*Stochastics: An International Journal of Probability and Stochastics Processes*,
Vol. 79, Nos. 3–4, June–August 2007, 233–251

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# Optimal stopping of Hunt and Lévy processes

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*(Received 27 April 2006; in final form 27 October 2006)*

Infinite horizon (perpetual) optimal stopping problems for Hunt processes on **R** are studied via the representation theory of excessive functions. In particular, we focus on problems with one-sided structure, that is, there exists a point $x^*$ such that the stopping region is of the form $[x^*, +\infty)$. The main result states that if it is possible to find a Radon measure such that the excessive function induced by this measure via the spectral representation has some very intuitive properties then the constructed excessive function coincides with the value function of the problem. Corresponding results for two-sided problems are also indicated. Specializing to Lévy processes, we obtain, by applying the Wiener–Hopf factorization, a general representation of the value function in terms of the maximum of the Lévy process. To illustrate the results, an explicit expression for the Green kernel of Brownian motion with exponential jumps is computed and some optimal stopping problems for Poisson process with positive exponential jumps and negative drift are solved.

*Keywords*: Optimal stopping problem; Markov processes; Hunt processes; Lévy processes; Green kernel; Diffusions with jumps

*AMS Classification*: 60G40; 60J25; 60J30; 60J60; 60J75

## 1. Introduction

Consider an optimal stopping problem for a real-valued Markov process $X = \{X_t : t \geq 0\}$ with reward function $g$ and discount rate $r \geq 0$. Denote by $V$ the value function of the problem, and by $\tau^*$ the optimal stopping time. In this paper, we analyze this optimal stopping problem departing from three main sources: (i) the characterization of the value function $V$ as the least excessive majorant of the reward function $g$, due to Snell [23] for discrete martingales and to Dynkin [6] for continuous time Markov processes; (ii) the representation of excessive functions as integrals of the Green kernel of the process, as exposed in Kunita and Watanabe [11] and Dynkin [7], and exploited by Salminen [20] in the framework of optimal stopping for diffusions; and (iii) recent results expressing the solution of some optimal stopping problems for Lévy processes and random walks in terms of the maximum of the process, see Darling *et al.* [5], Mordecki [15], Boyarchenko and Levendorskij [3],

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*Stochastics: An International Journal of Probability and Stochastics Processes*
ISSN 1744-2508 print/ISSN 1744-2516 online © 2007 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/17442500601100232

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Novikov and Shiryaev [18] and Kyprianou and Surya [12]. For papers on optimal stopping of Lévy processes using other methods, see, e.g. McKean [13], Gerber and Shiu [9], Chan [4] and Kou and Wang [10].

We then try to understand the structure of the solution of the optimal stopping problem in a regular enough framework of Markov processes, more precisely the class of Hunt processes, concluding that finding the solution of such a problem is equivalent to finding the representation of the value function in terms of the Green kernel. The Radon measure that appears in this representation is called the *spectral measure* corresponding to the excessive function $V$, and furthermore, it results that the support of this spectral measure is the stopping region for the problem. This is our main result, presented in Section 3.

Let us specialize to Lévy processes. Firstly, observe that in the case $r > 0$ the Green kernel is proportional to the distribution of the process stopped at an exponential time with parameter $r$, independent of the process. Secondly, relying on the Wiener–Hopf factorization for the Lévy process, this random variable is seen to be identical in law with the sum of two independent random variables, the first one having the distribution of the supremum of the process up to the exponential time and the second one with the distribution of the infimum of the process in the same random interval. Simple calculations taking into account this fact, and the one-sided structure of the solution of the optimal stopping problem, give a representation of the solution of an optimal stopping problem in terms of the maximum of the Lévy process—a result that has been obtained earlier in several particular cases. This analysis is carried out in Section 4.

The rest of the paper is as follows. In Section 2 the framework of Hunt processes in which we are working is described. Section 5 consists of two subsections. In the first one, we illustrate the made assumptions concerning the Hunt processes and Lévy processes by studying Brownian motion with exponential jumps. In the second one an optimal stopping problem for a compound Poisson process with negative drift and positive exponential jumps and the reward function $g(x) = (x^+)^\gamma$, $\gamma \geq 1$, is analyzed. Our interest in this particular reward function was arised by Alexander Novikov's talk in the Symposium on Optimal Stopping with Applications held in Manchester 22–27.1.2006 [17] where the optimal stopping problem for the same reward functions and general Lévy processes were considered.

## 2. Preliminaries on Hunt processes

Let $X = \{X_t\}$ be a transient Hunt process taking values in $\mathbf{R}$, where we omit $t \geq 0$ in the notation, as all the considered processes are indexed in the same set. In particular, $X$ is a strong Markov process, quasi left continuous on $[0, +\infty)$ and the sample paths of $X$ are right continuous with left limits (see Kunita and Watanabe [11] and Blumenthal and Getoor [2], p. 45). The notations $\mathbf{P}_x$ and $\mathbf{E}_x$ are used for the probability measure and the expectation operator, respectively, associated with $X$ when $X_0 = x$. The resolvent $\{G_r : r \geq 0\}$ of $X$ is defined via

$$G_r(x, A) := \int_0^\infty \mathrm{e}^{-rt}\, \mathbf{P}_x(X_t \in A)\, \mathrm{d}t, \tag{2.1}$$

where $x \in \mathbf{R}$ and $A$ is a Borel subset of $\mathbf{R}$. We assume also that there exists a dual resolvent $\{\hat{G}_r : r \geq 0\}$ with respect to some $\sigma$-finite (duality) measure $m$, that is, for all $f, g \in \mathcal{B}_o$ and

$r \geq 0$ it holds

$$\int m(\mathrm{d}x) f(x)\, G_r g(x) = \int m(\mathrm{d}x)\, \hat{G}_r f(x)\, g(x),$$

where $\mathcal{B}_o$ denotes the set of measurable bounded functions with compact support. Moreover, it is assumed that $\{\hat{G}_r : r \geq 0\}$ is a resolvent of a transient Hunt process $\hat{X}$ taking values in $\mathbf{R}$. Finally, we impose Hypothesis (B) from Ref. [11] p. 498:

- $(\mathrm{h}_1)$ $G_r(x, \mathrm{d}y) = G_r(x, y)\, m(\mathrm{d}y)$,
- $(\mathrm{h}_2)$ $\sup_{x \in A} G_0(x, B) < \infty$ for all compact $A$ and $B$,
- $(\mathrm{h}_3)$ $x \mapsto \hat{G}_r f(x), f \in \mathcal{B}_o$, is continuous and finite.

The assumption that the dual process is a Hunt process implies that $\hat{G}$ is regular (see Ref. [11], p. 494).

We remark that when $X$ is a Lévy process a dual resolvent always exists as the resolvent of the dual process $\hat{X} = \{-X_t\}$. Hereby the Lebesgue measure serves as the duality measure (see Section II.1 in Ref. [1]).

