POISSON BOUNDARIES OF RANDOM WALKS ON DISCRETE SOLVABLE GROUPS

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0. INTRODUCTION

Let G be a topological group, and μ – a probability measure on G. A function f on G is called *harmonic* if it satisfies the mean value property

$$f(g) = \int f(gx)\, d\mu(x)$$

for all $g \in G$. It is well known that under natural assumptions on the measure μ there exists a measure G-space Γ with a quasi-invariant measure ν such that the *Poisson formula*

$$f(g) = \langle \widehat{f}, g\nu \rangle$$

states an isometric isomorphism between the Banach space $H^\infty(G,\mu)$ of bounded harmonic functions with sup-norm and the space $L^\infty(\Gamma,\nu)$. The space (Γ,ν) is called the *Poisson boundary* of the pair (G,μ). Thus triviality of the Poisson boundary is equivalent to absence of non-constant bounded harmonic functions for the pair (G,μ) (the *Liouville property*).

Let $\{x_i\}$ be a sequence of independent μ-distributed G-valued random variables, and

$$y_n = x_1 x_2 \dots x_n$$

be the *random walk* corresponding to the pair (G,μ). Suppose that there exists a G-space B such that in a certain sence (which needs to be specified in each particular case)

$$y_n \to y_\infty = \varphi(y) \in B$$

for almost all paths $y = \{y_n\}$. Denote by λ the distribution of the limit point $y_\infty \in B$. Then the measure space (B,λ) is always a factor-space of the Poisson boundary (Γ,ν),

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and the Poisson formula permits one to identify the space $L^\infty(B,\lambda)$ with a certain closed subspace of the space $H^\infty(G,\mu)$ of bounded harmonic functions.

From the probabilistic point of view the Poisson boundary (Γ,ν) can be characterized as the factor-space of the path space of the random walk $\{y_n\}$ with respect to its *stationary σ-algebra* consisting of all the events in the path space which are invariant with respect to the time shift. Evidently, the map $y\mapsto y_\infty$ is measurable with respect to the stationary σ-algebra, and the spaces $L^\infty(B,\lambda)$ and $H^\infty(G,\mu)$ are isomorphic if and only if the limit points y_∞ generate the *whole* stationary σ-algebra.

For the random walks on Lie groups with transition probabilities absolutely continious with respect to the Haar measure the problem of describing the Poisson boundary was completely solved by Raugi with an extensive use of the structure theory of Lie groups [22], but topological methods used there can not be applied for discrete groups. Another method of identifying the Poisson boundary with a certain G-space B for discrete groups consists in finding the topological counterpart of the Poisson boundary – the Martin boundary, but it needs difficult estimates of the Green function of the random walk which as a rule can be done only for finitely supported measures μ.

In this paper we use the entropy approach [17], [20] for obtaining a description of the Poisson boundary for random walks on discrete solvable groups. It permits to impose only relatively mild condition on the group G and measure μ (e.g.,finiteness of the first moment). The solvable groups are a natural intermediate class between nilpotent groups (for which the Poisson boundary is always trivial) and non-amenable groups (for which the Poisson boundary is non-trivial for an arbitrary non-degenerate measure), which makes studying the Poisson boundary for these groups especially interesting. We show that the situation with the Poisson boundary for discrete solvable groups is similar to the Lie group case only for polycyclic groups, which are a natural discrete analogue of solvable Lie groups and in a sence can be characterized as "finite dimensional" discrete solvable groups. For other discrete solvable groups the behaviour of the Poisson boundary strikingly differs from the Lie case.

The structure of the paper is the following.

In Section 1 we introduce necessary notions from entropy theory of random walks. The main technical tool is Theorem 1.3 which says that for a finitely generated group G a boundary B is in fact isomorphic to the Poisson boundary Γ if there exists a family of mappings $\pi_n : B\to G$ such that $d(\pi_n(y_\infty),y_n)=o(n)$ for almost all paths $\{y_n\}$ of the random walk (G,μ) (here d is a left invariant word metric on G).

Section 2 is auxiliary. We introduce length functions and gauges on groups and prove some results connecting random walks on discrete groups with random walks on their subgroups of finite index.

In Section 3 we consider random walks on general semi-direct products and wreath products, and give necessary and sufficient conditions for triviality of the Poisson boundary for measures with a finite first moment. The exposition here is based on the papers [16], [20] by A.M.Vershik and the author.

In Section 4 we first state a global version of the law of large numbers for splittable solvable Lie groups (Theorem 4.2). It says that for almost all realizations of an arbitrary stationary sequence $\{x_k\}$ with a finite first moment there exists a group element $g = g(\{x_k\})$ such that $d(g^k, x_1 \dots x_k) = o(k)$, where d is a certain principal gauge (an analogoue of the word metric) on the group. With the help of this theorem we identify the Poisson boundary for random walks on polycyclic groups with certain contracting nilpotent Lie groups (Theorem 4.3). It is known that for solvable (and even more generally, for amenable) Lie groups the Poisson boundary is trivial for all compactly supported probability measures μ absolutely continious with respect to the Haar measure [13]. We prove that if the Poisson boundary (G, μ) is non-trivial for a certain symmetric probability measure μ with a finite first moment on a finitely generated solvable group G, then G contains an infinitely generated subgroup (Theorem 4.4). The results in this Section were first announced in the author's paper [18].

In Section 5 we consider random walks on the affine group of diadic rational line $\mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$ – the group of the transformations $x \mapsto a\,x + b$, where $a = 2^k$ and $b = m/2^n$. The one-dimensional affine group (which is often called the "$ax + b$" group) is the simplest non-trivial solvable group, and first results about its boundaries were obtained as early as in mid-60s. One can easily provide an example of non-trivial boundary behaviour for random walks on this group: if

$$\alpha = \int \log\, a\, d\mu(a, b) < 0\,,$$

then the transformations $x \mapsto a\,x + b$ are contracting in mean, and for almost all random products

$$(a_1, b_1) \dots (a_n, b_n) = (y_n, \varphi_n)$$

there exists the limit

$$\varphi_\infty = \lim_{n\to\infty} \varphi_n \in \mathbb{R}\,.$$

It is known that for measures with a finite first moment on the real affine group the Poisson boundary is non-trivial if and only if $\alpha < 0$, and in this case it is completely described by the limit points φ_∞ [9]. For the discrete affine group the situation is different: here a non-trivial boundary behaviour arises also for $\alpha > 0$ – in this case the limit φ_∞ also exists, but in 2-adic rather than ordinary topology. Theorem 5.1 completely describes the Poisson boundary for random walks on the group $\mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$ with a finite first moment: it is trivial for $\alpha = 0$, coincides with the real line $\mathbb{R}$ for $\alpha < 0$, and with the 2-adic line $\mathbb{Q}_2$ for $\alpha > 0$.

Finally, in Section 6 we discuss connections between random walks on wreath products and exchangeable σ-algebras of Markov chains and apply the techique developed in this paper to obtain a description of the exchangeable σ-algebra of random walks on $\mathbb{Z}$ with a finite first moment (Theorem 6.2).

Acknowledgements

The author gratefully acknowledges support received from the National Science Foundation via grant # 8802126 and from the Ohio State University Mathematical Research Institute, as well as the warm hospitality of the Ohio State University while this paper was being written.

1. POISSON BOUNDARIES OF RANDOM WALKS ON DISCRETE GROUPS

In this introductory Section we define the principal notions connected with random walks on discrete and formulate basic results from the entropy theory of random walks used in this paper.

1.1. Random walks and Poisson boundaries

Let G be a discrete group, and μ – a probability measure on G. The *random walk on G determined by the measure μ* is the Markov chain on G with the transition probabilities

$$p(g,h) = \mu(g^{-1}h)$$

which are invaraint with respect to the action of the group G on itself by left translations. Denote by $(G^\infty, \mathbf{P})$ the probability measure in the space of the sample paths of the random walk (G, μ) that start out from the identity e of the group G. This measure can be considered as the image of the Bernoulli measure μ^∞ in the space of sequences of independent μ-distributed *increments* x_n under the map

$$\{x_n\} \mapsto \{y_n\},$$

where

$$y_n = x_1 x_2 \cdots x_n$$

is the position of the random walk (G, μ) at time n. The distribution of y_n (one-dimensional distribution of the measure $\mathbf{P}$ at time n) is μ_n – the n-fold *convolution* of the measure μ.

