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PRICING DERIVATIVES ON TWO-DIMENSIONAL LÉVY PROCESSES

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The aim of this work is to use a duality approach to study the pricing of derivatives depending on two stocks driven by a bidimensional Lévy process. The main idea is to apply Girsanov's Theorem for Lévy processes, in order to reduce the posed problem to a problem with one Lévy driven stock in an auxiliary market, baptized as "dual market". In this way, we extend the results obtained by Gerber and Shiu [5] for two dimensional Brownian motion.

Keywords: Lévy processes; optimal stopping; dual market method; derivative pricing.

1. Introduction

Since Margrabe's [10] paper, many important extensions have been carrying on to study derivatives written on two stocks. Margrabe studied the pricing of European options for the case of two non-dividend-paying stocks driven by geometric Brownian motions. More precisely, the pricing of the right to change one asset for another at the end of some fixed period of time obtaining closed form formulas for this problem. We thus extend the Black and Scholes pricing model.

The American option pricing problem leads to the solution of an optimal stopping problem, that in general does not admit closed form solutions (see [6]). In the perpetual case, i.e., the option has no expiration date, Gerber and Shiu [5] obtain a closed form formula using the optional sampling theorem, assuming that stock prices are driven by geometric Brownian motions and stocks pay constant rate continuous dividends. They also study the pricing of the *Perpetual Maximum Option*, that is an option whose payoff is the maximum between two or more stocks and

has no expiration date. Finally, they study American perpetual options with more general payoffs which are homogeneous of degree one.

In this paper we consider the problem of pricing European and American type derivatives written on a two dimensional stock driven by a two dimensional Lévy processes (it can be said that the stock follows a *two dimensional geometric Lévy process*), with a payoff function homogeneous of an arbitrary degree.

The paper is organized as follows: In Sec. 2 we describe the market model and introduce the pricing problem, illustrating with several important examples of traded derivatives. In Sec. 3 we describe the *Dual Market Method*, a method which allows to reduce the two stock problem into a one stock problem. In Sec. 4 we derive some closed form formulas based on the proposed, method and known results for one-dimensional problems. The final section is the conclusions, followed by an appendix.

2. Market Model

2.1. *Multidimensional Lévy processes*

Let $X = (X^1, \dots, X^d)$ be a d-dimensional Lévy process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$. This means that X is a stochastically continuous stochastic process with independent increments, such that the distribution of $X_{t+s} - X_s$ does not depend on s, with $P(X_0 = 0) = 1$ and trajectories continuous from the left with limits from the right. The basis $\mathcal{B}$ is supposed to satisfy the usual assumptions, i.e., continuity from the right and $\mathcal{F}_0$ is P complete. For $z = (z_1, \dots, z_d)$ in C^d, when the integral is convergent (and this is always the case if $z = i\lambda$ with λ in $\mathbb{R}^d$), Lévy-Khinchine formula states, that $\mathsf{E}e^{zX_t} = \exp(t\psi(z))$ where the function ψ is the *characteristic exponent* of the process, and is given by

$$\psi(z) = (a, z) + \frac{1}{2}(z, \Sigma z) + \int_{\mathbb{R}^d} \big(e^{(z,y)} - 1 - (z, y)\mathbf{1}_{\{|y|\leq 1\}}\big)\Pi(dy), \tag{2.1}$$

where $a = (a_1, \dots, a_d)$ is a vector in $\mathbb{R}^d$, Π is a positive measure defined on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d}(|y|^2 \wedge 1)\Pi(dy)$ is finite, and $\Sigma = ((s_{ij}))$ is a symmetric nonnegative definite matrix, that can always be written as $\Sigma = A'A$ (where $'$ denotes transposition) for some matrix A.

The triplet (a, Σ, Π) completely determines the law of the process X. Particular interest has the case when $\alpha = \int_{\mathbb{R}^d} \Pi(dy)$ is finite, i.e., X is a diffusion with jumps. Introducing F by $\Pi(dy) = \alpha F(dy)$, Lévy-Khinchine formula is (changing the value of a if necessary)

$$\psi(z) = (a, z) + \frac{1}{2}(z, \Sigma z) + \int_{\mathbb{R}^d} \big(e^{(z,y)} - 1\big)\Pi(dy), \tag{2.2}$$

and the process $X = \{X_t\}_{t\geq 0}$ can be represented by

$$X_t = at + AW_t + \sum_{k=1}^{N_t} Y_k,$$

where W is a standard d-dimensional Brownian motion, $N = \{N_t\}_{t\geq 0}$ is a Poisson process with parameter α, and $\{Y_k\}_{k\geq 1}$ is a sequence of independent d-dimensional random vectors with identical distribution $F(dy)$.