Under these assumptions it can be proved (see Ref. [11] p. 498) that the function $G_r$ given in $(\mathrm{h}_1)$ constitutes a potential kernel (often called a Green kernel) of exponent $r$ associated with the pair $(X, \hat{X})$. This means that for each given $r \geq 0$ the function $(x, y) \mapsto G_r(x, y)$ is jointly measurable and has the properties

- $(\mathrm{p}_1)$ $\hat{G}_r(y, \mathrm{d}x) = G_r(x, y) m(\mathrm{d}x)$,
- $(\mathrm{p}_2)$ $x \mapsto G_r(x, y)$ is $r$-excessive for $X$ for each $y$,
- $(\mathrm{p}_3)$ $y \mapsto G_r(x, y)$ is $r$-excessive for $\hat{X}$ for each $x$.

Recall that a non-negative measurable function $f : \mathbf{R} \mapsto [0, +\infty]$ is called $r$-excessive for $X$ if for all $x \in \mathbf{R}$

- $(\mathrm{e}_1)$ $\mathrm{e}^{-rt}\, \mathbf{E}_x(f(X_t)) \leq f(x)$ for all $t \geq 0$,
- $(\mathrm{e}_2)$ $\mathrm{e}^{-rt} \mathbf{E}_x(f(X_t)) \rightarrow f(x)$ as $t \rightarrow 0$.

Notice that $r$-excessive functions of $X$ are 0-excessive for the process $X^0$ obtained from $X$ by exponential killing with rate $r$.

From the assumption $(\mathrm{h}_1)$ that the resolvent kernel of $X$ is absolutely continuous it follows that $r$-excessive functions are lower semi-continuous.

The Riesz decomposition of excessive functions is of key importance in our approach to optimal stopping. We state the decomposition relying on [11] Theorem 2 p. 505 and Proposition 13.1 p. 523. Indeed, it holds, under the made assumptions, that each $r$-excessive function $u$ locally integrable with respect to the duality measure $m$ can be decomposed uniquely in the form

$$u(x) = \int_{\mathbf{R}} G_r(x, y) \sigma_u(\mathrm{d}y) + h_r(x), \tag{2.2}$$

where $h_r$ is an $r$-harmonic function and $\sigma_u$ is a Radon measure on $\mathbf{R}$. We remark that the assumption that the dual process is a Hunt process implies that also the spectral measure $\sigma_u$ is unique (see Ref. [11] Proposition 7.11 p. 503).

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Conversely (see Ref. [11] Proposition 7.6 p. 501), given a Radon measure $\sigma$ on $(\mathbf{R}, \mathcal{B})$ the function $v$ defined via

$$v(x) := \int_{\mathbf{R}} G_r(x, y)\, \sigma(\mathrm{d}y)$$

is an $r$-excessive function (in fact, a potential).

An $r$-excessive function $u$ is said to $r$-harmonic on a Borel subset $A$ of $\mathbf{R}$ if for all open subsets $A_c$ of $A$ with compact closure

$$u(x) = \mathbf{E}_x\left(\mathrm{e}^{-rH(A_c)} u(X_{H(A_c)})\right), \tag{2.3}$$

where

$$H(A_c) := \inf\{t : X_t \notin A_c\}.$$

In our case (see Ref. [11] Proposition 6.2 p. 499) it holds that for each fixed $y \in \mathbf{R}$ the function $x \mapsto G_r(x, y)$ is $r$-harmonic on $\mathbf{R}\backslash\{y\}$. From equation (2.2) it follows that if there exists an open set $A$ such that the representing measure does not charge $A$, then the function $u$ is $r$-harmonic on $A$, i.e.

$$\sigma_u(A) = 0 \Rightarrow u \text{ is } r\text{-harmonic on } A. \tag{2.4}$$

In fact, implication (2.4) is an equivalence under mild assumptions; for this see Dynkin [7] Theorem 12.1.

## 3. Optimal stopping of Hunt processes

Let $X$ be a Hunt process satisfying the assumptions made in Section 2. For simplicity we consider non-negative continuous reward functions. Then the reward function $g$ has the smallest excessive majorant $V$ and

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x(\mathrm{e}^{-r\tau} g(X_\tau)), \tag{3.1}$$

where $\mathcal{M}$ denotes the set of all stopping times $\tau$ with respect to the natural filtration generated by $X$. In case $\tau = +\infty$ we define

$$\mathrm{e}^{-r\tau} g(X_\tau) := \limsup_{t \to \infty} \mathrm{e}^{-rt} g(X_t).$$

The result (3.1) can be found, for instance in Shiryayev [22] (Lemma 3 p. 118 and Theorem 1 p. 124) and holds for more general standard Markov processes, and for almost-Borel and $\mathcal{C}_0-$lower semicontinuous reward functions. We can express this result by saying that in an optimal stopping problem the value function and the smallest excessive majorant of the reward function coincide.

From equation (3.1) and the Riesz decomposition (2.2) we conclude that the problem of finding the value function is equivalent to the problem of finding the representing measure of the smallest excessive majorant (up to harmonic functions). Furthermore, based on implication (2.4), it is seen, roughly speaking, that the continuation region, that is, the region where it is not optimal to stop, is the "biggest" set not charged by the representing measure $\sigma_V$ of $V$. In short, the representing measure gives the value function via equation (2.2) and the

stopping region is by implication (2.4) the support of the representing measure. In the following result, we use the preceding considerations in order to express the solution of a particular type of optimal stopping problems

THEOREM 3.1. Consider a Hunt process $\{X_t\}$ satisfying the assumptions made in Section 2, a non-negative continuous reward function $g$, and a discount rate $r \geq 0$ such that

$$\mathbf{E}_x\left(\sup_{t\geq 0} \mathrm{e}^{-rt} g(X_t)\right) < \infty. \tag{3.2}$$

Assume that there exists a Radon measure $\sigma$ with support on the set $[x^*, \infty)$ such that the function

$$V(x) := \int_{[x^*,\infty)} G_r(x, y)\sigma(\mathrm{d}y) \tag{3.3}$$

satisfies the following conditions:

(a) $V$ is continuous,
(b) $V(x) \to 0$ when $x \to -\infty$.
(c) $V(x) = g(x)$ when $x \geq x^*$,
(d) $V(x) \geq g(x)$ when $x < x^*$.