Denote by $\mathcal{A}^\infty$ the *tail σ-algebra* of the random walk (G, μ), i.e.

$$\mathcal{A}^\infty = \bigcap_n \mathcal{A}_n^\infty,$$

where $\mathcal{A}_n^\infty$ are the coordinate σ-algebras generated by the random variables $\{y_k\}_{k \geq n}$. By $\mathcal{S}$ denote the *stationary σ-algebra* of the random walk (G, μ) – the subalgebra of $\mathcal{A}^\infty$ consisting of the measurable subsets of the path space which are invariant with respect to the time shift $\{y_n\} \mapsto \{y_{n+1}\}$. It is well known that for the random walks on groups these σ-algebras coincide $\mathbf{P}$ - $\mathrm{mod}\, 0$ [7], [20]. The quotient space (Γ, ν) of the path space $(G^\infty, \mathbf{P})$ corresponding to the stationary σ-algebra $\mathcal{S}$ is called the *Poisson boundary* of the random walk (G, μ). The Poisson boundary is endowed with a natural action of the group G, so that the measure ν (the *harmonic measure*) is *μ-stationary* with respect to this action:

$$\nu = \mu * \nu = \sum_{g \in G} \mu(g)\, g\nu\,.$$

In particular, if

$$\mathrm{sgr}\, \mu = \bigcup_{n \geq 0} \mathrm{supp}\, \mu_n$$

is the semigroup generated by the support of measure μ ($\equiv$ the set attainable by the random walk starting from the group identity e), then the measure $g\nu$ is absolutely continious with respect to the measure ν for all $g \in \mathrm{sgr}\, \mu$.

A function f on the group G is called *μ-harmonic*, if it is an invariant function of the *Markov operator* of the random walk (G, μ), i.e.

$$f(g) = \sum_{x \in G} \mu(x) f(gx) \qquad \forall\, g \in G\,.$$

The *Poisson formula*

$$f(g) = \langle \widehat{f}, g\nu \rangle$$

states an isometric isomorphism between the space $L^\infty(\Gamma, \nu)$ and the space $H^\infty(G, \mu)$ of bounded harmonic functions on $\operatorname{sgr} \mu$ with the sup-norm. If the Poisson boundary is trivial, then all bounded harmonic functions on the set $\operatorname{sgr} \mu$ as well as on its translations $g \operatorname{sgr} \mu\,, g \in G$ are constant. Thus, if G coincides with the group $\operatorname{gr} \mu$ generated by $\operatorname{supp} \mu$, then *triviality of the Poisson boundary (Γ, ν) is equivalent to absence of non-constant bounded μ-harmonic functions on the group G (Liouville property).*

The Poisson boundary is trivial for all measures on abelian and nilpotent groups (Blackwell [2], Choquet – Deny [5], Dynkin – Malyutov [8]). On the other hand, it is always non-trivial if the group $\operatorname{gr} \mu$ generated by $\operatorname{supp} \mu$ is non-amenable [20], [23]. Solvable groups present a natural intermediate class between these two extremities, and already within this class of groups the dependence of the boundary on measures bears much more complicated character (cf. also [16], [20] for examples of non-trivial Poisson boundaries for discrete amenable groups).

1.2. Entropy of random walks

Formulate now basic facts from the entropy theory of random walks on discrete groups which will be used below (see [6], [7], [17], [20] for more detailed exposition). The *entropy* of a discrete probability distribution $p = (p_i)\,, \sum p_i = 1$ is defined as

$$H(p) = -\sum p_i \log p_i\,.$$

Suppose that the entropy $H(\mu)$ of a probability measure μ on G is finite. Then the entropies of its convolutions $H(\mu_n)$ are also finite, and there exists the limit

$$h(G, \mu) = \lim_{n \to \infty} \frac{1}{n} H(\mu_n)\,,$$

which is called the *entropy of the random walk* (G, μ). The entropy $h(G, \mu)$ satisfies the following *Shannon – McMillan – Breiman type property*: for $\mathbf{P}$-a.e. path $\{y_n\}$ of the random walk (G, μ)

$$\lim_{n \to \infty} \frac{1}{n} \log\, \mu_n(y_n) = -h(G, \mu)\,.$$

Theorem 1.1 ([6],[20]). *Let μ be a probability measure on a discrete group G with finite entropy $H(\mu)$. Then the Poisson boundary (Γ, ν) of the random walk (G, μ) is trivial if and only if the entropy $h(G, \mu)$ is zero.*

Corollary. *The Poisson boundary of random walk* (G,μ) *is trivial if and only if there exist a sequence of finite sets* $A_n \subset G$ *and a positive number* $\varepsilon > 0$ *such that*

$$\log \operatorname{card} A_n = o(n)$$

and

$$\mu_n(A_n) > \varepsilon \qquad \forall\, n \ge 0\,.$$

Theorem 1.1 gives a necessary and sufficient condition for triviality of the Poisson boundary in entropy terms. The next results generalizes Theorem 1.1 and gives a criterion that a *certain* boundary is in fact *maximal*, i.e. coincides with the whole Poisson boundary.

Let $\pi : (G^\infty, \mathbf{P}) \to (B, \lambda)$ be a homomorphism of the path space $(G^\infty, \mathbf{P})$ onto a certain measurable G-space, so that the measure λ on B is an image of the measure $\mathbf{P}$ on G^∞. Suppose that π is measurable with respect to the stationary σ-algebra $\mathcal{S}$ in G^∞, and G-equivariant, i.e. $\pi(gy) = \pi(y)$ for $(gy)_n = gy_n$. Then the measure space (B, λ) coincides with the *factor-space* (Γ_ξ, ν_ξ) of the Poisson boundary (Γ, ν) with respect to a certain G-invariant measurable partition ξ. We shall say that the measure space (B, λ) is a *Furstenberg boundary* of the random walk (G,μ) (cf. [10]). Clearly, there exists a maximal Furstenberg boundary, which is the Poisson boundary itself, and *maximality* of a Furstenberg boundary means that it is isomorphic to the Poisson boundary. In particular, maximality of the trivial one-point Furstenburg boundary means that the Poisson boundary of the random walk (G,μ) is trivial.

The typical situation when a Furstenberg boundary (B,λ) can arise is when B is a certain topological or combinatorial boundary of the group G, and almost all paths $\{y_n\}$ of the random walk (G,μ) converge (in a certain sence which needs to be specified in each particular case) to a limit point $\varphi(y) = y_\infty \in B$. Then the space B considered as a measure space with the resulting hitting distribution λ (the *harmonic measure* on B) is a Furstenberg boundary of the random walk (G,μ).

Almost all points γ of a Furstenberg boundary $(B,\lambda) \cong (\Gamma_\xi, \nu_\xi)$ can be identified with (unbounded) μ-harmonic functions on $\operatorname{sgr}\mu$ by means of the *Radon–Nikodym transform*

$$\varphi_\gamma(g) = \frac{dg\nu_\xi}{d\nu_\xi}(\gamma)\,.$$

Denote by $\mathbf{P}^\gamma$ the *conditional measures* of the measure $\mathbf{P}$ in the path space conditioned by the points $\gamma \in \Gamma_\xi$. Then for almost all γ the measure $\mathbf{P}^\gamma$ is the measure in the path space of the Markov chain on G with the transition probabilities

$$p^\gamma(g,h) = \frac{\varphi_\gamma(h)}{\varphi_\gamma(g)}\mu(g^{-1}h)\,,$$

which is the *Doob transform* of the random walk (G,μ) corresponding to the harmonic function φ_γ. Let p_n^γ be the one-dimensional distributions of the conditional random walks at the time n:

$$p_n^\gamma(g) = \mathbf{P}^\gamma\big[\, y \in \Gamma^\infty : y_n = g \,\big] = \varphi_\gamma(g)\,\mu_n(g)\,.$$