Another important case is when the coordinates of X are independent processes. This happens if and only if Σ is a diagonal matrix (and A can be chosen to be diagonal also) and the measure Π has support on the set $\{x \in \mathbb{R}^d : \prod_{k=1}^d x_k = 0\}$, (i.e., it is concentrated on the union of the coordinate axes, see E 12.10 in Sato [13]). In this case $\psi(z) = \sum_{k=1}^d \psi_k(z_k)$, where ψ_k is the characteristic exponent of the k-coordinate of X, given by

$$\psi_k(z_k) = a_k z_k + \frac{1}{2} s_{kk} z_k^2 + \int_{\mathbb{R}} \left(e^{z_k y} - 1 - z_k y \mathbf{1}_{\{|y|\leq 1\}}\right) \Pi_k(dy),$$

where $\Pi_k(A) = \int_{\{x\in\mathbb{R}^d : x_k \in A\}} \Pi(dx)$.

2.2. *Market and problem*

Consider a market model with three assets (S^1, S^2, S^3) given by

$$S_t^1 = e^{X_t^1}, \quad S_t^2 = S_0^2 e^{X_t^2}, \quad S_t^3 = S_0^3 e^{X_t^3}, \tag{2.3}$$

where (X^1, X^2, X^3) is a three dimensional Lévy process, and for simplicity, and without loss of generality we take $S_0^1 = 1$. The first asset is the bond and is usually deterministic. Randomness in the bond $\{S_t^1\}_{t\geq 0}$ allows to consider more general situations, as for example the pricing problem of a derivative written in a foreign currency, referred as *Quanto option*.

Consider a function:

$$f : (0,\infty) \times (0,\infty) \to \mathbb{R},$$

homogenous of an arbitrary degree α; i.e., for any $\lambda > 0$ and for all positive x, y

$$f(\lambda x, \lambda y) = \lambda^\alpha f(x, y).$$

In the above market a derivative contract with payoff given by

$$\Phi_t = f(S_t^2, S_t^3)$$

is introduced.

Assuming that we are under a risk neutral martingale measure, thats to say, $\frac{S^k}{S^1}$ $(k = 2, 3)$ are P-martingales, i.e., P is an equivalent martingale measure (EMM),[1] we want to price the derivative contract just introduced. In the European case, the problem reduces to the computation of

$$E_T = E(S_0^2, S_0^3, T) = \mathsf{E}\left[e^{-X_T^1} f(S_0^2 e^{X_T^2}, S_0^3 e^{X_T^3})\right]. \tag{2.4}$$

In the American case, if $\mathcal{M}_T$ denotes the class of stopping times up to time T, i.e.,

$$\mathcal{M}_T = \{\tau : \ 0 \leq \tau \leq T, \ \tau \text{ stopping time}\},$$

[1]See appendix

for the finite horizon case, putting $T = \infty$ for the perpetual case, the problem of pricing the American type derivative introduced consists in solving an optimal stopping problem, more precisely, in finding the value function A_T and an optimal stopping time τ^* in $\mathcal{M}_T$ such that

$$A_T = A(S_0^2, S_0^3, T) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E}\big[e^{-X^1_\tau} f(S_0^2 e^{X^2_\tau}, S_0^3 e^{X^3_\tau})\big]$$
$$= \mathsf{E}\big[e^{-X^1_{\tau^*}} f(S_0^2 e^{X^2_{\tau^*}}, S_0^3 e^{X^3_{\tau^*}})\big]. \tag{2.5}$$

2.3. *Examples of derivatives*

In what follows we introduce some relevant derivatives as particular cases of the problem described.

2.3.1. *Option to default*

Consider the derivative which has the payoff

$$f(x, y) = \min\{x, y\},$$

if $X^1 = rt$, then the value of the *Option to Default* a promise S_T^2 backed by a collateral guarantee S_T^3, at the time T would be:

$$D = \mathsf{E}\big[e^{-rT} \min\{S_T^2, S_T^3\}\big].$$

2.3.2. *Margrabe's options*

Consider the following cases:

(a) $f(x, y) = \max\{x, y\}$, called the *Maximum Option*,
(b) $f(x, y) = |x - y|$, the *Symmetric Option*,
(c) $f(x, y) = \min\{(x - y)^+, ky\}$, the *Option with Proportional Cap*.