Let

$$\tau^* = \inf\{t \geq 0 : X_t \geq x^*\}. \tag{3.4}$$

Then $\tau^*$ is an optimal stopping time and $V$ is the value function of the optimal stopping problem for $\{X_t\}$ with the reward function $g$, in other words,

$$V(x) = \sup_{\tau\in\mathcal{M}} \mathbf{E}_x(\mathrm{e}^{-r\tau} g(X_\tau)) = \mathbf{E}_x(\mathrm{e}^{-r\tau^*} g(X_{\tau^*})), \quad x \in \mathbf{R}.$$

*Proof*. Since $V$ is an $r$-excessive function (see the discussion after the Riesz decomposition (2.2)) and, from conditions (c) and (d), a majorant of $g$, it follows by Dynkin's characterization of the value function as the least excessive majorant, that

$$V(x) \geq \sup_{\tau\in\mathcal{M}} \mathbf{E}_x(\mathrm{e}^{-r\tau} g(X_\tau)) \tag{3.5}$$

In order to conclude the proof, we establish the equality in equation (3.5). Indeed, consider for each $n \geq 1$ the stopping time

$$\tau_n = \inf\{t \geq 0 : X_t \notin (-n, x^* - 1/n)\}.$$

For $\omega \in \{\tau^* < \infty\}$ define $\bar{\tau} = \lim_{n\to\infty} \tau_n$. We have

$$\tau_1 \leq \tau_2 \leq \cdots \leq \bar{\tau} \leq \tau^*.$$

For $n$ large enough, $X_{\tau_n} \geq x^* - 1/n$, and, as the process is quasi-left continuous, $\lim_{n\to\infty} X_{\tau_n} = X_{\bar{\tau}}$, and, hence, $X_{\bar{\tau}} \geq x^*$. This give us that $\bar{\tau} = \tau^*$ a.s.

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As $V$ is $r$-excessive, the sequence $\{e^{-r\tau_n}V(X_{\tau_n})\}$ is a non-negative supermartingale, and, in consequence, it converges a.s. to a random variable. Because $X_{\tau_n} \to X_{\tau^*}$ a.s., and $V$ is continuous, we identify the limit as $e^{-r\tau^*}V(X_{\tau^*})$. From assumptions (a) and (b) it follows that

$$C_V := \sup_{x \le x^*} V(x) < \infty,$$

Furthermore, as

$$e^{-r\tau_n}V(X_{\tau_n}) = e^{-r\tau_n}V(X_{\tau_n})\mathbf{1}_{\{\tau_n<\tau_*\}} + e^{-r\tau^*}g(X_{\tau^*})\mathbf{1}_{\{\tau_n=\tau_*\}} \le C_V + \sup_{t\ge 0} e^{-rt}g(X_t)$$

we obtain, in view of condition (3.2), using the Lebesgue dominated convergence theorem

$$\mathbf{E}_x(e^{-r\tau_n}V(X_{\tau_n})) \downarrow \mathbf{E}_x(e^{-r\tau^*}V(X_{\tau^*})) \quad \text{as } n \to \infty.$$

Furthermore, as the representing measure $\sigma$ does not charge the open set $(-\infty, x^*)$, the function $V$ is harmonic on that set (cf. (2.4)), and, as $\tau_n$ are exit times from the open sets $(-n, x^* - 1/n)$, we conclude that

$$V(x) = \mathbf{E}\big(e^{-r\tau_n}V(X_{\tau_n})\big) \downarrow \mathbf{E}\big(e^{-r\tau^*}g(X_{\tau^*})\big),$$

and the proof is complete. □

Under the additional assumption (3.6), valid in many particular cases, we characterize now the optimal threshold $x^*$ as a solution, with a useful uniqueness property, of an equation derived from definition (3.3). Remember that by the definition of the support of a Radon measure, we have $\sigma(A) > 0$ for all open subsets $A$ of $[x^*, \infty)$ (see for instance page 215 in Folland [8]).

COROLLARY 3.2. Let $\{X_t\}, g, V, \sigma$ and $x^*$ be as in Theorem 3.1. Assume that

$$G_r(x, B) := \int_0^\infty e^{-rt}\mathbf{P}_x(X_t \in B)\, dt > 0 \tag{3.6}$$

for all $x$ and open subsets $B$ of $\mathbf{R}$. Then the equation

$$g(x) = \int_{[x,\infty)} G_r(x, y)\sigma(dy) \tag{3.7}$$

has no solution bigger than $x^*$.

*Proof.* Clearly, since $g(x^*) = V(x^*)$ it is immediate from definition (3.3) that $x^*$ is a solution of equation (3.7). Let now $x_o > x^*$ be another solution of equation (3.7), i.e.

$$g(x_o) = \int_{[x_o,\infty)} G_r(x_o, y)\sigma(dy).$$

From definition (3.3) we have

$$g(x_o) = V(x_o) = \int_{[x^*,\infty)} G_r(x_o, y)\sigma(dy).$$

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Consequently,

$$\int_{[x*,x_o)} G_r(x_o, y)\sigma(\mathrm{d}y) = 0. \tag{3.8}$$

But the function $y \mapsto G_r(x_o, y)$ is lower semi-continuous, and, hence, the set $\{y : G_r(x_o, y) > 0\}$ is open. From equation (3.8) it follows that $G_r(x_o, \cdot) \equiv 0$ on $(x*, x_o)$, but this violates assumption (3.6) and the claim is proved. □

The form of the optimal stopping time as in definition (3.4) appears in several optimal stopping problems, in particular in mathematical finance, where this sort of the problems are sometimes named *call-like* perpetual problems or options (see, e.g. [3]). Furthermore, as exposed in Section 4, one sided problems in the context of Lévy process admit a representation in terms of the maximum of the Lévy process.

The presented method works similarly when the stopping region is not a half line, i.e. when the problem is not a "one-sided" problem. Minor modifications in the proof of Theorem 3.1 give the following result, that can be considered as a "two-sided" optimal stopping problem.

THEOREM 3.3. Consider a Hunt process $\{X_t\}$ satisfying the assumptions made in Section 2 and a non-negative continuous reward function $g$, and a discount rate $r \geq 0$, such that condition (3.2) hold. Assume that there exists a positive Radon measure $\sigma$ with support on the set $S = (-\infty, x_*] \cup [x*, \infty)$ such that the function

$$V(x) := \int_S G_r(x, y)\sigma(\mathrm{d}y) = \int_{(-\infty, x_*]} G_r(x, y)\sigma(\mathrm{d}y) + \int_{[x*,\infty)} G_r(x, y)\sigma(\mathrm{d}y) \tag{3.9}$$

satisfies the following conditions:

(a) $V$ is continuous,
(b) $V(x) = g(x)$ when $x \in S$,
(c) $V(x) \geq g(x)$ when $x \notin S$.