Now define the *entropy of the conditional random walks* as the limit (provided that it exists)

$$h(G,\mu,\gamma) = -\lim_{n\to\infty} \frac{1}{n} H(p_n^\gamma) .$$

Theorem 1.2 ([17]). *Let $(B,\lambda) \cong (\Gamma_\xi, \nu_\xi)$ be a Furstenberg boundary of the random walk (G,μ). Then for almost all $\gamma \in \Gamma_\xi$ the entropy $h(G,\mu,\gamma)$ exists and is independent of γ. The Furstenberg boundary (Γ_ξ, ν_ξ) is maximal (i.e. coincides with the Poisson boundary) if and only if $h(G,\mu,\gamma) = 0$ for almost all $\gamma \in \Gamma_\xi$.*

Once again the entropy of the conditional random walks satisfies the Shannon – McMillan – Breiman property: under the conditions of Theorem 1.2 for almost all $\gamma \in \Gamma_\xi$ and for $\mathbf{P}^\gamma$- almost all paths $\{y_n\}$

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_n^\gamma(y_n) = -h(G,\mu,\gamma) .$$

Corollary. *A Furstenberg boundary $(B,\lambda) \cong (\Gamma_\xi, \nu_\xi)$ of random walk (G,μ) is maximal if and only if for almost all points $\gamma \in \Gamma_\xi$ there exist a positive number $\varepsilon(\gamma)$ and a sequence of finite sets $A_n^\gamma \subset G$ such that*

$$\log \operatorname{card} A_n^\gamma = o(n)$$

and

$$p_n^\gamma(A_n^\gamma) > \varepsilon(\gamma) \qquad \forall\, n \geq 0 .$$

Theorem 1.3([17]). *Suppose that G is a finitely generated group, and $\pi_n : B \to G$ is a family of measurable mappings from a Furstenberg boundary (B,λ) of a random walk (G,μ) to the group G such that*

$$d\,(\pi_n(y_\infty), y_n) = o(n)$$

for $\mathbf{P}$-almost all paths $y = \{y_n\} \in G^\infty$, where y_∞ is the point of the Furstenberg boundary B corresponding to the path y, and d is the distance on G corresponding to a certain principal gauge on G (see below). Then the Furstenberg boundary (B,λ) is maximal, i.e. coincides with the Poisson boundary of the random walk (G,μ).

2. GAUGES ON LOCALLY COMPACT GROUPS

This Section is auxiliary. We define notions of lenght functions and gauges for locally compact groups, and prove some results connecting random walks on discrete groups with random walks on their subgroups.

2.1. Length functions and gauges

Let G be a compactly generated locally compact group, $K \subset G$ - a compact generating subset, i.e.

$$G = \bigcup_{n\geq 0} K^n ,$$

where

$$K^n = \{g_1 \cdots g_n : g_i \in K\}\,.$$

The function

$$\delta_K(g) = \min\{n : g \in K^n\}$$

is called the *length function* corresponding to the set K. A non-negative function δ on G is called a *gauge* [13] if there exists a positive constant C such that

$$\delta(g_1 g_2) \le \delta(g_1) + \delta(g_2) + C \qquad \forall\, g_1, g_2 \in G\,.$$

We shall say that a gauge δ *dominates* another gauge δ' if there exist positive constants a, b such that

$$\delta'(g) \le a\,\delta(g) + b \qquad \forall\, g \in G\,.$$

Two gauges are *equivalent* if they dominate each other. Evidently, the length function δ_K is a gauge and all gauges δ_K are pairwise equivalent for different compact generating sets K. A gauge δ is a *principal gauge* if it is equivalent to the length function δ_K for a certain (or, equivalently, for every) compact generating set K. If δ is a gauge, let $d = d_\delta$ be the corresponding left-invariant function on the pairs $(g_1, g_2) \in G \times G$ measuring "how far" is g_1 from g_2:

$$d(g_1, g_2) = \delta(g_1^{-1} g_2)\,.$$

In the case when G is a finitely generated discrete group, the length function δ_K corresponding to a finite generating set K is the usual *word length* function on G, and $d_K(g_1, g_2) = \delta_K(g_1^{-1} g_2)$ is a left invariant metric on G if the set K is symmetric.

If μ is a probability measure on a compactly generated group G, we shall say that μ has a *finite first moment* if

$$\langle \delta, \mu \rangle = \int \delta(g)\, d\mu(g) < \infty$$

for a certain principal gauge δ on K. Evidently, this definition does not depend on the choice of the principal gauge δ. If μ is a measure with a finite first moment on a group G, then one can define the *rate of escape* of the random walk (G, μ) with respect to a gauge δ as

$$\lambda_\delta(G, \mu) = \lim_{n \to \infty} \frac{1}{n} \langle \delta, \mu_n \rangle < \infty\,.$$

As it follows from Kingman's subadditive ergodic theorem (cf. [6], [13]),

$$\lim_{n \to \infty} \frac{1}{n} \delta\,(y_n) = \lambda_\delta\,(G, \mu)$$

for almost all paths $\{y_n\}$ of the random walk (G, μ). In particular, if G is a finitely generated group, and δ is a principal gauge on G, then $\lambda_\delta(G, \mu) = 0$ implies trivialiy of the Poisson boundary of the random walk (G, μ) (see Theorem 1.1).

2.2. Random walks and gauges on normal subgroups

Lemma 2.1. *Let G be a finitely generated group with a principal gauge δ, and $G^0 \subset G$ be its normal subgroup of finite index. Then the group G^0 is also finitely generated, and the restriction of δ onto G^0 is a principal gauge on G^0.*

Proof. Denote by π the homomorphism $\pi : G \to G/G^0 = A$, where A is a finite group, and choose for each $a \in A$ a representative $[a] \in \pi^{-1}(a)$. In particular, put $[e_A] = e_G$, where e_A and e_G are identity elements of the groups A and G, respectively. Let $[A] = \{[a] : a \in A\}$ and $\overline{g} = [\pi(g)] \in [A]$ be the representative in $[A]$ of the image $\pi(g) \in A$ for $g \in G$. Now, every element x of the group G can be uniquely represented as the product $x = x^0\overline{x}$ with $x^0 = x(\overline{x})^{-1} \in G^0$. Remark that $x^0 = x$ for all $x \in G^0$.

Let $x_i \in G$ and the product $g = x_1 \cdots x_k$ belong to G^0. Then we can write $x_i = x_i^0 h_i$ with $h_i = \overline{x_i}$. Let

$$y_i = \overline{h_1 \cdots h_{i-2}}\, h_{i-1} x_i^0 \, (\overline{h_1 \cdots h_{i-1}})^{-1} ,$$

and

$$h = \overline{h_1 \cdots h_{k-1}}\, h_k ,$$

then

$$g = y_1 \cdots y_k \, h . \tag{1}$$

Now let $K \subset G$ be a finite set generating G. If $x_i \in K$ and $g \in G^0$, then the formula (1) means that g can be presented as the product of $k+1$ elements from a finite subset K^0 of the group G^0 defined as

$$K^0 = \{h_1 h_2 x^0 (\overline{h_1 h_2})^{-1} : x \in K \, , h_{1,2} \in [A]\}$$

($\overline{h_1 \cdots h_k} = e_G$ and $\overline{h_1 \cdots h_{k-1}} h_k \in K^0$, since $g \in G^0$).

Thus we have proved that G^0 is finitely generated and that the principal gauge on G^0 corresponding to the subset $K^0 \subset G^0$ is dominated by the restriction of δ onto G^0. On the other hand, if K contains a generating set K' for G^0, then one can easily see that the gauge $\delta_{K'}$ dominates the restriction of the gauge δ_K onto G_0. □

Lemma 2.2 (cf. [10])**.** *Let G be a discrete group, μ a probability measure on G, and G^0 – a subgroup of G, which is a recurrent set for the random walk (G, μ). Define a probability measure μ^0 on G^0 as the distribution of the point where the random walk issued from the identity of G returns for the first time to G^0. Then the Poisson boundaries $\Gamma(G, \mu)$ and $\Gamma(G^0, \mu^0)$ are isomorphic.*

Proof. State a one-to-one correspondence between the spaces of bounded harmonic functions $H^\infty(G, \mu)$ and $H^\infty(G^0, \mu^0)$. From the definition of the measure μ^0 follows that the restriction f^0 of any function $f \in H^\infty(G, \mu)$ onto G^0 is a μ^0-harmonic function on G^0. Conversely, let $f^0 \in H^\infty(G^0, \mu^0)$. Let

$$f(g) = \int f^0(y_\tau) \, d\mathbf{P}_g(y) ,$$

where $\tau = \tau(y)$ is the time when the path y for the first time hits the subgroup G^0, and $\mathbf{P}_g$ is the probability measure in the path space of the random walk (G, μ) corresponding to the initial distribution δ_g concentrated on a point $g \in G$. Then f is a μ-harmonic function on G and $f \equiv f^0$ on G^0. □

Remark. One can easily see that if the measure μ on G is symmetric, then the induced measure μ^0 on its recurrent subgroup G^0 is also symmetric.