2.3.3. *Swap options*

Consider

$$f(x, y) = (x - y)^+,$$

obtaining the option to exchange one risky asset for another.

2.3.4. *Quanto options*

Consider

$$f(x, y) = (x - ky)^+,$$

and take $S_t^3 = 1$, then

$$E_T = \mathsf{E} e^{X^1_T} (S_T^2 - k)^+,$$

where $e^{X_T^1}$ is the spot exchange rate (foreign units/domestic units) and S_T^1 is the foreign stock in foreign currency. Then we have the price of an option to exchange one foreign currency for another.

2.3.5. *Equity-linked foreign exchange option (ELF-X option)*

Take

$$S = S^1 : \text{ foreign stock in foreign currency,}$$

and Q is the spot exchange rate. We use foreign market risk measure, then an ELF-X is an investment that combines a currency option with an equity forward. The owner has the option to buy S_t with domestic currency which can be converted from foreign currency using a previously stipulated strike exchange rate R (domestic currency/foreign currency).

The payoff is:

$$\Phi_t = S_t(1 - RQ_t)^+,$$

then take $S_t^2 = 1$ and $f(x, y) = (y - Rx)^+$.

2.3.6. *Vanilla options*

Take

$$X_t^1 = rt,$$

then in the call case we have

$$f(x, y) = (x - ky)^+,$$

and

$$f(x, y) = (ky - x)^+,$$

in the put case, with $S_t^3 = S_0^3 e^{X_t}$ and $S_t^2 = 1$.

3. Dual Market Method

The main idea to solve the posed problems is the following: make a change of measure through Girsanov's Theorem for Lévy processes, in order to reduce the original problems to a pricing problems for an auxiliary derivative written on one Lévy driven stock in an auxiliary market with deterministic interest rate. This method was used in Shepp and Shiryaev [15] and Kramkov and Mordecki [9] with the purpose of pricing American perpetual options with path dependent payoffs. It was employed by Araujo and Oliveira [1] to consider the pricing of swaps, and is strongly related with the election of the *numéraire* (see Geman *et al.* [3]). This auxiliary market will be called the *Dual Market*.

More precisely, observe that

$$e^{-X_t^1} f\big(S_0^2 e^{X_t^2}, S_0^3 e^{X_t^3}\big) = e^{-X_t^1 + \alpha X_t^3} f\big(S_0^2 e^{X_t^2 - X_t^3}, S_0^3\big),$$

let $\rho = -\log \mathsf{E} e^{-X_1^1+\alpha X_1^3}$, that we assume finite. The process

$$Z_t = e^{-X_t^1+\alpha X_t^3+\rho t}, \tag{3.1}$$

is a density process (i.e., a positive martingale starting at $Z_0 = 1$) that allow us to introduce a new measure, the *dual martingale measure*, $\tilde{P}$ by its restrictions to each $\mathcal{F}_t$ by the formula

$$\frac{d\tilde{P}_t}{dP_t} = Z_t.$$

Denote now by $\tilde{X}_t = X_t^2 - X_t^3$, and $S_t = S_0^2 e^{\tilde{X}_t}$. Finally, let

$$F(x) = f(x, S_0^3).$$

With the introduced notations, under the change of measure we obtain

$$E_T = \tilde{\mathsf{E}}\big[e^{-\rho T} F(S_T)\big],$$
$$A_T = \sup_{\tau\in\mathcal{M}_T} \tilde{\mathsf{E}}\big[e^{-\rho\tau} F(S_\tau)\big].$$

The following step is to determine the law of the process X under the auxiliar probability measure $\tilde{P}$.