Let

$$\tau* = \inf\{t \geq 0 : X_t \in S\}.$$

Then $\tau*$ is an optimal stopping time and $V$ is the value function of the optimal stopping problem for $\{X_t\}$ with the reward function $g$, in other words,

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E}_x(\mathrm{e}^{-r\tau} g(X_\tau)) = \mathbf{E}_x(\mathrm{e}^{-r\tau*} g(X_{\tau*})), \quad x \in \mathbf{R}.$$

## 4. Optimal stopping and maxima for Lévy processes

As we have mentioned, in several papers explicit solutions to optimal stopping problems for general random walks or Lévy process, and some particular reward functions can be expressed in terms of the maximum of the process, killed at a constant rate $r$, the discount rate of the

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problem. The pioneer results in this direction are contained in the paper of Darling *et al.* [5], where solutions to optimal stopping problems for rewards $g(x) = (e^x - 1)^+$ and $g(x) = x^+$ are obtained in the whole class of random walks, in terms of the maximum of the random walk. These results are generalized for Lévy process by Mordecki in Refs [15] and [16], where it is also observed that similar results hold for solutions of optimal stopping problems in terms of the infimum of the process for the payoff $g(x) = (1 - e^x)^+$. Based on the technique of factorization of pseudo-differential operators, Boyarchenko and Levendorskij (see Ref. [3] and the references therein) obtain similar results, in a subclass of Lévy processes, called regular and of exponential type (RLPE), with the important feature that their results are not based on particular properties of the reward function $g$, and, hence, hold true in a certain class of rewards functions. Results in Ref. [3] suggest that any optimal stopping problem for a Lévy process with an increasing payoff can be expressed in terms of the maximum of the process. More recently, Novikov and Shiryaev [18] obtained the solution of the optimal stopping problem for a general random walk, in terms of the maximum, when the reward is $g(x) = (x^+)^n$, (and also when $g(x) = 1 - e^{-x^+}$). The respective generalization of this problem to the framework of Lévy processes has been carried out by Kyprianou and Surya [12]. In these cases the function $Q$ in Proposition 4.3 is an Appel polynomial. For the treatment of the reward $g(x) = (x^+)^\gamma$ with $\gamma \geq 1$ we refer to Ref. [17].

### *4.1 Lévy processes*

Let $X = \{X_t\}$ be a Hunt process with independent and stationary increments, i.e. a Lévy process. We denote $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{P} = \mathbf{P}_0$.

If $v \in \mathbf{R}$, Lévy–Khinchine formula states $\mathbf{E}(e^{ivX_t}) = e^{t\psi(iv)}$, where, for complex $z = iv$ the characteristic exponent of the process is

$$\psi(z) = az + \frac{1}{2}b^2z^2 + \int_{\mathbf{R}} (e^{zx} - 1 - zh(x))\Pi(dx). \tag{4.1}$$

Here the truncation function $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$ is fixed, and the parameters characterizing the law of the process are: the drift $a$, an arbitrary real number; the standard deviation of the Gaussian part of the process $b \geq 0$; and the Lévy jump measure $\Pi$, a non negative measure, defined on $\mathbf{R}\backslash\{0\}$ such that $\int(1 \wedge x^2)\Pi(dx) < +\infty$.

Denote by $\tau(r)$ an exponential random variable with parameter $r > 0$, independent of the process $X$. A key role in this section is played by the following random variables:

$$M_r = \sup_{0\leq t<\tau(r)} X_t \quad \text{and} \quad I_r = \inf_{0\leq t<\tau(r)} X_t \tag{4.2}$$

called the *supremum* and the *infimum* of the process, respectively, killed at rate $r$.

A relevant instrument to study these random variables is the Wiener–Hopf–Rogozin factorization, obtained by Rogozin [19], that states

$$\frac{r}{r - \psi(iv)} = \mathbf{E}(e^{ivM_r})\mathbf{E}(e^{ivI_r}) \tag{4.3}$$

In our first result, we give some simple sufficient conditions in order to hypothesis (3.2) to hold.

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LEMMA 4.1. Assume that a non-negative function $g$ satisfies

$$g(x) \le A_0 + A_1 \mathrm{e}^{\alpha x}, \tag{4.4}$$

for non-negative constants $A_0, A_1, \alpha$. Assume furthermore that

$$\mathbf{E}\left(\mathrm{e}^{\alpha X_1}\right) < \mathrm{e}^r. \tag{4.5}$$

Then, condition (3.2) holds.

*Proof*. Let us first verify that, for $r \ge 0$, the following three statements are equivalent:

(a) $\mathbf{E}(\mathrm{e}^{\alpha X_1}) < \mathrm{e}^r$.
(b) $\mathbf{E}(\mathrm{e}^{\alpha M_r}) < \infty$.
(c) $\mathbf{E}(\sup_{t\ge 0}(\mathrm{e}^{\alpha X_t - rt})) < \infty$.

First, (a) $\Leftrightarrow$ (b) is Lemma 1 in Ref. [15]. The equivalence (a) $\Leftrightarrow$ (c) is a particular case of (a) $\Leftrightarrow$ (b), when considering the Lévy process $\{\alpha X_t - rt\}$, the first constant equal to 1, and the second, the discount rate equal to 0. Now

$$\mathbf{E}_x\left(\sup_{t\ge 0} \mathrm{e}^{-rt} g(X_t)\right) \le \mathbf{E}\left(\sup_{t\ge 0} \mathrm{e}^{-rt}(A_0 + A_1 \mathrm{e}^{\alpha(x+X_t)})\right) \le A_0 + A_1 \mathrm{e}^{\alpha x} \mathbf{E}\left(\sup_{t\ge 0} \mathrm{e}^{(\alpha X_t - rt)}\right) < \infty$$

as condition (a) $\Rightarrow$ (c). □

*Remark* 4.2. Condition (4.4) is relatively natural in our context. For instance, if the function is increasing, and *submultiplicative* (as defined in Section 25 in Ref. [21]) it automatically satisfies our exponential growth condition (4.4). Nevertheless, the submultiplicative property does not seem to be appropiate for optimal stopping problems, as $g(x) = x^+$ is not submultiplicative. Furthermore, condition (4.5) is optimal in the following sense: For the reward function $g(x) = (\mathrm{e}^x - 1)^+$, if $\mathbf{E}\left(\mathrm{e}^{X_1}\right) = \mathrm{e}^r$, then condition (3.2) does not hold since (a) $\Leftrightarrow$ (c). We remark also that for the reward function $g(x) = (x^+)^n$ another sufficient condition for inequality (3.2) to hold can be found [18] and [17]. Moreover, in Ref. [12] a corresponding condition for Lévy processes is presented.

Our next result represents the value function of the optimal stopping problem for a Lévy process in terms of the maximum of the process and is a consequence of Theorem 3.1.

PROPOSITION 4.3. Assume that the conditions of Theorem 3.1 hold, and, furthermore, that $\{X_t\}$ is in fact a Lévy process. Then, there exists a function $Q : [x*, \infty) \to \mathbf{R}$ such that the value function $V$ in definition (3.3) satisfies

$$V(x) = \mathbf{E}_x(Q(M_r); M_r \ge x*), \quad x \le x*.$$

*Proof*. The key ingredient of the proof is formula (4.3), that can be also written as

$$X_{\tau(r)} = M_r + \tilde{I}_r \tag{4.6}$$

where $M_r$ and $\tilde{I}_r = X_{\tau(r)} - M_r$ are *independent* random variables, $M_r$ given in definition (4.2), and $\tilde{I}_r$ with the same distribution as $I_r$ in definition (4.2).