Lemma 2.3. *Let G be a finitely generated group, μ a probability measure on G with a finite first moment, and G^0 – a normal subgroup of finite index in G. Then the subgroup G^0 is recurrent for the random walk (G, μ), and the measure μ^0 on G^0 defined in Lemma 2.2 has a finite first moment in G^0.*

Proof. Since the factor-group $G/G^0 = A$ is finite, the random walk on A (image of the random walk on G under the homomorphism $\pi : G \to A$) is recurrent. Hence the subgroup G^0 is recurrent. Now, by Lemma 2.1, we have to verify that

$$\int \delta(y_\tau)\, d\mathbf{P}(y) < \infty ,$$

where $\tau = \tau(y)$ is the time when the path y for the first time hits the subgroup G^0, and δ is a certain principal gauge on G. Assume for the sake of simplicity that the gauge δ is subadditive (one can always achieve it by adding a positive constant to δ). Consider on the factor-group A the function

$$\lambda(a) = \frac{\sum_{x:\pi(x)=a} \delta(x)\mu(x)}{\sum_{x:\pi(x)=a} \mu(x)} ,$$

where π is the homomorphism $G \to G/G^0 = A$ (if $\sum_{x:\pi(x)=a} \mu(x) = 0$, put $\lambda(a) = 0$). The values of λ are conditional expectations of $\delta(x)$ provided that $\pi(x) = a$. Since μ has a finite first moment, and the group A is finite, there exists a positive constant C such that $\lambda(a) < C$ for all $a \in A$. Hence

$$\int \delta(y_\tau)\, d\mathbf{P}(y) \le C \sum_{n \ge 1} n\theta_n ,$$

where θ is the distribution of the first return time for the quotient random walk $(G/G^0, \pi(\mu))$. Since the group G/G^0 is finite, this random walk is positively recurrent, i.e. $\sum n\theta_n < \infty$. □

3. RANDOM WALKS ON SEMI-DIRECT PRODUCTS

In this Section we study the problem of boundary triviality for discrete solvable groups which are semi-direct products or wreath products. Examples show the striking difference between the solvable Lie groups and solvable discrete groups.

3.1. Semi-direct products

Let X and F be two discrete groups and

$$T : x \mapsto T^x \in \mathrm{Aut}(F)$$

be an action of the group X on the group F by its automorphisms. Then the groups X and F and the action T determine a new group $G = X \leftthreetimes F = X \overset{T}{\leftthreetimes} F$ which is called the *semi-direct product of the groups* X *and* F *with the action* T in the following way: as a set

$$G = \{(x, f) : x \in X, f \in F\}\,,$$

and the group operation in G is given by the formula

$$(x_1, f_1)\,(x_2, f_2) = (x_1 \cdot x_2, f_1 \cdot T^{x_1} f_2)\,. \tag{1}$$

In particular, when the action T is trivial, the semi-direct product coincides with the usual direct product of the groups X and F. Remark that from the formula (1) easily follows that

$$(x, f)^{-1} = (x^{-1}, (T^x)^{-1} f^{-1}) = (x^{-1}, T^{x^{-1}} f^{-1})\,.$$

We shall always assume that X and F are imbedded into G by the maps $x \mapsto (x, e_F)$ and $f \mapsto (e_X, f)$, where e_F and e_X are identities of the groups F and X, respectively. Then F is a normal subgroup in G, and $G/F \cong X$, so that we have an exact sequence

$$1 \to F \to G \to X \to 1\,.$$

If the group X is abelian, and F is nilpotent (in particular, abelian), then the semi-direct product $X \leftthreetimes F$ is solvable [21].

Let A_X and A_F be generating subsets for the groups X and F, respectively. Then the set $A = A_X \cup A_F$ generates the group G. In the case when the sets A_X and A_F are finite, denote by $|\cdot|_X\,, |\cdot|_F$ and $|\cdot|$ the corresponding length functions on X, F and G. Then, evidently

$$|x|_X \le |(x, f)| \qquad \forall\, x \in X\,, f \in F\,. \tag{2}$$

Since all the length functions corresponding to finite generating sets on finitely generated groups are equivalent (see Section 2), for all $x \in A_X$

$$|T^x f|_F \le C\,|f|_F \qquad \forall\, f \in F$$

for a certain positive constant $C = C(A_X, A_F)$. Hence

$$|T^x f|_F \le \exp\left(\log C\,|x|_X\right)|f|_F \qquad \forall\, x \in X\,, f \in F\,. \tag{3}$$

This in combination with (1) and (2) implies that

$$\log(1 + |f|_F) \le C_1\,|(x, f)| \qquad \forall\,(x, f) \in G\,. \tag{4}$$

Let μ be a probability measure on $G = X \leftthreetimes F$. Denote by (x_n, f_n) the increments of the random walk (G, μ), and by (y_n, φ_n) the random walk itself:

$$(y_n, \varphi_n) = (x_1, f_1) \cdots (x_n, f_n)\,. \tag{5}$$

Below we shall use the following formulas which follow from (1) and (5):

$$\begin{cases} y_n = x_1 \cdots x_n\,, \\ \varphi_n = f_1 \cdot T^{y_1} f_2 \cdot \dots \cdot T^{y_{n-1}} f_n\,, \end{cases} \tag{6}$$

or

$$\begin{cases} y_n = y_{n-1}\, x_n\,, \\ \varphi_n = \varphi_{n-1} \cdot T^{y_{n-1}} f_n\,. \end{cases} \tag{7}$$

One can easily see that the projection y_n of the random walk (G, μ) onto X is the random walk (X, μ_X) determined by the measure μ_X which is the projection of the measure μ onto X.

Theorem 3.1. *Let μ be a probability measure on a semi-direct product $G = X \leftthreetimes F$ such that the random walk (X, μ_X) is recurrent. If the group F is nilpotent (in particular, abelian), then the Poisson boundary of the random walk (G, μ) is trivial.*

Proof. Recurrence of the random walk (X, μ_X) means that the subgroup F is recurrent for the random walk (G, μ). Now Lemma 2.2 and triviality of the Poisson boundary for all measures on nilpotent groups [8] imply the desired statement. $\square$

The following result can be considered as a generalization of the previous theorem. It is obtained without making use of the structure theory for nilpotent groups – only from the fact that their growth is polynomial (cf. [12]).

Theorem 3.2. *Let $G = X \leftthreetimes F$ be a semidirect product of two finitely generated groups, and μ a probability measure on G with a finite first moment. If the group F is nilpotent, and for the random walk (X, μ_X) its rate of escape $l(X, \mu_X)$ is zero, then the Poisson boundary of the random walk (G, μ) is trivial.*

Proof. Since $l(X, \bar{\mu}) = 0$,

$$|y_n| = o(n)$$

for almost all trajectrories $\{(y_n, \varphi_n)\}$ of the random walk (G, μ). On the other hand, formula (4) and finiteness of the first moment of μ imply that for almost all increments $\{(x_n, f_n)\}$

$$\log(1 + |f_n|_F) = o(n)\,.$$

Thus from the formulas (3), (6) it follows that

$$\log(1 + |\varphi_n|_F) = o(n)$$

for almost all paths $\{(y_n, \varphi_n)\}$ of the random walk (G, μ). Now, since the growth of F is polynomial and $|y_n| = o(n)$, the entropy of the random walk (G, μ) is zero, and the Poisson boundary is trivial (cf. Theorem 1.1). $\square$

3.2. Wreath products

Wreath products are in a sence the simplest non-trivial case of semi-direct products, because essentialy they arise from the action of a group on itself by translations. Wreath products give a good illustration to dramatical distinctions between solvable Lie groups and solvable discrete groups, so that for the sake of completeness we included here some examples connected with the random walks on wreath products from [16], [20]. We shall see later that the wreath products of a special type are closely connected with discrete affine groups and can be applied to some purely probabilistic problems (see below Sections 5,6).