Lemma 3.1. *Let X be a Lévy process on $\mathbb{R}^d$ with characteristic exponent given in* (2.1)*. Let u and v be vectors in $\mathbb{R}^d$. Assume that $\mathsf{E}e^{(u,X_1)}$ is finite, and denote $\rho = -\log \mathsf{E} e^{(u,X_1)}$. In this conditions, introduce the probability measure $\tilde{P}$ by its restrictions $\tilde{P}_t$ to each $\mathcal{F}_t$ by*

$$\frac{d\tilde{P}_t}{dP_t} = \exp[(u, X_t) + \rho t].$$

Then

(a) *the law of the unidimensional Lévy process $\{(v, X_t)\}_{t\geq 0}$ under $\tilde{P}$ is given by the triplet*

$$\begin{cases} \tilde{a} = (a, v) + \dfrac{1}{2}[(v, \Sigma u) + (u, \Sigma v)] + \displaystyle\int_{\mathbb{R}^d} e^{(u,y)}(v, y)\mathbf{1}_{\{|(v,y)|\leq 1, |x|>1\}}\Pi(dx), \\ \tilde{\sigma}^2 = (v, \Sigma v), \\ \tilde{\pi}(A) = \displaystyle\int_{\mathbb{R}^d} \mathbf{1}_{\{(v,y)\in A\}} e^{(u,y)}\Pi(dy). \end{cases} \tag{3.2}$$

(b) *In the particular case when X is a diffusion with jumps which characteristic exponent given in* (2.2) *the law of the unidimensional Lévy process $\{(v, X_t)\}_{t\geq 0}$ under $\tilde{P}$ is given by the triplet*

$$\begin{cases} \tilde{a} = (a, v) + \dfrac{1}{2}[(v, \Sigma u) + (u, \Sigma v)], \\ \tilde{\sigma}^2 = (v, \Sigma v), \\ \tilde{\pi}(A) = \displaystyle\int_{\mathbb{R}^d} \mathbf{1}_{\{(v,y)\in A\}} e^{(u,y)}\Pi(dy). \end{cases} \tag{3.3}$$

Furthermore, the intensity of the Poisson process under $\tilde{P}$ is given by

$$\tilde{\alpha} = \int_{\mathbb{R}^d} e^{(u,y)}\Pi(dy) = \alpha \int_{\mathbb{R}^d} e^{(u,y)}F(dy).$$

(c) *Assume* (b), *and let $\Pi(dy) = \alpha F(dy)$ where F is the common distribution of the random variables $\{Y_k\}_{k\geq 1}$, and has characteristic function (under P) given by*

$$\phi(z) = \int_{\mathbb{R}^d} e^{(z,y)}F(dy).$$

Then, the characteristic function of the same random variables under $\tilde{P}$ is given by

$$\tilde{\phi}(\theta) = \frac{\phi(\theta v + u)}{\phi(u)}. \tag{3.4}$$

Remark 3.1. Consider a diffusion with gaussian jumps, in what can be considered as a multidimensional extension of the jump-diffusion model proposed by Merton [11]. Then, the characteristic function corresponding to the distribution of the jumps is given by

$$\phi(z) = \exp\left[(z,\mu) + \frac{1}{2}(z,\Delta z)\right],$$

where the d-dimensional vector μ is the drift of the jumps, and the nonnegative definite matrix Δ is the covariance. According to (3.4), the characteristic exponent of the jumps of the process $\{(v, X_t)\}_{t\geq 0}$ under the probability measure $\tilde{P}$ in the Lemma 3.1 is given by

$$\tilde{\phi}(\theta) = \frac{\phi(\theta v + u)}{\phi(u)} = \exp\left\{\theta((v,\mu) + \frac{1}{2}[(v,\Delta u) + (u,\Delta v)]) + \frac{1}{2}\theta^2(v,\Delta v)\right\}. \tag{3.5}$$

In conclusion, jumps under $\tilde{P}$ are also gaussian, with mean and variance obtained in (3.5).

Proof. First compute the expectation under $\tilde{P}$ as an expectation under P.

$$\tilde{\mathsf{E}}e^{\theta(v,X_t)} = \mathsf{E}e^{(u+\theta v,X_t)+\rho t} = \exp\{t[\psi(u+\theta v) - \psi(u)]\}.$$