From the definition of the Green kernel (2.1), it is clear that

$$rG_r(x, \mathrm{d}y) = \mathbf{P}_x(X_{\tau(r)} \in \mathrm{d}y),$$

and, in view of equation (4.6), assuming that $M_r$ and $I_r$ have respective densities $f_M$ and $f_I$ (only for simplicity of exposition), we obtain that

$$rG_r(x, y) = \begin{cases} \int_{-\infty}^{y-x} f_I(t) f_M(y - x - t)\,\mathrm{d}t, & \text{if } y - x < 0, \\ \int_{y-x}^{\infty} f_M(t) f_I(y - x - t)\,\mathrm{d}t, & \text{if } y - x > 0. \end{cases} \tag{4.7}$$

Using this formula for the Green kernel in definition (3.3) with $x < x*$ yields

$$\begin{aligned} V(x) &= \int_{x*}^{\infty} G_r(x, y)\sigma(\mathrm{d}y) = r^{-1} \int_{x*}^{\infty} \left[ \int_{y-x}^{\infty} f_M(t) f_I(y - x - t)\,\mathrm{d}t \right] \sigma(\mathrm{d}y) \\ &= r^{-1} \int_{x*-x}^{\infty} f_M(t) \left[ \int_{x*}^{x+t} f_I(y - x - t)\sigma(\mathrm{d}y) \right] \mathrm{d}t \\ &= \int_{x*-x}^{\infty} f_M(t) Q(x + t)\,\mathrm{d}t = \mathbf{E}_x(Q(M_r); M_r \geq x*), \end{aligned}$$

where

$$Q(z) = r^{-1} \int_{x*}^{z} f_I(y - z)\sigma(\mathrm{d}y), \quad z \geq x*. \tag{4.8}$$

This concludes the proof. □

The following results use Theorem 3.3 to provide a representation of the value function in terms of both the supremum and the infimum of the Lévy process.

PROPOSITION 4.4. Assume that the conditions of Theorem 3.3 hold, and that $\{X_t\}$ is a Lévy process. Then, there exist two functions

$$Q_* : (-\infty, x_*] \to \mathbf{R}, \quad Q* : [x*, \infty) \to \mathbf{R}$$

such that the value function $V$ in definition (3.3) satisfies

$$V(x) = \mathbf{E}_x(Q_*(I_r); I_r \leq x_*) + \mathbf{E}_x(Q*(M_r); M_r \geq x*), \quad x_* \leq x \leq x*.$$

*Proof.* The proof consists of rewriting each summand in equation (3.9) in terms of the maximum and infimum of the process, respectively. For the second term we use Proposition 4.3, and obtain

$$\int_{[x*,\infty)} G_r(x, y)\sigma(\mathrm{d}y) = \mathbf{E}_x(Q*(M_r); M_r \geq x*), \tag{4.9}$$

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where $Q*$ coincides with $Q$ as defined in equation (4.8). The first term is analyzed via (4.9) by considering the dual Lévy process $\hat{X}$, as follows:

$$\begin{aligned}\int_{(-\infty, x_*]} G_r(x,y)\sigma(\mathrm{d}y) &= \int_{[-x_*,\infty)} \hat{G}_r(-x,y)\sigma(-\mathrm{d}y)\\ &= \hat{\mathbf{E}}_{-x}\big(\hat{Q}*(\hat{M}_r); \hat{M}_r \geq -x_*\big)\\ &= \mathbf{E}_x(Q_*(I_r); I_r \leq x_*),\end{aligned}$$

where

$$Q_*(z) = \hat{Q}*(-z) = r^{-1}\int_{-x_*}^{-z} f_{\hat{I}}(y+z)\sigma(-\mathrm{d}y) = r^{-1}\int_z^{x_*} f_M(y-z)\sigma(\mathrm{d}y),$$

and the proof is complete. □

## 5. A case study

### 5.1 Brownian motion with exponential jumps

Here we illustrate the assumptions made in Section 2 and, in particular, the concept of Green kernel by taking $X$ to be a Brownian motion with drift and compounded with two-sided exponentially distributed jumps.

To introduce $X$, consider a standard Wiener process $W = \{W_t : t \geq 0\}$, $N^\lambda = \{N_t^\lambda : t \geq 0\}$ and $N^\mu = \{N_t^\mu : t \geq 0\}$ two Poisson processes with intensities $\lambda$ and $\mu$, respectively, $Y^\alpha = \{Y_i^\alpha : i = 1, 2, \dots\}$ and $Y^\beta = \{Y_i^\beta : i = 1, 2, \dots\}$ two sequences of independent exponentially-distributed random variables with parameters $\alpha$ and $\beta$, respectively. Moreover, $W, N^\lambda, N^\mu, Y^\alpha$ and $Y^\beta$ are assumed to be independent. The process $X = \{X_t\}$ is now defined via

$$X_t = at + bW_t + \sum_{i=1}^{N_t^\lambda} Y_i^\alpha - \sum_{i=1}^{N_t^\mu} Y_i^\beta, \tag{5.1}$$

where $a$ and $b \geq 0$ are real parameters. Clearly, $X$ is a Lévy process and its Lévy exponent is given by

$$\mathbf{E}(\exp(zX_t)) = \exp(t\psi(z)) \tag{5.2}$$

with

$$\psi(z) = az + \frac{1}{2}b^2z^2 + \lambda\frac{z}{\alpha - z} - \mu\frac{z}{\beta + z}. \tag{5.3}$$

It is enough for our purposes to take hereby $z$ real, and then the representation in equation (5.2) holds for $z \in (-\beta, \alpha)$.

Next, we compute the Green kernel of $X$ when all parameters in equation (5.3) are positive. It is easily seen that for $r \geq 0$ the equation $\psi(z) = r$ has exactly four solutions $\rho_k, k = 1, 2, 3, 4$. These satisfy

$$\rho_1 < -\beta < \rho_2 \leq 0 < \rho_3 < \alpha < \rho_4 \tag{5.4}$$

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and

$$\psi'(\rho_1) < 0, \quad \psi'(\rho_2) < 0, \quad \psi'(\rho_3) > 0, \quad \psi'(\rho_4) > 0. \tag{5.5}$$

Notice that $\rho_2 = 0$ if and only if $r = 0$, in which case it is assumed $\psi'(0) < 0$ implying

$$\lim_{t\longrightarrow\infty} X_t = -\infty \quad \text{a.s.}$$

Using the general definition (2.1) of the resolvent, we have for $z \in (\rho_2, \rho_3)$

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{zx} G_r(0, \mathrm{d}x) &= \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-rt}\, \mathbf{E}(\exp(zX_t)) = \frac{1}{r - \psi(z)} \\ &= \frac{\psi'(\rho_1)^{-1}}{\rho_1 - z} + \frac{\psi'(\rho_2)^{-1}}{\rho_2 - z} + \frac{\psi'(\rho_3)^{-1}}{\rho_3 - z} + \frac{\psi'(\rho_4)^{-1}}{\rho_4 - z}.\end{aligned}$$

Consequently, inverting the right hand side yields

$$G_r(0, \mathrm{d}x) = \begin{cases} -\psi'(\rho_1)^{-1}\mathrm{e}^{-\rho_1 x}\mathrm{d}x - \psi'(\rho_2)^{-1}\mathrm{e}^{-\rho_2 x}\mathrm{d}x, & x < 0, \\ \psi'(\rho_3)^{-1}\mathrm{e}^{-\rho_3 x}\mathrm{d}x + \psi'(\rho_4)^{-1}\mathrm{e}^{-\rho_4 x}\mathrm{d}x, & x > 0. \end{cases} \tag{5.6}$$

and, hence, the resolvent is absolutely continuous with respect to Lebesgue measure. With slight abuse of notation, we let $G_r(0, x)$ denote also the Green kernel, i.e. the density of the resolvent $G_r$ with respect to the Lebesgue measure. From the spatial homogeneity of $X$ it follows that $G_r(x, 0) = G_r(0, -x)$.

