## EXPLANATION FOR THE PLURAL FILES

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### 1. Computations

We present how we used the computer algebra system Singular::Plural [15] to construct simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules. This should be read as complement to either [16] or [17], where the details of the theory can be found. The system Singular::Plural allows us to do computations on algebras given by generators and rewriting relations of a particular form, allowing Gröbner basis computations to be done. For details on these algebras, we refer the reader to [2] and [12].

Let $\mathcal{B}'$ be the subalgebra of $U_{r,s}(\mathfrak{sl}_3)$ generated by $\{f_1, f_2, \omega_1', \omega_2'\}$. Adding the element $\mathcal{F}_{21} = f_2 f_1 - s f_1 f_2$ to the generating set, $\mathcal{B}'$ is generated by $\{f_1, \mathcal{F}_{21}, f_2, \omega_1', \omega_2'\}$ subject to the relations

(1) $\mathcal{F}_{21} f_1 = r f_1 \mathcal{F}_{21}$ and $f_2 \mathcal{F}_{21} = r \mathcal{F}_{21} f_2$,
(2) $f_2 f_1 = s f_1 f_2 + \mathcal{F}_{21}$,
(3) $\omega_1' \mathcal{F}_{21} = s^{-1} \mathcal{F}_{21} \omega_1'$ and $\omega_2' \mathcal{F}_{21} = r \mathcal{F}_{21} \omega_2'$,
(4) (a) $\omega_1' f_1 = r s^{-1} f_1 \omega_1'$,
  (b) $\omega_2' f_1 = s f_1 \omega_2'$,
  (c) $\omega_1' f_2 = r^{-1} f_2 \omega_1'$,
  (d) $\omega_2' f_2 = r s^{-1} f_2 \omega_2'$, and
(5) $\omega_1' \omega_2' = w_2' \omega_1'$.

Therefore $\mathcal{B}'$ is generated by $\{x_1 = f_1,\ x_2 = \mathcal{F}_{21},\ x_3 = f_2,\ x_4 = \omega_1',\ x_5 = \omega_2'\}$, subject to relations $\{x_j x_i = C_{ij} x_i x_j + D_{ij}, 1 \le i < j \le 5\}$ where the coefficients $C_{ij}$ and polynomials $D_{ij}$ are given by the relations above; that is $D_{ij} = 0$ if $(i,j) \neq (1,3)$ and

(1) $C_{12} = r$ and $C_{23} = r$,
(2) $C_{13} = s$ and $D_{13} = \mathcal{F}_{21}$,
(3) $C_{24} = s^{-1}$ and $C_{25} = r$,
(4) (a) $C_{14} = r s^{-1}$,
  (b) $C_{15} = s$,
  (c) $C_{34} = r^{-1}$,
  (d) $C_{35} = r s^{-1}$, and
(5) $C_{45} = 1$.

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From this presentation it follows that $\mathcal{B}'$ is of the form required by SINGULAR::PLURAL [15]. Let $I$ be the two-sided ideal of $\mathcal{B}'$ generated by the set

$$\left\{(\omega_1')^\ell - 1,\, (\omega_2')^\ell - 1,\, f_1^\ell,\, \mathcal{F}_{21}^\ell,\, f_2^\ell\right\},$$

we have that $H_{r,s} = (\mathfrak{b}')^{\mathrm{coop}} = \mathcal{B}'/I$.

For $\ell$, $y$ and $z$ positive integers with $\gcd(y-z,\ell) = 1$, we define the ring `B`. We write the code in terms of parameters `l`, `y` and `z`; the values of these parameters can be fixed in a preamble.

```
ring B = (0,Q), (F(1), F(21), F(2), W(1), W(2)), Dp;
minpoly = rootofUnity(l);
```

The underlying coefficient field has characteristic 0 and it contains `Q`, which is a primitive $\ell$th root of unity and is generated by the elements `F(1), F(21), F(2), L(1), L(2)` (which correspond to $f_1, \mathcal{F}_{21}, f_2, \omega_1'$ and $\omega_2'$ respectively). The monomial ordering `Dp` is the degree lexicographical order. We write the numbers $C_{ij}$ and $D_{ij}$ that define the relations in $\mathcal{B}'$; these are given with upper-triangular matrices `C` and `D`, and only the non-zero elements need to be given.

```
matrix C[5][5];
matrix D[5][5];
C[1,2] = Q^y; C[1,3] = Q^z; C[1,4] = Q^(y-z); C[1,5] = Q^z;
C[2,3] = Q^y; C[2,4] = Q^(-z); C[2,5] = Q^y;
C[3,4] = Q^(-y); C[3,5] = Q^(y-z);
C[4,5] = 1;
D[1,3] = F(21);
```

The command `ncalgebra(C,D)` creates the G-algebra with the relations given by `C` and `D`, and sets it as the base ring. I then give the generators of the ideal $I$.

```
ncalgebra(C,D);
option(redSB); option(redTail);
ideal I = F(1)^l, F(2)^l, W(1)^l - 1 , W(2)^l - 1, (F(21))^l;
qring B = twostd(I);
```