Now, compute the characteristic exponent of (v, X),

$$\begin{aligned}
\psi(u+\theta v) - \psi(u) &= (a, u+\theta v) - (a,u) + \frac{1}{2}[(u+\theta v, \Sigma u + \theta v) - (u, \Sigma u)] \\
&\quad + \int_{\mathbb{R}^d} \left(e^{(u+\theta v,y)} - 1 - (u+\theta v, y)\mathbf{1}_{\{|y|\leq 1\}}\right)\Pi(dy) \\
&\quad - \int_{\mathbb{R}^d} \left(e^{(u,y)} - 1 - (u,y)\mathbf{1}_{\{|y|\leq 1\}}\right)\Pi(dy) \\
&= \theta\left\{(a,v) + \frac{1}{2}[(v,\Sigma u) + (u,\Sigma v)]\right\} + \frac{1}{2}(v,\Sigma v) \\
&\quad + \int_{\mathbb{R}^d} \left(e^{(u+\theta v,y)} - e^{(u,y)} - (\theta v, y)\mathbf{1}_{\{|y|\leq 1\}}\right)\Pi(dy)
\end{aligned}$$

$$= \theta\Big\{(a,v) + \frac{1}{2}[(v,\Sigma u) + (u,\Sigma v)] + \int_{\mathbb{R}^d} e^{(u,y)}(v,y)\mathbf{1}_{\{|(v,y)|\leq 1, |x|>1\}}\Pi(dx)\Big\} + \frac{1}{2}(v,\Sigma v) + \int_{\mathbb{R}^d}\big(e^{(\theta v,y)} - 1 - (\theta v,y)\mathbf{1}_{\{|y|\leq 1\}}\big)e^{(u,y)}\Pi(dy),$$

giving (3.2).

In what concerns (3.3), similar calculations give the result.

Let us see (c). As the distribution of the jumps under $\tilde{P}$ is given by $\frac{1}{\tilde{\alpha}}\tilde{\pi}(dy)$,

$$\tilde{\phi}(\theta) = \frac{1}{\tilde{\alpha}}\int_R e^{\theta x}\tilde{\pi}(dx) = \frac{\alpha}{\tilde{\alpha}}\int_{\mathbb{R}} e^{(\theta v+u,y)}F(dy) = \frac{\phi(\theta v+u)}{\phi(u)}.$$

□

4. Examples

1. Let $X_t^1 = rt$ and (X_t^2, X_t^3) be a bidimensional Lévy process. We show how to obtain a formula for the value of an option to exchange one risky asset for another at the end of a determined period, as was considered by Margrabe (1978). Let S_T^2 and S_T^3 be two risky assets, a contract with payoff $(S_T^2 - S_T^3)^+$ can be priced using *The Dual Market Method*:

$$D = \mathsf{E}\big[e^{-rT}(S_T^2 - S_T^3)^+\big] = \int_{\mathcal{A}} e^{-rT}(S_0^2 e^{X_T^2} - S_0^3 e^{X_T^3})dP.$$

Assuming for simplicity $S_0^2 = S_0^3 = 1$, Then $\mathcal{A} = \{\omega\in\Omega : X_T^2(\omega) > X_T^3(\omega)\}$, we proceed to applied the method:

$$D = \int_{\mathcal{A}} e^{-rT}(e^{X_T^2} - e^{X_T^3})dP = \int_{\{S_T>1\}} e^{-rT}e^{X_T^3}(S_T - 1)dP,$$

where $S_T = e^{X_T}$ and $X = X^2 - X^3$. Now, to use the dual measure, observe that $\rho = -\log \mathsf{E}e^{-r+X_1^3} = r - \log \mathsf{E}e^{X_1^3}$, then:

$$d\tilde{P} = \frac{e^{X_T^3}}{\mathsf{E}e^{X_T^3}}dP. \tag{4.1}$$

With all this:

$$D = e^{-\rho T}\int_{\{S_T>1\}}(S_T - 1)d\tilde{P},$$

$$D = e^{-\rho T}\int_{\{S_T>1\}} S_T d\tilde{P} - e^{-\rho T}\int_{\{S_T>1\}} d\tilde{P}.$$

Now to reduce this expression we need to assume a distribution for X and then apply the Lemma 3.1 to obtain the density of S_T under $\tilde{P}$. For example, assume a multidimensional version of the model proposed by Merton (1976), by part (c) of Lemma 3.1 and taking $u = (-1, 0, 1)$, $v = (0, 1, -1)$, $\mu = (\mu_1, \mu_2, \mu_3)$ and $\Delta = [\sigma_{ij}]$, we have that jumps under $\tilde{P}$ are also gaussian with mean $(\mu_2 - \mu_3) + (\sigma_{13} + \sigma_{23} - \sigma_{12} - \sigma_3^2)$ and variance $\sigma_2^2 + \sigma_3^2 - 2\sigma_{23}$. In this way an analogous formula to the one obtained by Merton [11] can be obtained.

