 2, for the second term on the right-hand side of (3.13), we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{\tau=1}^{t+1}\|b[\tau]\|\prod_{\sigma=\tau+1}^{t+1}(1-ha[\sigma])\right]\\
&\quad=\mathbb{E}\left[\sum_{\tau=1}^{t+1}\|b[\tau]\|\mathbb{E}_\tau\left[\prod_{\sigma=\tau+1}^{t+1}(1-ha[\sigma])\right]\right]\\
&\quad\le C'(t+1)^{\ell-3}\sum_{\tau=1}^{t+1}e^{-\kappa((A_0+B_0t)^{1-\alpha}-(A_0+B_0\tau)^{1-\alpha})}e^{-\kappa(A_0+B_0(\tau-1))^{1-\alpha}}\\
&\quad\le K(t+1)^{\ell-2}e^{-\kappa t^{1-\alpha}}
\end{aligned}$$

for a convenient constant K. Then, the induction thesis (3.10) for ℓ follows with $C_\ell = C + K$. This concludes the proof of the theorem. □

Proof of Corollary 1. We have

$$\mathbb{P}(\|\nu[t]\|_\infty \ge \delta_t) \le \frac{1}{\delta_t}\mathbb{E}\|\nu[t]\|_\infty \le \frac{1}{\delta_t}C_k t^{k-2}e^{-\kappa(A_0+B_0t^{1-\alpha})},$$

and, since the series in (3.1) converges, the result follows by application of the Borel–Cantelli lemma. □

Proof of Theorem 1 (case $\alpha = 1$). The proof is carried by induction as for the case $\alpha < 1$. We begin with $\ell = 2$.

From (3.9) and Lemma 2, we have

$$\mathbb{E}\|\nu_2[t+1]\| \le \|\nu_2[0]\|\mathbb{E}\prod_{\tau=1}^{t+1}(1-ha_{21}[\tau]) \le \|\nu_2[0]\|\left(\frac{A_0}{A_0+hw_{0\ell}(t+1)}\right)^{\gamma_2}.$$

In view of Remark 3, under the condition $\|\nu_2[0]\| < \varepsilon$, the series of the velocities in (3.4) is convergent, and we have flocking of the first two birds.

Consider now the ℓth bird. For the velocity, we have

$$\begin{aligned}
\nu_\ell[t+1] &= \nu_\ell[t] + h\sum_{j\in\mathcal{L}(\ell)} a_{\ell j}[t+1]\,(\nu_j[t]-\nu_\ell[t])\\
&= (1-ha[t+1])\,\nu_\ell[t] + b[t+1], \qquad (3.14)
\end{aligned}$$

where $a[t+1]$ and $b[t+1]$ are defined as in (3.12).

With this notation, we have

$$\nu_\ell[t+1] = \prod_{\tau=1}^{t+1}(1-ha[\tau])\,\nu_\ell[0] + \sum_{\tau=1}^{t+1}b[\tau]\prod_{\sigma=\tau+1}^{t+1}(1-ha[\sigma]),$$

where we agree that $\prod_{\sigma=t+2}^{t+1}(1-ha[\sigma]) = 1$. By Lemma 2 we obtain

$$\mathbb{E}\prod_{\sigma=\tau+1}^{t+1}(1-ha[\tau]) \le \left(\frac{A_0+hw_{0\ell}\tau}{A_0+hw_{0\ell}(t+1)}\right)^{\gamma_\ell}.$$

Remark 4. The case $w_{0\ell} = 0$ is not considered in the previous computations. It can be checked with slightly different arguments that in this case the convergence still holds.

Define recursively $\delta_\ell = \delta_{\ell-1} \wedge \gamma_\ell - 1$. To estimate $b[t+1]$ in (3.14), we have

$$\begin{aligned}\mathbb{E}\|b[\tau]\| &= h \sum_{j\in\mathcal{L}(\ell)} \mathbb{E}\left(a_{\ell j}[\tau]\|\nu_j[\tau-1]\|\right) \le h \sum_{j\in\mathcal{L}(\ell)} \mathbb{E}\|\nu_j[\tau-1]\| \\ &\le h\left(\sum_{j\in\mathcal{L}(\ell)} \|\nu_j[0]\|\right)\left(\frac{A_0}{A_0 + hw_{0\ell}\,(\tau-1)}\right)^{\delta_{\ell-1}} \\ &\le h\left(\sum_{j\in\mathcal{L}(\ell)} \|\nu_j[0]\|\right)\left(\frac{A_0+hw_{0\ell}}{A_0}\right)^{\delta_{\ell-1}}\left(\frac{A_0}{A_0+hw_{0\ell}\tau}\right)^{\delta_{\ell-1}}.\end{aligned}$$

With this information, we conclude that

$$\begin{aligned}\mathbb{E}\|\nu_\ell[t+1]\| &\le \|\nu_\ell[0]\|\left(\frac{A_0}{A_0+hw_{0\ell}(t+1)}\right)^{\gamma_\ell} \\ &\quad + h\left(\frac{A_0+hw_{0\ell}}{A_0}\right)^{\delta_{\ell-1}}\left(\sum_{j\in\mathcal{L}(\ell)}\|\nu_j[0]\|\right)(t+1)\left(\frac{A_0}{A_0+hw_{0\ell}(t+1)}\right)^{\delta_{\ell-1}\wedge\gamma_\ell} \\ &\le k_\ell\left(\frac{A_0}{A_0+hw_{0\ell}(t+1)}\right)^{\delta_\ell},\end{aligned}$$

where k_ℓ is a convenient constant that depends on the initial positions and velocities of the flock. If $\delta_\ell > 1$, we have flocking of the first ℓ birds. It is easy to see that this condition is equivalent to (3.2) with $\ell = k$. This concludes the proof. □

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