The last command sets the base ring to be the quotient of the previous ring by the ideal `I`. We now have $\mathfrak{b}'$ as the base ring. Next we describe how we generate the simple $\mathfrak{u}_{r,s}(\mathfrak{sl}_3)$-modules. Combining the definition of the $\bullet_\beta$ action, together with the coproduct formulas in $H = (\mathfrak{b}')^{\mathrm{coop}}$ we have that for all $x \in H$ and $g \in G(H)$,

$$f_i \bullet_\beta x = -xS^{\mathrm{op}}(f_i) + \beta(\omega_i')f_i x(\omega_i')^{-1} = -xf_i(\omega_i')^{-1} + \beta(\omega_i')f_i x(\omega_i')^{-1} \tag{1.1}$$

and

$$\omega_i' \bullet_\beta g = \beta(\omega_i')w_i' g(\omega_i')^{-1} = \beta(\omega_i')g.$$

The second equation shows that if $g \in G(H)$, then $H \bullet_\beta g$ is generated by

$$\left\{(f_1^k \mathcal{F}_{21}^t f_2^m) \bullet_\beta g \,:\, 0 \le k,\, t,\, m < \ell\right\}.$$

Using Equation (1.1) recursevely, we define the procedure `Beta` so that if $0 \leq k,\, t,\, m < \ell$, $h \in H$ and $\beta : H \to \mathbb{K}$ is an algebra map given by $\beta(f_1) = \theta^a$ and $\beta(f_2) = \theta^b$, then `Beta(a,b,k,t,m,h)` gives $(f_1^k \mathcal{F}_{21}^t f_2^m) \bullet_\beta h$. Fix a group-like element $g = (\omega_1')^c (\omega_2')^d \in H$. In what follows we will construct a basis and compute the dimensions for the module $H \bullet_\beta g$. Let

$$\mathcal{F}_\ell = \{f_1^k \mathcal{F}_{21}^t f_2^m : 0 \leq k,\, t,\, m < \ell\}$$

(so that $H \bullet_\beta g = \mathbb{K}\{f \bullet_\beta g : f \in \mathcal{F}_\ell\}$). The basic idea is to consider the linear map $T_\beta : \mathbb{K}\mathcal{F}_\ell \to H$ given by $T_\beta(f) = f \bullet_\beta g$, and construct the matrix $M$ representing $T_\beta$ in the basis $\mathcal{F}_\ell$ and $\{fh : f \in \mathcal{F}_\ell, h \in G(H)\}$ of $\mathbb{K}\mathcal{F}_\ell$ and $H$ respectively. Then $\dim(H \bullet_\beta g) = \operatorname{rank}(M)$, and the non-zero columns of the column-reduced Gauss normal form of $M$ give the coefficients for the elements of a basis of $H \bullet_\beta g$. The problem with this method is that since $\dim(H) = \ell^5$ and $\dim(\mathbb{K}\mathcal{F}_\ell) = \ell^3$, the size of $M$ is $\ell^5 \times \ell^3$ and computing the Gauss normal form of these matrices is an expensive calculation even for small values of $\ell$ such as $\ell = 5$. However, by some reordering of $\mathcal{F}_\ell$ and of the PBW basis of $H$, $M$ is block diagonal. We proceed to show how this is done.

For a monomial $h = f_1^{\alpha_1} \mathcal{F}_{21}^{\alpha_2} f_2^{\alpha_3} (\omega_1')^{\alpha_5} (\omega_2')^{\alpha_6}$ let $\deg_1(h) = \alpha_1 + \alpha_2$ and $\deg_2(h) = \alpha_2 + \alpha_3$. Note that Equation (1.1) implies that $h \bullet_\beta x$ is a linear combination of monomials $m$ with $\deg_i(m) = \deg_i(h) + \deg_i(x)$. For all $0 \leq u, v < 2\ell$, let

$$D_{(u,v)} = \{h \in \mathcal{F}_\ell : \deg_1(h) = u \text{ and } \deg_2(h) = v\}$$

and

$$R_{(u,v)} = \{f (\omega_1')^{-u} (\omega_2')^{-v} g : f \in D_{(u,v)}\}.$$

Then for all $h \in D_{(u,v)}$, $h \bullet_\beta g \in \mathbb{K}R_{(u,v)}$. The possible pairs $(u, v)$ are such that $0 \leq u, v \leq 2(\ell - 1)$ and since $|v - u|$ is the maximum power of $\mathcal{F}_{21}$ that can be a factor of a monomial in $D_{(u,v)}$, we must have $|v - u| \leq \ell - 1$; that is $u - (\ell - 1) \leq v \leq u + \ell - 1$. Another way of describing the sets $D_{(u,v)}$ and $R_{(u,v)}$ is as follows.

