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Market symmetry in time-changed Brownian models

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ABSTRACT

In this paper we examine which Brownian subordination with drift exhibits the symmetry property introduced by Fajardo and Mordecki [2006. Quantitative Finance 6, 219–227]. We find that when the subordination results in a Lévy process, a necessary and sufficient condition for the symmetry to hold is that the drift must be equal to-1/2. Also, we derive explicit conditions to test whether the NIG, CGMY and Meixner processes are symmetric or not. Finally, we perform some tests with real financial data.

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1. Introduction

Asset returns have been studied for many years. One of the main findings is the presence of many small jumps in a finite time interval. To deal with this fact, more realistic jump structures have been suggested, as for example the CGMY model of Carr et al. (2002), the NIG model of Barndorff-Nielsen (1995) and the Meixner model of Schoutens (2001). All these models are considered Lévy market models.

Also, we know that many Lévy processes can be represented as time-changed Brownian motion. This fact is very useful for the pricing of multiasset derivatives, since one can correlate assets by correlating the Brownian motions.

On the other hand, it is not easy to express explicitly the time change used. In the particular case of the NIG process the time change was obtained by Barndorff-Nielsen (1995). More recently, Madan and Yor (2008) have been obtained the time changes used for the CGMY and the Meixner process.

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In this paper, we study Lévy processes that are time-changed Brownian motions plus drift and find that a necessary and sufficient condition for this process to be symmetric is that drift must be equal to-0.5. Then, based on the representations obtained for the NIG, CGMY and Meixner models, we derive explicit and simple conditions to test whether these processes are symmetric or not. Moreover, we perform tests with real data.

Although, duality methods have been previously applied to Time-changed Lévy process in Fajardo and Mordecki (2008), they do not obtain such explicit and simple symmetry condition. While in our case, due to elegant properties of the Brownian motion, we obtain and test.

It is important to note that when market symmetry holds, the relationship known as put-call symmetry is valid, as Fajardo and Mordecki (2006) and Carr and Lee (2009) have shown. So, we can construct semi-static hedging for barrier options and some classes of bidimensional options, as Carr and Lee (2009) have verified. Also, very recently Tehranchi (2009) have studied the implications of a very similar symmetry property for continuous local martingales, finding that if a market model of the implied volatility that is initially flat remains symmetric for all future times then it must be the Black and Scholes model.

The paper is organized as follows: in Section 2, we introduce time-changed Brownian motion and some particular cases. In Section 3, we describe the market model. In Section 4, we describe symmetry and obtain symmetry results. In Section 5, we obtain the explicit symmetry tests and perform some empirical tests. In the last section we conclude.

2. Time-changed Brownian motion

Let $t \mapsto \mathcal{T}_t,\ t \geqslant 0$, be an increasing cádlág process, such that for each fixed t, $\mathcal{T}_t$ is a stopping time with respect to $\mathbf{F}$. Furthermore, suppose $\mathcal{T}_t$ is finite and positive $P-a.s.,\quad \forall\, t \geqslant 0$ and $\mathcal{T}_t \to \infty$ as $t \to \infty$. Then $\{\mathcal{T}_t\}$ defines a random change on time and we can also impose $E\mathcal{T}_t = t$.

Now let $Y_t = X_{\mathcal{T}_t},\ X_t = \mu t + W_t$, i.e. Y_t is a time-changed Brownian motion with drift μ.

Henceforth, we assume $\mathcal{T}_t$ is a subordinator, independent of X_t, i.e. a Lévy process with a non decreasing trajectory, the resulting process will be a Lévy process.[1] with characteristic function:

$$\phi_{Y_t}(z) = \mathcal{L}_{\mathcal{T}_t}(\psi(z)) = \mathcal{L}_{\mathcal{T}_t}\left(-\frac{z^2}{2} + i\mu z.\right), \tag{1}$$

where $\mathcal{L}_{\mathcal{T}_t}$ is the Laplace transform of $\mathcal{T}_t$.

2.1. NIG, CGMY and Meixner processes

The representation of NIG as a time-changed Brownian motion was obtained by Barndorff-Nielsen (1995) and more recently, Madan and Yor (2008) have shown, it is possible to represent the CGMY and Meixner processes as time-changed Brownian motions. We now describe these processes.