The absolute continuity of the resolvent can alternatively be verified by checking that the condition (ii) in Theorem II.5.16 in Bertoin [1] holds. We recall also the general result (see Ref. [1] p. 25) which says that the absolute continuity of the resolvent is equivalent with the property that $x \mapsto G_r f(x)$ is continuous for all essentially bounded measurable functions $f$.

As we have noticed, the process $\hat{X} = \{-X_t\}$ may be viewed as the dual process associated with $X$. Let $\hat{G}_r$ denote the resolvent of $\hat{X}$. Then the duality relationship

$$\int \mathrm{d}x f(x) G_r g(x) = \int \mathrm{d}x\, \hat{G}_r f(x)\, g(x)$$

holds the duality measure being the Lebesgue measure. The Green kernel of the dual process is given by:

$$\hat{G}_r(x, y) = G_r(y, x).$$

Notice that the value of $x \mapsto G_r(0, x)$ at 0 is chosen so that the resulting function is lower semi-continuous (since the Green kernel when considered as a function of the second argument should be excessive for the dual process).

To conclude the above discussion, we have verified Hypothesis (B) in Ref. [11], that is, $(\mathrm{h}_1)$, $(\mathrm{h}_2)$ and $(\mathrm{h}_3)$ in Section 2 are fulfilled. Consequently, also $(\mathrm{p}_1)$, $(\mathrm{p}_2)$ and $(\mathrm{p}_3)$ in Section 2 are valid and the Riesz decomposition (2.2) holds. Moreover, it can be proved, e.g. using the Martin boundary theory, as presented in Ref. [11], that the harmonic function $h_r$ appearing in equation (2.2) is of the form

$$h_r(x) = c_1 \mathrm{e}^{\rho_2 x} + c_2 \mathrm{e}^{\rho_3 x},$$

where $c_1$ and $c_2$ are non-negative constants.

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It is interesting to note that when multiplying both sides of equation (5.6) by $z$ and letting $z \to \infty$ we obtain, in case $b > 0$ (cf. (5.3)),

$$\frac{1}{\psi'(\rho_1)} + \frac{1}{\psi'(\rho_2)} + \frac{1}{\psi'(\rho_3)} + \frac{1}{\psi'(\rho_4)} = 0,$$

which implies that the Green kernel is continuous at $x = 0$. But, when $b = 0$, the Green kernel may be discontinuous. This happens, for instance, when $X$ is a compound Poisson process with negative drift and exponentially distributed positive jumps. More precisely, taking $b = \mu = 0$, and $a < 0$ in equation (5.1) the characteristic exponent reduces to

$$\psi(z) = az + \lambda \frac{z}{\alpha - z}.$$

Now there are only two roots $\rho_1$ and $\rho_2$ and these satisfy

$$\rho_1 \leq 0 < \rho_2.$$

Consequently,

$$\frac{1}{r - \psi(z)} = \frac{\psi'(\rho_1)^{-1}}{\rho_1 - z} + \frac{\psi'(\rho_2)^{-1}}{\rho_2 - z}, \tag{5.7}$$

and we have the Green kernel

$$G_r(0, x) = \begin{cases} -\psi'(\rho_1)^{-1} \mathrm{e}^{-\rho_1 x}, & x < 0, \\ \psi'(\rho_2)^{-1} \mathrm{e}^{-\rho_2 x}, & x \geq 0. \end{cases} \tag{5.8}$$

From equation (5.7) it is seen that

$$\psi'(\rho_1)^{-1} + \psi'(\rho_2)^{-1} = \frac{1}{a},$$

and, hence, $x \mapsto G_r(0, x)$ is discontinuous at 0 (but lower semi-continuous since $a < 0$ implies $-\psi'(\rho_1) < \psi'(\rho_2)$).

### *5.2 Optimal stopping of processes with two sided exponential Green kernel*

We consider here a subclass of processes introduced in Section 5.1 the aim being to apply results in Theorem 3.1 and 3.3. Indeed, let $X$ be a Lévy process having a Green kernel with the following simple exponential structure:

$$G_r(x) := G_r(0, x) = \begin{cases} -A_1 \mathrm{e}^{-\rho_1 x}, & x < 0, \\ A_2 \mathrm{e}^{-\rho_2 x}, & x \geq 0, \end{cases} \tag{5.9}$$

where $\rho_{1,2}$ are the roots of the equation $\psi(z) = r$ such that $\rho_1 \leq 0 < \rho_2$ and $A_{1,2} = 1/\psi'(\rho_{1,2})$. In the case $r = 0$ it is assumed that the process drifts to $-\infty$ and, hence, we have $\rho_1 = 0$.

The Green kernel of the exponential form (5.9) appears in two basic cases which, using the notation in equations (5.2) and (5.3), are:

- Wiener process with drift, i.e. $b > 0$, $\lambda = \mu = 0$,
- compound Poisson process with negative drift and positive exponential jumps, i.e. $a < 0$, $b = 0$, $\lambda > 0$ and $\mu = 0$.

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The point we want to make here is that our approach to optimal stopping treats these processes similarly. Recall that in the case of Wiener process usually smooth pasting is valid when moving from the continuation region to the stopping region but in the compound Poisson case there is "only" continuous pasting. In other words, our approach does not use smooth pasting as a tool, but this property can, of course, be checked (when valid) from the calculated explicit form of the value function.

PROPOSITION 5.1. For a given $x* \in \mathbf{R}$ let $\sigma$ be a measure on $[x*, +\infty)$ with a continuously differentiable density $\sigma'$ on $(x*, +\infty)$. Then the function

$$V(x) := \int_{x*}^{\infty} G_r(y - x)\sigma(dy)$$

is two times continuously differentiable on $D := \{x \in \mathbf{R} : x \neq x*\}$ and satisfies on $D$ the ordinary differential equation (ODE)

$$V''(x) - (\rho_2 + \rho_1)V'(x) + \rho_1\rho_2 V(x) = -(A_2 + A_1)\sigma''(x) + (\rho_2 A_1 + \rho_1 A_2)\sigma'(x), \quad (5.10)$$

where $\sigma''(x) = \sigma'(x) = 0$ for $x \in (-\infty, x*)$.