Let X and A be two discrete groups. Denote by $\mathrm{fun}(X, A)$ the direct sum of isomorphic copies of the group A indexed by the elements from X:

$$\mathrm{fun}(X, A) = \sum_{x \in X} A\,.$$

It will be convenient to consider $\mathrm{fun}(X, A)$ as the group of *finitely supported* A-valued *configurations* on X with the operation of pointwise multiplication. Let

$$\mathrm{Fun}(X, A) = \prod_{x \in X} A$$

be the direct product of X isomorphic copies of the group A, i.e. the group of *all* (not necessarily finitely supported) A-valued configurations on X. By $f(x) \in A$ denote the value of a configuration f at a point $x \in X$, and by $\mathrm{supp}\, f$ its support:

$$\mathrm{supp} f = \{x \in X : f(x) \neq e\}\,.$$

We shall say that a sequence of configurations f_n *converges* to a configuration f_∞, if

$$\lim_{n \to \infty} f_n(x) = f_\infty(x) \qquad \forall\, x \in X\,,$$

i.e., for all $x \in X$ the sequence $f_n(x)$ stabilizes. The group $\mathrm{fun}(X, A)$ is endowed with the natural action of the group X by translations:

$$T^x f(y) = f(x^{-1}y) \qquad f \in \mathrm{fun}(X, A);\ x, y \in X\,.$$

Definition (e.g.,see [21])**.** The semidirect product $X \leftthreetimes \mathrm{fun}(X, A)$ corresponding to the action T of the group X on $\mathrm{fun}(X, A)$ by translations is called the *(restricted) wreath product of the group A by the group X with passive group A and active group X.*

For the sake of simplicity in this Section we consider mainly the groups of *dynamical configurations* (the term is proposed by A.M.Vershik)

$$G_k = \mathbb{Z}^k \leftthreetimes \mathrm{fun}(\mathbb{Z}^k, \mathbb{Z}_2)\,,$$

where $\mathbb{Z}^k$ is the free abelian group with k generators, and $\mathbb{Z}_2 = \{0, 1\}$ is the cyclic group of order 2. All the groups G_k are solvable of degree 2, finitely generated and have exponential growth.

Denote by δ_x , $x \in \mathbb{Z}_k$ the configuration supported at x:

$$\delta_x(y) = \begin{cases} 1, & y = x\,, \\ 0, & y \neq x\,. \end{cases}$$

The set $\delta_0 \cup \{z_i\}$, where z_i are generators of $\mathbb{Z}^k$, is a generating set for the group G_k. Denote by $|\cdot|$ the corresponding length function on G_k. One can easily see that

$$|(x,f)| \geq |\operatorname{supp} f| \qquad \forall\,(x,f) \in G_k\,,$$

where $|\operatorname{supp} f|$ is the number of elements in $\operatorname{supp} f$.

Theorem 3.3. *Let $k \geq 1$, and μ be a probability measure on the group G_k with a finite first moment. Then the Poisson boundary of the random walk (G_k, μ) is non-trivial if and only if the random walk $(\mathbb{Z}^k, \mu_{\mathbb{Z}^k})$ is transient and the subgroup* $\operatorname{gr}\mu$ *generated by* $\operatorname{supp}\mu$ *is non-abelian. In this case for almost all paths $\{(y_n, \varphi_n)\}$ of the random walk (G_k, μ) there exists the limit*

$$\lim_{n\to\infty} \varphi_n = \varphi_\infty \in \operatorname{Fun}(\mathbb{Z}^k, \mathbb{Z}_2)\,. \tag{8}$$

Proof. If the random walk $(\mathbb{Z}^k, \mu_{\mathbb{Z}^k})$ is recurrent, then the Poisson boundary is trivial as it follows from Theorem 3.1.

Show that if the random walk $(\mathbb{Z}^k, \bar{\mu})$ is transient, then the limit (8) exists. Fix a point $z \in \mathbb{Z}_k$ and using the formula (7) estimate the probability that $\varphi_n(z) \neq \varphi_{n+1}(z)$:

$$\begin{aligned} \mathbf{P}[\varphi_n(z) \neq \varphi_{n+1}(z)] &= \mathbf{P}[f_{n+1}(z - y_n) \neq 0] \\ &= \sum_{u\in\mathbb{Z}^k} (\mu_{\mathbb{Z}^k})_n(u) \sum_{(x,f)\in G_k} |f(z-u)|\mu(x,f)\,. \end{aligned}$$

The random walk $(\mathbb{Z}^k, \mu_{\mathbb{Z}^k})$ is transient, hence its Green function

$$\mathfrak{G}(u) = \sum_{n\geq 0} (\mu_{\mathbb{Z}^k})_n(u)$$

is bounded [25]. Thus

$$\begin{aligned} \sum_{n\geq 0} \mathbf{P}[\varphi_n(z) \neq \varphi_{n+1}(z)] &\leq \sum_{u\in\mathbb{Z}^k} \mathfrak{G}(u) \sum_{(x,f)\in G_k} |f(z-u)|\,\mu(x,f) \\ &\leq \text{Const} \sum_{(x,f)\in G_k} \mu(x,f) \sum_{u\in\mathbb{Z}^k} |f(z-u)| \\ &= \text{Const} \sum_{(x,f)\in G_k} \mu(x,f)\,|\operatorname{supp} f| \\ &\leq \text{Const} \sum_{(x,f)\in G_k} \mu(x,f)\,|(x,f)| < \infty\,. \end{aligned}$$

Now the Borel–Cantelli Lemma implies that the values $\varphi_n(z)$ a.e. change only a finite number of times, hence the limit in question exists.

Show that the limit configuration φ_∞ can not be the same for a.e. paths. Indeed, suppose the contrary. Then

$$(x,f)\,\varphi_\infty = T^x\,\varphi_\infty + f = \varphi_\infty \qquad \forall\,(x,f)\in \operatorname{supp}\mu\,.$$

Since the group generated by the support of the measure μ is non-abelian, there exist at least two non-commuting elements in $\operatorname{supp}\mu$. Their commutator has the form $(0,f)$ for a certain non-zero configuration f, and also preserves φ_∞, i.e.

$$\varphi_\infty = \varphi_\infty + f\,,$$

which is impossible. □

Corollary. *Let μ be a non-degenerate symmetric measure with a finite first moment on a group G_k. Then the Poisson boundary of the random walk (G_k,μ) is trivial for $k=1,2$, and non-trivial for $k\ge 3$.*

Remarks. 1.In fact, the proof of Theorem 3.3 given here can be almost *verbatim* carried over to the case of the wreath products $G=\mathbb{Z}^k \leftthreetimes \operatorname{fun}(\mathbb{Z}^k,A)$ with an arbitrary finitely generated group A.

2. For measures without a finite first moment Theorem 3.3 is not true. Namely, there exist measures μ on the group G_1 such that the random walk $(\mathbb{Z},\mu_\mathbb{Z})$ is transient, and the Poisson boundary $\Gamma(G_1,\mu)$ is non-trivial, but the configurations φ_n do not stabilize. There exist also measures on G_1 such that the random walk $(\mathbb{Z},\mu_\mathbb{Z})$ is transient, but nonetheless the Poisson boundary $\Gamma(G_1,\mu)$ is trivial.

3. The problem of proving that the limit configurations φ_n describe the whole Poisson boundary even for measures with a finite first moment on groups G_k seems to be very intriguing. We can prove it only for finitely supported measures and some special measures with a finite first moment on groups G_1 (see below Section 6).