2.1.1. NIG process

The NIG process is a pure jump Lévy process with Lévy measure given by:

$$v_{\mathrm{NIG}}(x) = \frac{\delta\alpha K_1(\alpha|x|)e^{\beta x}}{\pi x},$$

where $\mu \in IR,\ \alpha,\ \delta > 0,\ 0 \leqslant |\beta| \leqslant \alpha$ and K_1 denotes the modified Bessel function of third order with index 1. Barndorff-Nielsen (1995) obtained the following representation for NIG process:

$$X_t = \beta\mathcal{T}_t + W_{\mathcal{T}_t},$$

where $\mathcal{T}_t = \delta^2 \mathrm{IG}(t, 1, \delta\sqrt{\alpha^2 - \beta^2})$ is a subordinator and IG is the inverse Gaussian distribution.

[1] See Kyprianou (2006)[Lemma 2.15].

2.1.2. CGMY process

The CGMY process is a pure jump Lévy process with Lévy measure given by:

$$\nu_{CGMY}(x) = C\left[\frac{e^{-G|x|}}{|x|^{1+Y}}\mathbf{1}_{x<0} + \frac{e^{-Mx}}{x^{1+Y}}\mathbf{1}_{x>0}\right],$$

where $C > 0,\ Y < 2$, and $G \geqslant 0, M \geqslant 0$. We can obtain $\phi_{\mathrm{CGMY}_t}(z) = \mathcal{L}_{\mathcal{T}_t}\left(-\frac{z^2}{2} + i\mu z\right)$. Madan and Yor (2008) have found a time change that is absolutely continuous with respect to the one-side stable $Y/2$ process and the drift $\frac{G-M}{2}$ that make both characteristic functions equal.

2.1.3. Meixner process

The Meixner process is a Lévy process with Lévy measure given by:

$$\nu_{\mathrm{Meixner}}(dx) = d\frac{e^{\frac{b}{a}x}}{x\sinh(\pi x/a)}dx,$$

where a, b, d and m are parameters of the Meixner density, such that $a > 0, -\pi < b < \pi, d > 0$ and $m \in IR$.

The subordinator that allows us to obtain the Meixner process as a time-changed Brownian motion with drift $\mu = \frac{b}{a}$, is absolutely continuous with respect to the one-side stable 1/2 process. As Madan and Yor (2008) also have proved.

3. Market model

Consider a time-changed Brownian market with a riskless asset, with price process denoted by $B = \{B_t\}_{t\geqslant 0}$, we take $B_0 = 1$ for simplicity, and a risky asset, with price process denoted by $S = \{S_t\}_{t\geqslant 0}$:

$$B_t = e^{rt}, \quad r \geqslant 0, \text{ and } S_t = S_0 e^{Y_t}, \quad S_0 = e^y > 0, \tag{2}$$

where Y_t is a time-changed Brownian motion with a drift μ and independent subordinator $\mathcal{T}_t$. Denote by (b, σ, ν) the characteristics of the time-changed Brownian with drift.

We also assume that the stock pays dividends, with constant rate $\delta \geqslant 0$, and we assume that the probability measure P is the chosen equivalent martingale measure. To avoid arbitrage opportunities we have to restrict our attention to time-changed Brownian motion such that the exponential process $e^{Y_t-(r-\delta)t}$ is a P–martingale.

Now, consider a market model with two assets, a deterministic savings account $\widetilde{B}$ and a stock $\widetilde{S}$, given by

$$\widetilde{B}_t = e^{\delta t}, \quad r \geqslant 0, \quad \text{and } \widetilde{S}_t = Ke^{\widetilde{Y}_t}, \quad S_0 = e^x > 0,$$

where $\widetilde{Y} = \{\widetilde{Y}_t\}_{t\geqslant 0}$ is a Lévy process and with characteristics $(\tilde{b}, \tilde{\sigma}, \tilde{\nu})$.

4. Symmetry

In Fajardo and Mordecki (2006) a market is defined to be *symmetric* when

$$\mathcal{L}\left(e^{-(r-\delta)t+Y_t}|P\right) = \mathcal{L}\left(e^{-(\delta-r)t-Y_t}|\widetilde{P}\right), \tag{3}$$

meaning equality in law. Fajardo and Mordecki (2006) derived the characteristics of the *dual process* $\widetilde{Y}_t$:

$$\begin{aligned}
&\widetilde{b} = \delta - r - \sigma^2/2 - \int_{IR}\left(e^x - 1 - x\mathbf{1}_{|x|\leqslant 1}\right)\widetilde{\nu}(dx),\\
&\widetilde{\sigma} = \sigma,\\
&\widetilde{\nu}(dx) = e^{-x}\nu(-dx).
\end{aligned}$$

In particular, they found that a necessary and sufficient condition for (3) to hold is

$$\nu(dx) = e^{-x}\nu(-dx). \tag{4}$$

This ensures $\tilde{\nu} = \nu$, and from this $b - (r - \delta) = \tilde{b} - (\delta - r)$ follows, giving (3), since we always have $\tilde{\sigma} = \sigma$.