$$\begin{aligned} D_{(u,v)} &= \{f_1^{u-i} \mathcal{F}_{21}^i f_2^{v-i},\ \forall i \in \mathbb{N} : 0 \leq u - i,\, i,\, v - i \leq \ell - 1\} \\ &= \{f_1^{u-i} \mathcal{F}_{21}^i f_2^{v-i},\ \forall i \in \mathbb{N} : n_{u,v} \leq i \leq m_{u,v}\} \end{aligned}$$

where $n_{u,v} = \max(0, \ell - 1 - u, \ell - 1 - v)$ and $m_{u,v} = \min(\ell - 1, u, v)$. Since $(\omega_i')^{-1} = (\omega_i')^{\ell - 1}$, if $g = (\omega_1')^c (\omega_2')^d$ we also have

$$R_{(u,v)} = \{f (\omega_1')^{(\ell-1)u + c} (\omega_2')^{(\ell-1)v + d} : f \in D_{(u,v)}\}.$$

**Remark 1.2.** *It is clear that* $\mathcal{F}_\ell = \bigcup D_{(u,v)}$*, the union disjoint, and that* $H \bullet_\beta g = \oplus \mathbb{K}R_{(u,v)}$*. Therefore a basis for* $H \bullet_\beta g$ *is a disjoint union of the bases for* $\mathbb{K}D_{(u,v)} \bullet_\beta g$ *for all possible pairs* $(u, v)$*, and* $\dim(H \bullet_\beta g) = \sum_{(u,v)} \dim(\mathbb{K}D_{(u,v)} \bullet_\beta g)$.

We define the procedure `Submod`, where the output of `Submod(a,b,c,d,u,v)` is a list `L`, where the first component of the list is a basis for $D_{(u,v)} \bullet_\beta g$ and the second component is $\dim(D_{(u,v)} \bullet_\beta g)$ (for $\beta$ given by $a$ and $b$ and $g = (\omega_1')^c)(\omega_2')^d)$.

```
proc Submod(int a, int b, int c, int d, int u, int v)
{ list L;
  ideal D;
  ideal R;
  list e = u-(l-1),v-(l-1),0; int n= Max(e);
  list f = u,v, l-1; int m= Min(f);
  int a = (z-y)*c+ y*d; int b= -z*c+(z-y)*d;
for(int i= n; i<= m; i++)
{
D[i+1-n] = Beta(a, b , u-i, i, v-i , W(1)^c * W(2)^d);
R[i+1-n] = F(1)^(u-i)* F(21)^i* F(2)^(v-i)*
           W(1)^(((l-1)*u+c) mod l)* W(2)^(((l-1)*v+d) mod l);}
matrix M = coeffs(D,R);
matrix N = gauss_col(M);
matrix K[1][m-n+1] = R;
matrix S = K*N;
L[1] = compress(S);
L[2] = mat_rk(N);
return(L);}
```

The procedure `Totalbasis(a,b, c,d)` returns $\dim(H \bullet_\beta g)$ and a basis for $H \bullet_\beta g$, and is defined using Remark 1.2.

```
proc Totalbasis(int a, int b, int c , int d)
 { list T; matrix A; int t; t = 0;
   for(int u = 0; u<=2*(l-1); u++)
     { list e = 0, u-(l-1);
       list f = u+(l-1), 2*(l-1);
       for(int v = Max(e); v <= Min(f); v++)
         { list M = Submod(c,d, u,v);
           A = compress(concat(A, M[1]));
           t = t + M[2];
         }
     }
   T[1] = A; T[2] = t; return(T);
 }
```

**Example 1.3.** *For $\ell = 5$, $y = 1$ and $z = 4$, for $g = (\omega_1')^4(\omega_2')^2$ and $\beta(\omega_1') = \theta^4$ and $\beta(\omega_2') = 1$, we construct the module $H \bullet_\beta g$ as follows. To give* Singular:Plural *the values of $\ell$, $y$ and $z$, we write at the beginning of the code*

```
ring r0 = 0,x,dp;
int l = 1;
int y = 4;
int z = 1;
```

*Then the command*

```
Totalbasis(4,0,4,2);
```

*returns*

```
// [1]:
//    _[1,1]=W(1)^4*W(2)^2
//    _[1,2]=F(1)*W(1)^3*W(2)^2
//    _[1,3]=(-Q^3-Q^2-2*Q-1)*F(1)*F(2)*W(1)^3*W(2)+F(21)*W(1)^3*W(2)
// [2]:
//     3
```

*which tells us that* $dim(H\bullet_\beta((\omega_1')^4(\omega_2')^2)) = 3$. *A basis for* $H\bullet_\beta g$ *is* $\{1\bullet_\beta g, f_1\bullet_\beta g, \mathfrak{F}_{21}\bullet_\beta g\}$ *since*

```
Beta(4,0,0,0,0,W(1)^4*W(2)^2);
Beta(4,0,1,0,0,W(1)^4*W(2)^2)/(-Q^3-Q^2-2*Q-1);
Beta(4,0,0,1,0,W(1)^4*W(2)^2)/(-Q^3-Q^2-2*Q-1);
```

*returns*

```
// W(1)^4*W(2)^2
// F(1)*W(1)^3*W(2)^2
// (-Q^3-Q^2-2*Q-1)*F(1)*F(2)*W(1)^3*W(2)+F(21)*W(1)^3*W(2)
```

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