3.3. Locally finite solvable wreath products

Denote by $D=\operatorname{fun}(\mathbb{N},\mathbb{Z}_2)$ the countable direct sum of the cyclic groups $\mathbb{Z}_2=\{0,1\}$ indexed with natural numbers $1,2,3,\dots\in\mathbb{N}$, and consider the wreath product

$$\mathcal{D} = D \leftthreetimes \operatorname{fun}(D,\mathbb{Z}_2)\,.$$

Let $\delta_n\in D$ be the point configuration at a point $n\in N$:

$$\delta_n(k) = \begin{cases} 1, & k=n\,,\\ 0, & k\ne n\,,\end{cases}$$

then the set $\{\delta_n\}$ generates the group D.

Denote the identities of the groups D and $\operatorname{fun}(D,\mathbb{Z}_2)$, respectively, as

$$\phi\in D:\phi(n)=0 \qquad \forall\,n\in N\,,$$

and

$$\Phi\in\operatorname{fun}(D,\mathbb{Z}_2):\Phi(f)=0 \qquad \forall\,f\in D\,.$$

Fix yet another element of the group $\mathrm{fun}(D, \mathbb{Z}_2)$:

$$\omega \in \mathrm{fun}(D, \mathbb{Z}_2) : \omega(f) = \begin{cases} 1, & f = \phi\,, \\ 0, & f \neq \phi\,. \end{cases}$$

By e denote the identity (ϕ, Φ) of the group $\mathcal{D}$.

The group $\mathcal{D}$ is a locally finite solvable group of degree 2. Formulate several more special properties of $\mathcal{D}$.

Lemma 3.1. *The set $\{\delta_n\}_{n=1}^{\infty} \cup \{\omega\}$ generates the group $\mathcal{D}$.*

Proof. Since $\{\delta_n\}_{n=1}^{\infty}$ is a generating set for the group D, it is sufficient to show that the D-orbit of the element ω generates the group $\mathrm{fun}\,(D, \mathbb{Z}_2)$. Indeed, let

$$F \in \mathrm{fun}\,(D, \mathbb{Z}_2)\,, \qquad \mathrm{supp}\, F = \{f_i\}_{i=1}^{k}\,,$$

then, evidently,

$$F = T^{f_1}\omega + T^{f_2}\omega + \cdots + T^{f_k}\omega\,.$$

□

Lemma 3.2. *Orders of all elements of the group $\mathcal{D}$ don't exceed* 4.

Proof. Let $g = (f, F) \in \mathcal{D}$, then

$$g^2 = (f + f, F + T^f F) = (\phi, F + T^f F)\,,$$

since $f + f = \phi$ and $F + F = \Phi$ for all $f \in D\,, F \in \mathrm{fun}(D, \mathbb{Z}_2)$. Hence

$$g^4 = (\phi, F + T^f F)^2 = (\phi, F + T^f F + F + T^f F) = (\phi, \Phi) = e\,.$$

□

Lemma 3.3. *Orders of all finitely generated subgroups of $\mathcal{D}$ with not more than k generators are uniformly bounded for all $k \geq 1$.*

Proof. Fix a certain k-subset T of the group $\mathcal{D}$. Without loss of generality we can assume that T is symmetric and contains the identity e, so that the group $\mathrm{gr}\, T$ generated by T coincides with the union $\bigcup_{n \geq 0} T^n$, and the sets T^n form a non-decreasing sequence. Estimate the cardinality of the set T^n for a given n.

Let $\{(f_i, F_i)\}_{i=1}^{n}$ be a set of elements from T, then

$$(f_1, F_1) \cdot \ldots \cdot (f_n, F_n) = (f_1 + \cdots + f_n, F_1 + T^{f_1} F_2 + \cdots + T^{f_1 + \cdots + f_{n-1}} F_n)\,.$$

The subgroup of D consisting of the sums $f_1 + \cdots + f_n$ has not more than k generators of order 2, hence it consists of not more than 2^k elements. Thus the subgroup of

$\mathrm{fun}(D, \mathbb{Z}_2)$ generated by the elements of the form $T^{f_1+\dots+f_{n-1}}F_n$ has not more than $k\,2^k$ generators of order 2 and consists of not more than $2^{k\,2^k}$ elements. Finally,

$$\mathrm{card}\, T^n \le 2^k\, 2^{k\,2^k}$$

for all n. □

Recall that a group G is said to be a group of *uniformly polynomial growth* [4], [20], if there exists a sequence of polynomials p_k such that

$$\mathrm{card}\, T^n \le p_k(n)$$

for all subsets $T \subset G$ with no more than k elements. Thus the group $\mathcal{D}$ has uniformly polynomial growth.

Theorem 3.4. *There exists a non-degenerate symmetric measure μ with finite entropy $H(\mu)$ on the group $\mathcal{D}$ such that the Poisson boundary $\Gamma(\mathcal{D}, \mu)$ is non-trivial.*

Proof. The proof goes along the same lines as in Theorem 3.3. Let μ be the following probability measure on the group $\mathcal{D}$:

$$\mu(\delta_n) = p_n\,,$$
$$\mu(\omega) = q\,,$$
$$\mu(e) = r\,,$$

where $p_n, q, r > 0$ and $\sum p_n + q + r = 1$. Then the measure μ is symmetric and non-degenerate. Let $\{(h_n, H_n)\}_{n=0}^{\infty}$ be the random walk determined by the measure μ, i.e.

$$(h_n, H_n) = (f_1, F_1) \cdot \ldots \cdot (f_n, F_n)\,,$$

where (f_i, F_i) are independent $\mathcal{D}$-valued random variables with the distribution μ (increments of the random walk). Since the supports of all the configurations F_i are either empty or consist of the single point ω, transience of the random walk $\{h_n\}$ on D would imply non-triviality of the Poisson boundary $\Gamma(\mathcal{D}, \mu)$. It is well known (e.g., see [25]), that if

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1\,,$$

then the random walk (h_n) on D is indeed transient. □

Remarks. 1. Recall that for finitely generated groups of polynomial growth the Poisson boundary is trivial for all probability measures as follows from the Gromov theorem [12] in combination with triviality of the Poisson boundary for an arbitrary measure on a nilpotent group [8]. This Theorem shows that for infinitely generated groups the sitiuation is much more complicated.

2. A bit more "sophisticated group" $D \leftthreetimes \mathrm{fun}(D, D)$ was considered by Hulanicki (see [15]) who proved that its group algebra is non-symmetric. The result of Theorem 3.4 can be extended to this group too.

4. GLOBAL LAW OF LARGE NUMBERS ON SOLVABLE LIE GROUPS AND POISSON BOUNDARY FOR POLYCYCLIC GROUPS

In this Section we first consider a global version of the law of large numbers for solvable Lie groups, and then apply it to obtain a description of the Poisson boundary for random walks on polycyclic groups.

4.1. Global law of large numbers on solvable Lie groups

Let $R = A \leftthreetimes N$ be a semi-direct product of an abelian simply connected Lie group $A = \mathbb{R}^d$ by a simply connected nilpotent Lie group N. Denote by $\mathfrak{N}$ the Lie algebra of the group N, and by $\mathfrak{N}^{\mathbb{C}}$ its complexification. Let $\Lambda \subset \mathrm{Hom}(A, \mathbb{C})$ be the set of *weights of the adjoint representation* of the group A in $\mathfrak{N}^{\mathbb{C}}$, and $\mathfrak{N}^{\mathbb{C}}_\lambda \subset \mathfrak{N}^{\mathbb{C}}$ – the *weight subspace* corresponding to a weight $\lambda \in \Lambda$. For a vector $\alpha \in A$ let

$$\Lambda_-(\alpha) = \{\lambda \in \Lambda : \langle \Re\lambda, \alpha\rangle < 0\}\,,$$
$$\Lambda_+(\alpha) = \Lambda \setminus \Lambda_-(\alpha)\,,$$

where $\Re\lambda$ is the real part of the weight λ, and put

$$\mathfrak{N}^{\mathbb{C}}_-(\alpha) = \sum_{\lambda\in\Lambda_-(\alpha)} \mathfrak{N}^{\mathbb{C}}_\lambda\,,$$
$$\mathfrak{N}^{\mathbb{C}}_+(\alpha) = \sum_{\lambda\in\Lambda_+(\alpha)} \mathfrak{N}^{\mathbb{C}}_\lambda\,.$$

Lemma 4.1([22], Section 3)**.** *For an arbitrary $\alpha \in A$ the subspaces $\mathfrak{N}^{\mathbb{C}}_-(\alpha)\,, \mathfrak{N}^{\mathbb{C}}_+(\alpha) \subset \mathfrak{N}^{\mathbb{C}}$ are complexifications of subalgebras of the Lie algebra $\mathfrak{N}$. If $N_-(\alpha)$ and $N_+(\alpha)$ are the corresponding subgroups of the simply connected nilpotent Lie group N, then every element $n \in N$ can be uniquely decomposed as*

$$n = n_-n_+$$

with $n_- \in N_-\,, n_+ \in N_+$.