Now lets see how symmetry works explicitly in our time-changed Brownian market. Denote by $(\gamma, 0, \rho)$ the characteristics of $\mathcal{T}_t$. Then, as X_t has characteristics $(\mu, 1, 0)$, applying Sato (1999)[Th. 30.1] we have that the characteristics of $Y_t = \mathcal{T}_t\mu + W(\mathcal{T}_t)$ are given by (b, γ, ν), where:

$$b = \gamma\mu + \int_{|x|\leqslant 1}\left[\int_0^\infty \frac{x}{\sqrt{2\pi y}} e^{-\frac{(x-\mu y)^2}{2y}}\rho(dy)\right]dx,$$

and

$$\nu(dx) = \left[\int_0^\infty \frac{1}{\sqrt{2\pi y}} e^{-\frac{(x-\mu y)^2}{2y}}\rho(dy)\right]dx.$$

We can express this last equation as

$$\nu(dx) = e^{\mu x} f(x)dx, \text{ with } f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2y}(x^2+\mu^2 y^2)}\rho(dy). \tag{5}$$

This density is even, i.e., $f(x) = f(-x)$. Now we can state our symmetry result.

Proposition 1. *A time-changed Brownian market is symmetric if and only if the drift is equal to-1/2.*

Proof. By condition (4) we have that a market is symmetric if and only if

$$e^{\mu x} f(x)dx = e^{-x}[e^{-\mu x} f(-x)]dx,$$

whence here $\mu = -1/2$. □

An immediate consequence of Proposition 1 is the following corollary:

Corollary 4.1.

(a) The CGMY market model is symmetric if and only if

$$G - M = -1.$$

(b) The Meixner market model is symmetric if and only if

$$2b + a = 0.$$

(c) The NIG market model is symmetric if and only if

$$\beta = -1/2.$$

Proof. Since, Madan and Yor (2008) have obtained the explicit representations of the CGMY model and Meixner model as time-changed Brownian motions with drift $\frac{G-M}{2}$ and $\frac{b}{a}$, respectively and for NIG model we have the drift β obtained by Barndorff-Nielsen (1995), the result follows. □

This corollary is a very useful result, since it allows us to construct a statistical test to see whether the market models based on the estimated NIG, CGMY or Meixner processes have the symmetry property.

5. Testing symmetry

To perform the symmetry test we need to estimate the parameters in a risk-neutral world. In the following $\widetilde{a}$ and $\widehat{a}$ are used to denote estimations of any parameter a obtained under a risk-neutral and statistical measure, respectively.

We choose the market risk-neutral measure and the risk-neutral measure given by the Esscher transform with Esscher parameter θ. We know that the Lévy measure under the Esscher transform, denoted by ν^θ, is given by: $\nu^\theta(dx) = e^{\theta x}\nu(dx) = e^{\theta+\mu}f(x)dx$, where we used Eq. (5) to obtain the last equality. Then, the risk-neutral drift μ^θ is equal to $\mu + \theta$. Now we have all the ingredients to carry out our symmetry tests.

5.1. CGMY test

For the CGMY model the symmetry test is given by

$$\mathrm{H}_0: \ G - M = -1.$$

In Madan and Yor (2008), option prices in the S&P 500 index are used to obtain the following estimates: $\widetilde{G} = 5$ and $\widetilde{M} = 10$, so $\widetilde{G} - \widetilde{M} = -5$. Moreover, in Carr et al. (2002) the drift is estimated under the market risk- neutral measure and in the majority of cases $\widetilde{G} - \widetilde{M}$ is negative and less than-1. Unfortunately, the standard econometric model admits arbitrage with probability one. So we cannot apply our test with the market risk-neutral measure because it is unclear how to compute the standard deviations. If we use the Esscher transform risk-neutral measure, the resulting test will be given by:

$$\mathrm{H}_0: \ (G - M) = -2(\theta + 0.5).$$

Carr et al. (2002) obtained maximum likelihood estimations from historical data using the period January 1, 1994 to December 31, 1998 with daily returns and also reported standard errors of the variance-gamma parameters σ and Θ. So we can use the relationship[2] $G - M = \frac{2\Theta}{\sigma^2}$ to obtain the following hypothesis test:

$$\mathrm{H}_0: \ \Theta + (\theta + 0.5)\sigma^2 = 0.$$

Unfortunately, we do not know the distribution of $\widehat{\Theta} + (\theta + 0.5)\widehat{\sigma}^2$. But we can find evidence of symmetry indirectly by first computing θ, assuming a fixed interest rate $r = 0.01$, and then testing whether Θ is negative or positive.

