A Graph which is Edge Transitive but not Arc Transitive

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ABSTRACT

A graph having 27 vertices is described, whose automorphism group is transitive on vertices and undirected edges, but not on directed edges.

The object of this Note is to provide an example of a finite graphs whose automorphism group is transitive on vertices and undirected edges, but not on directed edges. In other words, the graph is edge transitive but not arc transitive. The question of the existence of such a graph was raised by Tutte on p. 60 of [2], and he showed that it must have even valency if it exists. An infinite family of such graphs was constructed by Bouwer in [1], one for each valency $2n(n \geq 2)$. His smallest example has 54 vertices and valency 4. The example constructed here has 27 vertices and valency 4, and so it appears to be the smallest known example at present. It has diameter 4 and girth 5, and it is not bipartite. I have been informed by the referee that a further example of such a graph has been constructed by P. Kornya.

We adopt the following procedure. Start with a transitive permutation group on a finite set Ω, denoted by G^Ω. Let Δ be an orbit of G on $\Omega \times \Omega$, other than the diagonal orbit, and let Δ' be the paired orbit $\{(\beta,\alpha) \mid (\alpha,\beta) \in \Delta\}$. Then G acts transitively on the arcs of the digraph having point set Ω and arc set Δ. Now let D be the digraph with point set Ω and arc set $\Delta \cup \Delta'$, and let Γ be the graph obtained from D by viewing each symmetric pair of arcs $\{(\alpha,\beta),(\beta,\alpha)\}$ as an undirected edge $\{\alpha,\beta\}$. If $\Delta \neq \Delta'$, then G acts transitively on the edges of Γ, but not on the arcs of D, which are the directed edges of Γ. The problem is that this does not automatically yield an example of the type that we are seeking, because the full automorphism group of Γ may be larger than G, and may act transitively on the arcs of D. Indeed, this seems to happen most of the time. The smallest example that I have been able to find in which this does not occur is one in which G^Ω is a group of order 54 and degree 27, having a regular nonabelian subgroup. The relevant Δ has

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order 54, and so the graph on $\Delta \cup \Delta'$ has valency 4. G^Ω is an imprimitive group, and it would be interesting to find an example of a primitive group which exhibits these properties.

It is a routine matter to compute the graph once G^Ω is chosen, and so we present the results in a form in which the reader can easily check the relevant properties of the graph. In the notation below, Δ is the orbit of (1,2) and Δ' is that of (1,4).

The vertices of Ω are numbered from 1 to 27, and the edges are as follows:

(i) {1,2}, {1,3}, {1,4}, {1,5};

(ii) {2,6}, {2,7}, {2,8}; {3,9}, {3,10}, {3,11}; {4,12}, {4,13}, {4,14}; {5,15}, {5,16}, {5,17};

(iii) {6,11}, {7,12}, {7,16}, {8,10}, {8,17}, {9,14}, {9,15}, {11,13}, {12,17}, {13,15};

(iv) {6,19}, {6,26}, {7,23}, {8,18}, {9,25}, {10,19}, {10,22}, {11,18}, {12,20}, {13,21}, {14,22}, {14,23}, {15,24}, {16,25}, {16,26}, {17,27};

(v) {18,20}, {18,24}, {19,23}, {19,25}, {20,21}, {20,22}, {21,23}, {21,26}, {22,27}, {24,26}, {24,27}, {25,27}.

The edges in groups (i), (ii), and (iv) are shown in Figure 1, and the remaining edges in Figure 2.

Let H be the automorphism group of Γ. Then the following facts can be checked:

(i) H contains permutations x,y,z, where
$x = (1,25,8,26,12,15,22,11,23)(2,16,17,24,20,13,14,3,19)$