Remark 4.1. It is worth noting that instead of assuming a distribution for X, we can make the following change of measure:

$$d\hat{P} = \frac{e^{X_T^2}}{\mathsf{E}e^{X_T^2}} dP,$$

we obtain:

$$D = e^{-\rho T} \int_{\{S_T > 1\}} e^{X_T^2 - X_T^3} \frac{e^{X_T^3}}{\mathsf{E}e^{X_T^3}} dP - e^{-\rho T} \tilde{P}(S_T > 1)$$

$$D = e^{-\rho T} \frac{\mathsf{E}e^{X_T^2}}{\mathsf{E}e^{X_T^3}} \hat{P}(S_T > 1) - e^{-\rho T} \tilde{P}(S_T > 1),$$

which is analogous to the formula obtained by Gerber and Shiu [4]. The original formula obtained by Margrabe [10] was extended by Stulz [18] to contracts with payoffs:

$$(\min\{S_2(T), S_3(T)\} - K)^+,$$

and finally extended to the case of several assets by Johnson [8]. Unfortunately our method does not applied, since the payoffs are not longer homogenous of any degree in any of its arguments, unless $K = 0$.

2. Now consider the ELF-X options $f(S_T^2, S_T^3) = S_T^2(S_T^3 - K)^+$, observe that although f is no more homogenous in both arguments, we can still adapt our method, since this function is homogenous in its first argument, that is we can make a change of measure, reducing in this way the dimension of the problem, in this case the new measure is given by:

$$d\tilde{P} = \frac{e^{X_T^2}}{\mathsf{E}e^{X_T^2}} dP,$$

then

$$D = \int_{\{S_T^3 > K\}} e^{-rT} S_0^2 e^{X_T^2} (S_T^3 - K) dP$$

$$= S_0^2 \int_{\{S_T^3 > K\}} e^{-rT} e^{X_T^2} (S_T^3 - K) dP.$$

Taking $\rho = r - \log \mathsf{E} e^{X_1^2}$, then:

$$D = e^{-\rho T} S_0^2 \int_{\{S_T^3 > K\}} (S_T^3 - K) d\tilde{P},$$

$$D = e^{-\rho T} S_0^2 S_0^3 \int_{\{X_T^3 > \ln(K/S_0^3)\}} e^{X_T^3} d\tilde{P} - e^{-\rho T} K \tilde{P}(X_T^3 > \ln(K/S_0^3)).$$

To resolve the integral we need the distribution of X_T^3 under $\tilde{P}$. To this end take $u = (0, 1, 0)$ and $v = (0, 0, 1)$ in Lemma 3.1.

3. In the case of an asset or nothing option $f(S_T^2, S_T^3) = S_T^3 \mathbf{1}_{\{S_T^2 > K\}}$, we can also adapt our method as in item 2, under the change of measure

$$d\tilde{P} = \frac{e^{X_T^3}}{\mathsf{E} e^{X_T^3}} dP,$$

we have:

$$\begin{aligned} D &= \int e^{-rT} S_0^3 e^{X_T^3} \mathbf{1}_{\{S_T^2 > K\}} dP \\ &= S_0^3 \int_{\{S_T^2 > K\}} e^{-rT} e^{X_T^3} dP, \end{aligned}$$

then,

$$D = e^{-\rho T} S_0^3 \tilde{P}(X_T^2 > \ln(K/S_0^2)).$$

To compute the probability we need the distribution of X_T^2 under $\tilde{P}$ and it can be obtained by taking $u = (0, 0, 1)$ and $v = (0, 1, 0)$ in Lemma 3.1.