The estimates obtained for SPX are: $\widehat{C} = 24.79, \widehat{G} = 94.45, \widehat{M} = 95.79$ and $\widehat{Y} = 0.2495$. Then, $\theta^{\mathrm{SPX}} = 0.8071$. The hypothesis test here is $\mathrm{H}_0: \ \Theta + 1.3071\sigma^2 = 0$.

The estimates and s.d. of $\widehat{\Theta}$ and $\widehat{\sigma}$ are -0.0042 (0.00066) and 0.0739 (0.0311), respectively. Now, since $\sigma^2 \geqslant 0$, we first perform the following test:

$$\mathrm{H}_0: \ \Theta = 0 \quad \text{and} \quad \mathrm{H}_1: \ \Theta < 0$$

Then, $\widehat{\Theta}/\mathrm{sd} = -6.363 < -1.64$. We can thus reject the null hypothesis at 95% of significance.

5.2. Meixner test

For the and Meixner model we have the following test:

$$\mathrm{H}_0: \ b + 0.5a = 0.$$

Madan and Yor (2008), under the market risk-neutral measure, obtained the following values for the S&P 500: $\widetilde{a} = 0.25$ and $\widetilde{b} = -1.5$, so $\widetilde{b} + 0.5\widetilde{a} = -1.375$. Under the Esscher risk-neutral measure we have:

$$\mathrm{H}_0: \ b + a(\theta + 0.5) = 0.$$

Recently, Grigoletto and Provassi (2009) have shown how maximum likelihood estimators can be obtained for the Meixner process. We consider DAX and SPX estimates, obtained using the period January 2002 to December 2006 with weekly returns.

[2] See Carr et al. (2002).

Table 1
Risk neutral estimation for EuroUSD.

EuroUSD/	$\widetilde{\beta}$
06/05/2005–06/05/2009	−105.2899
06/05/2004–06/05/2009	−109.1379
All sample	−119.1508

For the DAX index the estimations obtained were: $\widehat{a} = 5.7616$ (5.5278), $\widehat{b} = -0.4794$ (1.1382), $\widehat{d} = 0.5411$ (0.9004) and $\widehat{m} = 0.8448$ (1.6947). Then $\theta^{\text{DAX}} = -0.4335$. The test results in $\text{H}_0: \ b + 0.0665a = 0$, with the s.d. of the statistic given by $\sqrt{1.1382^2 + 0.0665^2 \times 5.5278^2} = 1.196$, so we have $|\widehat{b} + 0.0665\widehat{a}/\text{sd}| = |-0.0805| < 1.64$. Therefore, we cannot reject the null hypothesis at 95% of significance.

For the S&P500 index we have: $\widehat{a} = 3.9324$ (3.6518), $\widehat{b} = -0.3714$ (1.1272), $\widehat{d} = 0.4547$ (0.7144) and $\widehat{m} = 0.4183$ (0.9253). Then $\theta^{\text{SPX}} = -0.4798$. The test results in $\text{H}_0: \ b + 0.0202a = 0$, with the s.d. of the statistic given by $\sqrt{1.1272^2 + 0.0202^2 \times 3.6518^2} = 1.1296$, and we have: $|\widehat{b} + 0.0202\widehat{a}/\text{sd}| = |-0.2585| < 1.64$. Therefore, we cannot reject the null hypothesis at 95% of significance.

5.3. NIG test

For the NIG model the symmetry test is given by

$$\text{H}_0: \ \beta = -0.5.$$

Now we use the period 06/05/2003 to 06/05/2009 to obtain NIG estimates for the EuroUSD exchange rate, the results are presented in the Table 1 below. As we can see EuroUSD exchange rate is very asymmetric. Using estimates obtained by Fajardo and Farias (2009) for other assets, we can find more evidence against market symmetry.

6. Conclusions

In a time-changed Brownian market we have shown that the market is symmetric if and only if the drift is equal to-1/2. From there, we have developed specific test for NIG, CGMY and Meixner market symmetry. Finally, the statistical tests carried out with real data, have shown evidence against market symmetry for the NIG and CGMY models and in favor of market symmetry for the Meixner model. Of course, in the latter case the estimations were obtained with weekly data, while in the former cases they were obtained with daily data.

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