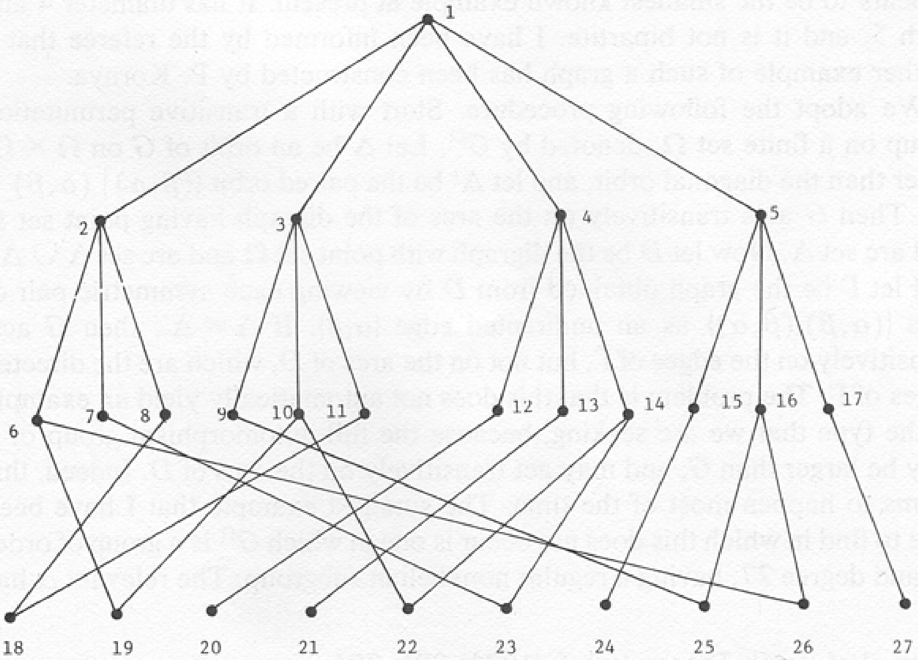

FIGURE 1

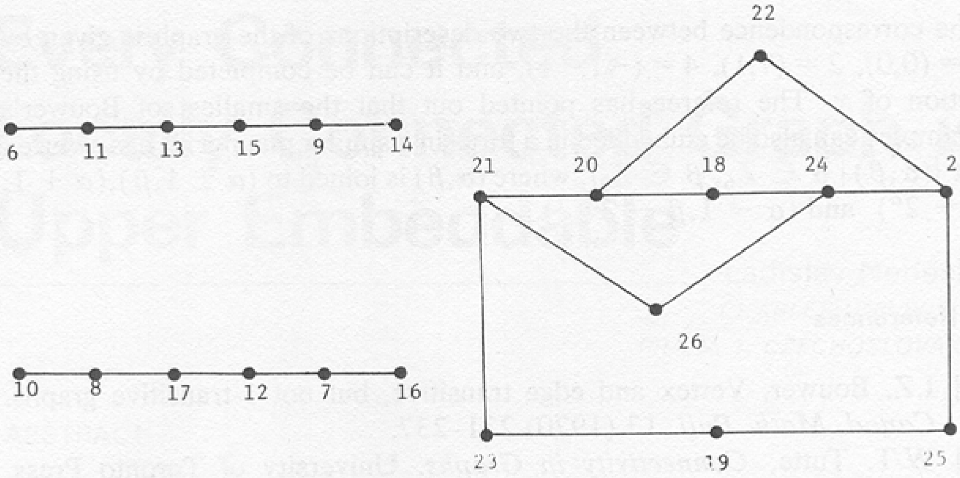

FIGURE 2

$$(4,9,10,6,7,5,27,18,21),$$
$$y = (1,19,18)(2,10,11)(3,6,8)(4,23,20)(5,25,24)(7,22,13)$$
$$(9,26,17)(12,14,21)(15,16,27),$$
$$z = 1,18,19,(2,3)(4,5)(6,10)(7,9)(8,11)(12,15)(13,17)(14,16)$$
$$(20,24)(21,27)(22,26)(23,25);$$

(ii) $G = \langle x,y,z \rangle$ has order 54, with $x^9 = y^3 = z^2 = 1$, $y^{-1}xy = x^4$, $z^{-1}xz = x^{-1}$ and $zy = yz$;

(iii) $\langle x,y \rangle$ has order 27 and acts transitively on the set of vertices;

(iv) z interchanges the edges $\{1,2\}$ and $\{1,3\}$ and the edges $\{1,4\}$ and $\{1,5\}$, whereas $y^{-1}x^2$ maps the edge $\{1,2\}$ to $\{4,1\}$.

Now observe that H is transitive on both vertices and undirected edges. For, by (iv), H is transitive on edges incident with 1, but, by (iii), it is transitive on vertices, and hence H must be transitive on all undirected edges.

Let $Y = \{y \mid 6 \leq y \leq 17\}$ denote the set of vertices at distance two from 1. Let $\Gamma(Y)$ be the graph whose vertex set is Y, and whose edges are those edges of Γ that join two vertices of Y. Then $\Gamma(Y)$ consists of two disjoint paths: (6,11,13,15,9,14) and (10,8,17,12,7,16). If $g \in H$ maps the directed edge (1,2) to (1,4), then g must be an automorphism of $\Gamma(Y)$, and it follows from $2^g = 4$ that $6^g = 14$, and therefore $11^g = 9$, so that 10 has to be left fixed by g. This in turn implies that 8 and therefore 2 are left fixed by g, which is a contradiction. Hence no such g exists, and so H cannot be transitive on directed edges. In fact, if $g \in H$ fixes the directed edge (1,2), then it is easily seen that g fixes each vertex of $\Gamma(Y)$, and hence each vertex of Γ, so $H = G$.

I am grateful to Professor W. Brisley for bringing this problem to my attention. He has suggested the following concise description of the graph. The vertex set is $\{(\alpha,\beta) \mid \alpha \in \mathbb{Z}_9, \beta \in \mathbb{Z}_3\}$, where (α,β) is joined to $(4\alpha \pm 1, \beta - 1)$ and to $(7\alpha \pm 7, \beta + 1)$. x,y and z are given by $y:(\alpha,\beta) \to (\alpha,\beta+1)$, $z:(\alpha,\beta) \to (-\alpha,\beta)$, and $x:(\alpha,\beta) \to (\alpha + q,\beta)$, where $q4^{\beta} = 1$.

The correspondence between the two descriptions of the graph is given by $1 = (0,0)$, $2 = (7,1)$, $4 = (-1,-1)$, and it can be completed by using the action of x. The referee has pointed out that the smallest of Bouwer's examples can also be embedded in a torus in a similar manner. It has a vertex set $\{(\alpha,\beta) \mid \alpha \in \mathbb{Z}_6, \beta \in \mathbb{Z}_9\}$, where (α,β) is joined to $(\alpha \pm 1,\beta)$, $(\alpha + 1, \beta + 2^{\alpha})$ and $(\alpha - 1, \beta - 2^{\alpha-1})$.

References

[1] I.Z. Bouwer. Vertex and edge transitive, but not 1-transitive graphs. *Canad. Math. Bull.* 13 (1970) 231–237.

[2] W.T. Tutte, *Connectivity in Graphs*. University of Toronto Press, Toronto (1966).