If the decomposition $N = N_-N_+$ corresponding to a certain $\alpha \in A$ is fixed, the components of a group element $g = (a, n_-n_+) \in R$ in the groups A, N, N_-, N_+ will be denoted as

$$a(g) = a \in A\,,$$
$$n(g) = n_-n_+ \in N\,,$$
$$n_-(g) = n_- \in N_-\,,$$
$$n_+(g) = n_+ \in N_+\,,$$

respectively. Denote by δ_R and δ_N principal gauges in the groups R and N.

Lemma 4.2. *Let $\{(a_k, n_k)\}_{k=1}^\infty$ be a sequence of elements of the group $R = A \leftthreetimes N$ such that*

(i) $\log \delta_N(n_k) = o(k)$*;*
(ii) $\lim_{k\to\infty}(a_1 + \cdots + a_k)/k = \alpha$ *exists;*

and $N = N_-(\alpha)N_+(\alpha)$ *be the corresponding decomposition of* N. *Then for the sequence of products*

$$y_k = (a_1, n_1) \cdots (a_k, n_k)$$

there exists the limit

$$n_- = \lim_{k\to\infty} n_-(y_k)\,,$$

and

$$\log \delta_N\left(n\left(g^{-k} y_k\right)\right) = o(k)\,,$$

for

$$g = (0, n_-)\,(\alpha, 0)\,(0, n_-)^{-1}\,.$$

Proof. For the sake of simplicity consider first the case when the simply connected nilpotent group N is abelian, i.e. N is the additive group of a finite dimensional real vector space. In this situation

$$N_-(\alpha) = \{0\} \cup \{a \in N \setminus \{0\} : \lim_{k\to\infty} \frac{1}{k} \log \|T^{k\alpha} a\| < 0\}\,,$$

and

$$N_+(\alpha) = \{0\} \cup \{a \in N \setminus \{0\} : \lim_{k\to\infty} \frac{1}{k} \log \|T^{-k\alpha} a\| \le 0\}\,,$$

i.e. $N_-(\alpha)$ coincides with the contracting subspace of the operator T^α in N (here T is the action of A in N, and $\|\cdot\|$ is a norm in N). Since the space N is finite dimensional, there exists a number $\varepsilon < 0$ such that

$$N_-(\alpha) = \{0\} \cup \{a \in N \setminus \{0\} : \lim_{k\to\infty} \frac{1}{k} \log \|T^{k\alpha} a\| < \varepsilon < 0\}\,.$$

Denote by P_- and P_+ the projectors onto subspaces N_- and N_+, respectively, arising form the decomposition $n = n_- + n_+$. For the principal gauge δ_N on N one can take the norm $\|\cdot\|$ on N.

Let $\alpha_k = a_1 + a_2 + \cdots + a_k = a(y_k)$ and $\alpha_0 = 0$. Projection of the vector

$$n(y_k) = \sum_{i=1}^{k} T^{\alpha_{i-1}}\, n_i$$

onto N_- gives

$$\begin{aligned} n_-(y_k) = P_-\, n(y_k) = P_- \sum_{i=1}^{k} T^{\alpha_{i-1}}\, n_i \\ = \sum_{i=1}^{k} P_-\, T^{\alpha_{i-1}}\, n_i \\ = \sum_{i=1}^{k} P_-\, T^{\alpha_{i-1}-(i-1)\alpha}\, T^{(i-1)\alpha}\, n_i\,. \end{aligned}$$

Since $\log\left(1 + \|n_k\|\right) = o(k)$, and $\alpha_k/k \to \alpha$, this immediately implies existence of the limit

$$n_- = \sum_{i=1}^{\infty} P_-\, T^{\alpha_{i-1}}\, n_i\,.$$

If

$$g = (0, n_-)\,(\alpha, 0)\,(0, n_-)^{-1} = (\alpha, n_- - T^\alpha n_-)\,,$$

then

$$g^{-k} = (-k\alpha, n_- - T^{-k\alpha} n_-)\,,$$

and

$$\begin{aligned} n(g^{-k} y_k) &= n_- - T^{-k\alpha} n_- + T^{-k\alpha} n(y_k) \\ &= n_- + T^{-k\alpha}(n(y_k) - n_-)\,. \end{aligned}$$

Show that

$$\log(1 + \|T^{-k\alpha}(n(y_k) - n_-)\|) = o(k)\,.$$

Indeed,

$$\begin{aligned} T^{-k\alpha}\big(n(y_k) - n_-\big) &= T^{-k\alpha}\left(\sum_{i=1}^{k} T^{\alpha_{i-1}}\, n_i - \sum_{i=1}^{\infty} P_-\, T^{\alpha_{i-1}}\, n_i\right) \\ &= T^{-k\alpha}\left(\sum_{i=1}^{k} T^{\alpha_{i-1}}\, n_i - \sum_{i=1}^{k} P_-\, T^{\alpha_{i-1}}\, n_i - \sum_{i=k+1}^{\infty} P_-\, T^{\alpha_{i-1}}\, n_i\right) \\ &= T^{-k\alpha}\left(\sum_{i=1}^{k} P_+\, T^{\alpha_{i-1}}\, n_i - \sum_{i=k+1}^{\infty} P_-\, T^{\alpha_{i-1}}\, n_i\right)\,. \end{aligned}$$

The operators T^α and P_+ (as well as T^α and P_-) commute, hence

$$T^{-k\alpha} \sum_{i=1}^{k} P_+\, T^{\alpha_{i-1}}\, n_i = \sum_{i=1}^{k} T^{(-k+i-1)\alpha}\, P_+\, T^{\alpha_{i-1}-(i-1)\alpha}\, n_i\,,$$

and

$$\log\big(1 + \|T^{-k\alpha} \sum_{i=1}^{k} P_+\, T^{\alpha_{i-1}}\, n_i\|\big) = o(k)\,.$$

On the other hand,

$$T^{-k\alpha} \sum_{i=k+1}^{\infty} P_-\, T^{\alpha_{i-1}}\, n_i = \sum_{i=k=1}^{\infty} T^{(-k+i-1)\alpha}\, P_-\, T^{\alpha_{i-1}-(i-1)\alpha}\, n_i\,,$$

and

$$\log\big(1 + \|T^{-k\alpha} \sum_{i=k+1}^{\infty} P_-\, T^{\alpha_{i-1}}\, n_i\|\big) = o(k)$$

(cf. above the proof of the existence of the limit n_-).

The general case can be treated along the same lines with some techical sophistications caused by the non-commutativity of N as in [22],Théorème 9.2. □

Definition ([13])**.** Action T of an abelian connected Lie group A on a connected nilpotent Lie group N is called *dilating* if the Lie subalgebra in $\mathfrak{N}^{\mathbb{C}}$ generated by the weight spaces $\{\mathfrak{N}^{\mathbb{C}}_{\lambda} : \Re\lambda \neq 0\}$ coincides with $\mathfrak{N}^{\mathbb{C}}$.

Lemma 4.3 ([13])**.** *Let T be a dilating action of an abelian simply connected Lie group A on a simply connected nilpotent Lie group N, δ_N – a principal gauge on N, and δ – a principal gauge on the semi-direct product $A \leftthreetimes N$ determined by the action T. Then $\log(1+\delta_N)$ is equivalent to the restriction of δ on N.*

Combination of Lemmas 4.2 and 4.3 gives now the following result.