*Proof*. From the definition of $V$, taking into account the form of the Green kernel, we have for $x > x*$

$$V(x) = -A_1 e^{\rho_1 x}\int_{x*}^{x} e^{-\rho_1 y}\sigma(\mathrm{d}y) + A_2 e^{\rho_2 x}\int_{x}^{\infty} e^{-\rho_2 y}\sigma(\mathrm{d}y).$$

The right hand side of this equation can be differentiated twice proving that $V''$ exist in $D$, and the claimed ODE is obtained after some straightforward manipulations. □

COROLLARY 5.2. Let $X$ be a Wiener process with drift. Then the ODE in equation (5.10) takes the form

$$\frac{b^2}{2}V''(x) + aV'(x) - rV(x) = -(a^2 + 2b^2 r)\sigma'(x). \quad (5.11)$$

*Proof*. The quantities needed to derive equation (5.11) from equation (5.10) are

$$\rho_1 = -\frac{1}{b^2}\left(\sqrt{a^2 + 2b^2 r} + a\right), \quad \rho_2 = \frac{1}{b^2}\left(\sqrt{a^2 + 2b^2 r} - a\right)$$

and

$$A_1 = -\sqrt{a^2 + 2b^2 r}, \quad A_2 = \sqrt{a^2 + 2b^2 r}.$$

In particular, notice that $A_2 + A_1 = 0$ which reflects the fact that the Green kernel is continuous. □

In Novikov and Shiryayev [18] the optimal stopping problem for a general random walk with reward function $\max\{0, x^n\}$, $n = 1, 2, \ldots$, is considered, and the solution is characterized via the Appell polynomials associated with the distribution of the maximum of

the process. The extension of this result for Lévy processes is done in Kyprianou and Surya [12]. In the next example, we present explicit results for a more general reward function, that is, $\max\{0, x^\gamma\}$, $\gamma \geq 1$, but for a more particular Lévy process studied in the subsection.

*Example* 5.3. Let $X$ denote a compound Poisson process with negative drift and positive exponential jumps, i.e. take $a < 0$, $b =$ and $\mu = 0$ in equation (5.1). For simplicity, we consider optimal stopping problem without discounting:

$$\sup_{\tau \in \mathcal{M}} \mathbf{E}_x(g(X_\tau)),$$

where $g(x) := \max\{0, x^\gamma\}$ with $\gamma \geq 1$. For $r = 0$ the Green kernel of $X$ is

$$G(x,0) := G_0(x,0) = \begin{cases} A_2 \mathrm{e}^{\rho x} \mathrm{d}x, & x \leq 0, \\ -A_1, & x > 0, \end{cases} \tag{5.12}$$

where

$$\rho := \rho_2 = \alpha + \frac{\lambda}{a} > 0 \tag{5.13}$$

and

$$A_1 = \frac{\alpha}{\lambda + a\alpha} < 0, \quad A_2 = \frac{\lambda}{a(\lambda + a\alpha)} > 0.$$

Notice that $\rho > 0$ means that a.s. $\lim_{t \to \infty} X_t = -\infty$.

Our aim is to find a measure $\sigma$ and a number $x*$ such that the function $V$ defined via

$$V(x) = \int_{[x^*, +\infty)} G(x,y)\sigma(\mathrm{d}y) \tag{5.14}$$

has properties (a)–(d) given in Theorem 3.1.

To begin with, consider equation (5.10) for $x > x*$ and $V(x) = x^\gamma$, i.e.

$$-\sigma''(x) + \alpha\sigma'(x) = a\gamma(\gamma - 1)x^{\gamma - 2} - (a\alpha + \lambda)\gamma x^{\gamma - 1}.$$

Assuming $\lim_{x \to +\infty} \mathrm{e}^{-\alpha x}\sigma'(x) = 0$ we obtain the solution

$$\sigma'(x) = -a\gamma x^{\gamma - 1} - \lambda \mathrm{e}^{\alpha x} \int_x^\infty \mathrm{e}^{-\alpha y} \gamma y^{\gamma - 1} \mathrm{d}y.$$

If $\gamma = 1$ then $\sigma'(x) = -a - (\lambda/\alpha) > 0$. For $\gamma > 1$ it is easily seen that $\sigma'(0) < 0$ and $\sigma'(x) \to +\infty$ as $x \to \infty$.

The claim is that the equation $\sigma'(x) = 0$, i.e.

$$x^{\gamma - 1} = \frac{\lambda}{(-a)} \mathrm{e}^{\alpha x} \int_x^\infty \mathrm{e}^{-\alpha z} z^{\gamma - 1} \mathrm{d}z, \tag{5.15}$$

has a unique solution for $x > 0$, which we denote by $x^*_{\gamma - 1}$. Equation (5.15) is equivalent to

$$F(x, \gamma - 1) = 1, \tag{5.16}$$

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if we define

$$F(x;u) := \frac{\lambda}{(-a)} \int_0^\infty \mathrm{e}^{-\alpha y} \Big(1 + \frac{y}{x}\Big)^u \mathrm{d}y. \tag{5.17}$$

We revise some properties of the function just introduced.

LEMMA 5.4. The equation $F(\varphi(u), u) = 1, u \geq 1$, where $F(x, u)$ is as given in definition (5.17), defines an implicit function $\varphi : [1, \infty) \to \mathbf{R}$. Moreover, the function $\varphi$ is increasing, and satisfies the inequality

$$\varphi(1) < \varphi(u) < \frac{u}{\rho}. \tag{5.18}$$

*Proof*. It is not difficult to verify that, for fixed $u > 0$, the function $F$ is decreasing in $x$, and that

$$\lim_{x \to 0+} F(x, u) = \infty, \quad \lim_{x \to \infty} F(x, u) = \frac{\lambda}{(-a)\alpha} < 1.$$

This means that for any $u \geq 1$ the equation $F(x, u) = 1$ has a unique solution $x := \varphi(u)$. Furthermore, it is also clear that, for fixed $x > 0$, the function $F(x, u)$ is increasing in $u$. This means, that $\varphi$ is increasing, as

$$\frac{\partial \varphi}{\partial u} = -\frac{\partial F}{\partial x} \left(\frac{\partial F}{\partial u}\right)^{-1} > 0.$$

Finally, multiplying the inequality

$$\Big(1 + u\frac{y}{x}\Big) \leq \Big(1 + \frac{y}{x}\Big)^u \leq \mathrm{e}^{uy/x}$$

by $\mathrm{e}^{-\alpha y}$ and integrating we obtain that

$$F_1(x, u) := \frac{\lambda}{(-a)\alpha} \Big(1 + \frac{u}{\alpha x}\Big) < F(x, u) < \frac{\lambda}{(-a)} \frac{1}{\alpha - u/x} =: F_2(x, u).$$

and inequality (5.18) follows as the bounds are the respective roots of the equations $F_1(x, u) = 1$, $F_2(x, u) = 1$, and, in particular, the root $x_1$ of the first equation is

$$x_1 = \frac{\gamma\lambda}{(-a)\alpha\rho},$$

and $x_1^* = -\lambda/a\alpha\rho$. This last value can be computed from the equation $F(x, 1) = 1$, and was found in Ref. [14]. This concludes the proof of the Lemma. □