4. Now we consider a Quanto Call: $f(S_T^2, S_T^3) = S_T^1 (S_T^2 - S_T^3)^+$, we assume that $S_0^i = 1$, for $i = 1, 2, 3$. Then

$$\begin{aligned} D &= \mathsf{E}\big[e^{-X_T^1} (S_T^2 - S_T^3)^+\big] \\ &= \int_{\mathcal{A}} e^{-X_T^1} (e^{X_T^2} - e^{X_T^3}) dP, \end{aligned}$$

where $\mathcal{A} = \{\omega \in \Omega : X_T^2(\omega) > X_T^3(\omega)\}$, applying the method:

$$\begin{aligned} D &= \int_{\mathcal{A}} e^{-X_T^1} (e^{X_T^2} - e^{X_T^3}) dP \\ &= \int_{\{S_T > 1\}} e^{-X_T^1} e^{X_T^3} (S_T - 1) dP, \end{aligned}$$

where $S_T = e^{X_T}$ and $X = X^2 - X^3$. Observe that $\rho = -\log \mathsf{E} e^{-X_1^1 + X_1^3}$, then:

$$d\tilde{P} = e^{-X_T^1 + X_T^3 + \rho T} dP, \tag{4.2}$$

then

$$D = e^{-\rho T} \int_{\{S_T > 1\}} (S_T - 1) d\tilde{P},$$
$$D = e^{-\rho T} \int_{\{X_T > 0\}} e^{X_T} d\tilde{P} - e^{-\rho T} \tilde{P}(X_T > 0).$$

To obtain the distribution of X_T under $\tilde{P}$ we take $v = (0, 1-1)$ and $u = (-1, 0, 1)$ in Lemma 3.1.

5. Now consider an American perpetual swap, it is a derivative with the payoff function at any time t given by

$$f(S_t^2, S_t^3) = (S_t^2 - S_t^3)^+,$$

then using the Dual market method, the pricing problem of this derivative would be:

$$A_T = \sup_{\tau \in \mathcal{M}_T} \tilde{\mathsf{E}}\big[e^{-\rho\tau}(S_\tau - S_0^3)^+\big] = \tilde{\mathsf{E}}\big[e^{-\rho\tau^*}(S_{\tau^*} - S_0^3)^+\big],$$

and this problem can be solved using the following proposition.

Proposition 4.1. *Let $M = \sup_{0 \le t \le \tau} X_t$ with τ an independent exponential random variable with parameter ρ, then $\tilde{\mathsf{E}} e^M < \infty$ and*

$$A(S_0^2, S_0^3) = \frac{\tilde{\mathsf{E}}\big[S_0^2 e^M - S_0^3 \tilde{\mathsf{E}}(e^M)\big]}{\tilde{\mathsf{E}}(e^M)}$$

the optimal stopping time is

$$\tau_c^* = \inf\{t \ge 0, S_t \ge S_0^3 \tilde{\mathsf{E}}(e^M)\}$$

Proof. See Mordecki [12]. □

5. Conclusions

In this paper we have extended the results obtained by Gerber and Shiu [5] for the bidimensional Geometric Brownian Motion to the case of bidimensional Geometric Lévy motion. We have shown that using the Dual market method it is possible to price many derivatives written in terms of two assets driven by geometric Lévy motions, with payoffs homogenous of any degree and in some particular cases we can adapt the method for payoffs homogeneous just in one argument. These results holds for European case and for the American perpetual case. Another important result in this paper is the possibility of having a stochastic discount. This allow us to consider derivatives as quanto derivatives.

Appendix: How to Obtain an EMM

The procedure introduced in this section is in spirit of Gerber and Shiu [4]. Take the original probability measure P and suppose that relative prices $\left\{\frac{S^j}{S^1}\right\}$ are not martingales under P, then define the following probability:

$$\frac{dP^\theta}{dP} = \frac{e^{\theta\cdot X'}}{\mathsf{E}e^{\theta\cdot X'}}.$$

Now we need that $\left\{\frac{S^j_t}{S^1_t}\right\}$ be a martingale under P^θ for some θ, as $S^1_0 = 1$, then is enough to prove that

$$\begin{aligned} S^j_0 &= \mathsf{E}^\theta\left(\tfrac{S^j_t}{S^1_t}\right), \qquad \forall\, j,\ \forall\, t,\\ 1 &= \mathsf{E}^\theta(e^{X^j_t - X^1_t}), \quad \forall\, j,\ \forall\, t. \end{aligned}$$

Defining $\bar{1}_j = (-1, 0, \ldots, \underbrace{1}_{j-position}, \ldots, 0)$, we have

$$1 = M(\bar{1}_j, 1; \theta), \tag{A.1}$$

where $M(z,t;\theta) = \frac{M(z+\theta,t)}{M(\theta,t)}$ and $M(\theta,t) = \mathsf{E}(e^{\theta\cdot X'_t})$. From the solution of (A.1) we have the EMM.

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