Observe now that for $x > x^*_{\gamma-1} =: x^o_\gamma$ the function $\sigma'$ induces a positive Radon measure on $(x^0_\gamma, \infty)$. However, since, for any constant $c$, the function $x^\gamma + c$ induces the same measure as the function $x^\gamma$ it remains to find, the support of $\sigma$ of the form $(x*, \infty)$ such that for all $x > x*$

$$x^\gamma = \int_{[x*, +\infty)} G(x, y)\sigma'(y)\mathrm{d}y.$$

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Therefore, consider

$$\int_x^\infty G(x,y)\sigma(\mathrm{d}y) = A_2 \mathrm{e}^{\rho x}\int_x^\infty \mathrm{e}^{-\rho y}\sigma'(y)\mathrm{d}y$$
$$= \gamma A_2 \mathrm{e}^{\rho x}\int_x^\infty \mathrm{e}^{-\rho y}\left(-a y^{\gamma-1} - \lambda \mathrm{e}^{\alpha y}\int_y^\infty \mathrm{e}^{-\alpha z} z^{\gamma-1}\mathrm{d}z\right)\mathrm{d}y.$$

Applying Fubini's theorem for the latter term yields

$$\int_x^\infty \mathrm{d}y\, \mathrm{e}^{(\alpha-\rho)y}\int_y^\infty \mathrm{d}z\, \mathrm{e}^{-\alpha z} z^{\gamma-1} = \frac{1}{\alpha-\rho}\left(\int_x^\infty \mathrm{e}^{-\rho z} z^{\gamma-1}\mathrm{d}z - \mathrm{e}^{(\alpha-\rho)x}\int_x^\infty \mathrm{e}^{-\alpha z} z^{\gamma-1}\mathrm{d}z\right).$$

Observing that $-a = \lambda/(\alpha-\rho)$ we have

$$\int_x^\infty G(x,y)\sigma(\mathrm{d}y) = \frac{\alpha-\rho}{\rho}\mathrm{e}^{\alpha x}\int_x^\infty \mathrm{e}^{-\alpha z}\gamma z^{\gamma-1}\mathrm{d}z$$

Consequently, after an integration by parts, the equation

$$x^\gamma = \frac{\alpha-\rho}{\rho}\mathrm{e}^{\alpha x}\int_x^\infty \mathrm{e}^{-\alpha z}\gamma z^{\gamma-1}\mathrm{d}z$$

is seen to be equivalent with

$$x^\gamma = (\alpha-\rho)\mathrm{e}^{\alpha x}\int_x^\infty \mathrm{e}^{-\alpha z} z^\gamma \mathrm{d}z \tag{5.19}$$

which coincides with equation (5.15) if therein $\gamma - 1$ is changed to $\gamma$. Hence, equation (5.19) has a unique solution which is, using the notation introduced above, $x^*_\gamma$. As the function $x = \varphi(u)$ is increasing, we know that $x^0_\gamma = \varphi(\gamma-1) < \varphi(\gamma) = x^*_\gamma$.

Next step is to verify that the value function obtained from equation (5.14) is continuous and satisfies $V(x) > x^\gamma$ for $x < x^*_\gamma$. Therefore, consider for $x < x^*_\gamma$

$$V(x) = \int_{[x^*_\gamma,+\infty)} G(x,y)\sigma(\mathrm{d}y) = \mathrm{e}^{\rho(x-x^*_\gamma)}\left(x^*_\gamma\right)^\gamma. \tag{5.20}$$

Consequently, $V$ is continuous and

$$V(x) > x^\gamma \Leftrightarrow \mathrm{e}^{-\rho x^*_\gamma}\left(x^*_\gamma\right)^\gamma > \mathrm{e}^{-\rho x} x^\gamma. \tag{5.21}$$

The right hand side of equivalence (5.21) holds if $x \mapsto G(x) := \mathrm{e}^{-\rho x}x^\gamma$ is increasing for $x < x^*_\gamma$. Clearly, $G'$ is positive if

$$-\rho x + \gamma > 0,$$

and this holds since $x^*_\gamma < \gamma/\rho$ by the second inequality in (5.18).

To conclude, the optimal stopping time $\tau^*$ is given by:

$$\tau^* := \inf\left\{t : X_t \geq x^*_\gamma\right\},$$

and the value function $V$ is for $x < x^*_\gamma$ as in equation (5.20). Since $V'\left(x^*_\gamma-\right) = \rho\left(x^*_\gamma\right)^\gamma$ and $g'\left(x^*_\gamma\right) = \gamma\left(x^*_\gamma\right)^{\gamma-1}$ there is no smooth fit at $x^*_\gamma$.

.

| $\alpha$ | $\rho$ | $-\lambda/\rho$ | $\gamma$ | $\gamma/\rho$ | $x^*_\gamma$ | $x^0_\gamma$ |
|---|---|---|---|---|---|---|
| 10 | 1 | 9 | 20 | 20 | 19.8896 | 18.8896 |
| 10 | 1 | 9 | 10 | 10 | 9.8902 | 8.8904 |
| 10 | 1 | 9 | 5 | 5 | 4.8915 | 3.8921 |
| 10 | 1 | 9 | 2.5 | 2.5 | 2.3939 | 1.3968 |
| 10 | 1 | 9 | 1 | 1 | 0.9 | – |
| 10 | 9 | 1 | 20 | 2.2222 | 1.7613 | 1.6579 |
| 10 | 9 | 1 | 10 | 1.1111 | 0.7511 | 0.6547 |
| 10 | 9 | 1 | 5 | 0.5555 | 0.2881 | 0.2045 |
| 10 | 9 | 1 | 2.5 | 0.2789 | 0.0917 | 0.0319 |
| 10 | 9 | 1 | 1 | 0.1111 | 0.0111 | – |
| 1 | 0.5 | 0.5 | 20 | 40 | 38.1592 | 36.166 |
| 1 | 0.5 | 0.5 | 10 | 20 | 18.2726 | 16.2942 |
| 1 | 0.5 | 0.5 | 5 | 10 | 8.4369 | 6.5011 |
| 1 | 0.5 | 0.5 | 2.5 | 5 | 3.6529 | 1.8398 |
| 1 | 0.5 | 0.5 | 1 | 2 | 1 | – |

We conclude by presenting the above table with some numerical results. The computations are done with Mathematica-package where one can find a subroutine for incomplete gamma-function and programs for numerical solutions of equations based on standard Newton–Raphson's method and the secant method. A good starting value for Newton–Raphson's method seems to be $\gamma/\rho$. It is interesting to notice from the table that if $\rho \ll \alpha$ then $x^*_\gamma \simeq \gamma/\rho$.

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## Acknowledgements

Ernesto Mordecki thanks Åbo Akademi (Åbo, Finland), and the National Visitors Program (Finland) for the support and hospitality, that made possible the initiation of this work. The research of Paavo Salminen was supported by the Academy of Finland (grant no. 105849).

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