Theorem 4.1. *Let T be a dilating action of an abelian simply connected Lie group A on a simply connected nilpotent Lie group N, δ_N – a principal gauge on N, and δ – a principal gauge on the semi-direct product $R = A \leftthreetimes N$ determined by the action T. Let $\{(a_k, n_k)\}_{k=1}^{\infty}$ be a sequence of elements of the group R such that*

(i) $\log \delta_N(n_k) = o(k)$;

(ii) $\lim_{k\to\infty}(a_1 + \cdots + a_k)/k = \alpha$ *exists.*

Then there exists an element $g \in R$ such that

$$d\,(g^k, (a_1, n_1)\cdots(a_k, n_k)) = o(k)\,,$$

where $d\,(g_1, g_2) = \delta\,(g_1^{-1}g_2)$.

Theorem 4.2. *Let $R = A \leftthreetimes N$ be the semidirect product of an abelian connected Lie group A and a connected nilpotent Lie group N corresponding to a dilating action of A on N. Then for almost all realizations $\{x_k\}$ of a stationary sequence of R-valued random variables with a finite first moment (with respect to a principal gauge in R), there exists a group element $g = g(\{x_k\})$ such that*

$$d(g^k, x_1 \dots x_k) = o(k)\,.$$

Proof. Passing, if necessary, to the universal covering group, we may assume that the groups A and N are both simply connected (if a measure μ has a finite first moment on a group G, then there exists a lift $\widetilde{\mu}$ to a covering group $\widetilde{G}$ also with a finite first moment). Now, since the increments x_k have a finite first moment with respect to a principal gauge δ on R, the condition (i) of Theorem 4.1 is satisfied. Being an image of the sequence $\{x_k = (a_k, n_k)\}$ under the homomorphism $R = A \leftthreetimes N \to A$, the sequence $\{a_k\}$ is a stationary sequence with a finite first moment in $A \cong \mathbb{R}^d$, hence the condition (ii) of Theorem 4.1 is also satisfied. □

Remark. Usually, by the law of large numbers for a non-commutative group G one means a statement on the convergence of some numerical functionals of the random product $y_k = x_1 \cdots x_k$ (e.g., see [11], [13], [14]). Here we present a "global" formulation suitable for an arbitrary connected locally compact group. Unlike the classical law of large numbers, in the general case the "mean" need not to be unique and may depend on the realization of the sequence of increments $\{x_k\}$. The non-uniqueness is due to the fact that, in general, the *classes of asymptotic equivalence relation*

$$g_1 \sim g_2 \iff \delta(g_1^k, g_2^k) = o(k)$$

in a Lie group contain more than one element. The dependence of the "mean" $g = g(\{x_k\})$ on the sequence of increments $\{x_k\}$ is connected with non-trivial tail behaviour for the sequence of products $y_k = x_1 \cdots x_k$ and carries the same character as in the usual ergodic theorem for non-ergodic stationary sequences: the "mean" $g = g(\{x_k\})$ is measurable with respect to the tail σ-algebra of the sequence of partial products $y_k = x_1 \cdots x_k$. In addition to the semi-direct products referred to in the Theorem 4.2 above, this "global law of large numbers" holds true for arbitrary stationary sequences in semi-simple Lie groups with finite center (in this situation it is essentially equivalent to the Oseledec multiplicative ergodic theorem, see [19]), and for connected nilpotent Lie groups. It would be interesting to find out when the tail σ-algebra of the sequence of products $y_k = x_1 \cdots x_k$ is *generated* by the means $g = g(\{x_k\})$. It is the case both for the semi-direct products and semi-simple groups when the increments $\{x_k\}$ are independent and their distribution is either spread out, or concentrated on a discrete subgroup (see [17], [22]) and Theorem 4.3 below).

4.2. Poisson boundary for random walks on polycyclic groups

Definition. A discrete group G is called *polycyclic* if it admits a normal series with cyclic factors, i.e. a series of subgroups

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$$

such that each G_i is a normal subgroup in G_{i+1} and all the factor-groups G_{i+1}/G_i are cyclic.

In a certain sence, polycyclic groups can be called "finite dimensional" solvable groups. Here is some evidence for this statement (see [21], [24]):

(1) Polycyclic groups can be characterized as solvable groups with finitely generated subgroups, or, even more, as solvable groups with finitely generated abelian subgroups;
(2) Solvable groups of integer matrices are polycyclic, and, conversely, every polycyclic group has a faithful representation in $GL(n, \mathbb{Z})$.

Lemma 4.4 (*semi-simple splitting*, see [24], Theorem 7.2 and [1]). *Every polycyclic group G contains a normal polycyclic subgroup $G^0 \subset G$ of finite index such that there exists a polycyclic group S containing G^0 with the following properties*

(1) *$S = A \leftthreetimes N$, where A is finitely generated free abelian, and N is finitely generated torsion free nilpotent;*
(2) *The action of the group $A \cong \mathbb{Z}^d$ on N can be extended to an action of A on a simply connected nilpotent Lie group $\widetilde{N}$ containing N as a discrete subgroup (the Lie hull of the group N) by semi-simple automorphisms.*

Theorem 4.3. *Let G be a polycyclic group with a semi-simple splitting*

$$G \supset G^0 \subset S = A \leftthreetimes N$$

as in Lemma 4.4, and μ – a probability measure on G with a finite first moment. Let μ^0 be the probability measure on S constructed in Lemma 2.2, $\alpha \in \mathbb{R}^d$ – the mean of the *projection of the measure* μ^0 *onto* $A \cong \mathbb{Z}^d$*, and* $\widetilde{N} = \widetilde{N}_-(\alpha)\,\widetilde{N}_+(\alpha)$ *– the corresponding decomposition of the Lie group* $\widetilde{N}$*. For* $g = (a, n) \in G^0$ *let* $n_-(g) \in N_-(\alpha)$ *be the corresponding term in the decomposition* $N \ni n(g) = n_-(g)\,n_+(g)$*. Denote by* $\tau_k = \tau_k(y)$ *the times when a path* $y = \{y_n\}$ *of the random walk* (G, μ) *visits the subgroup* G^0*. Then for almost all paths* $\{y_n\}$ *of the random walk* (G, μ) *there exists the limit*

$$n_-(y) = \lim_{k \to \infty} n_-(y_{\tau_k}) \in \widetilde{N}_-(\alpha)\,,$$

and the space $\widetilde{N}_-(a)$ *with the resulting limit distribution is isomorphic to the Poisson boundary of the random walk* (G, μ)*.*

Proof. Lemma 2.2 implies that the Poisson boundaries of random walks (G, μ) and (G^0, μ^0) coincide, and μ^0 has a finite first moment in G^0 by Lemma 2.3. Using semi-simplicity, extend the action of $A \cong \mathbb{Z}^d$ to an action of $\mathbb{R}^d$ in the complexification $\mathfrak{N}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{N}$ of the Lie hull $\widetilde{N}$ (not necessarily by Lie algebra automorphisms). Theorem 4.2 holds true also in this slightly more general situation. Now Theorem 1.3 implies the desired result. ☐

If the measure μ is symmetric, then the measure μ^0 is also symmetric, hence the mean α is zero and the subgroup $\widetilde{N}_-$ is trivial. Thus we have

Corollary. *If* μ *is a symmetric measure with a finite first moment on a polycyclic group* G*, then the Poisson boundary* $\Gamma(G, \mu)$ *is trivial.*

Remarks.

1. Another way of obtaining a description of the Poisson boundary of random walks on a polycyclic group G consists in embedding G into the matrix group $GL(n, \mathbb{Z})$ and using the descriprion of the Poisson boundary for random walks on discrete subgroups of semi-simple Lie groups [17]. In this approach the boundary $\widetilde{N}_-(\alpha)$ can be naturally identified with a certain flag space in $\mathbb{R}^n$.
2. Triviality of the Poisson boundary for symmetric measures with a finite first moment on polycyclic groups can be also proved in a more direct way, by proving that the entropy of the random walk $\{y_{\tau_k}\}$ on G^0 determined by the measure μ^0 is zero. It follows from the fact that a.e. $|a(y_{\tau_k})| = o(k)$ and $\log|n(y_{\tau_k})| = o(k)$ (cf. Section 4.1).
3. Obtained results show (as one could expect), that the boundary theory for polycyclic groups is parallel to that for solvable Lie groups. The description of the Poisson boundary for polycyclic groups obtained in Theorem 4.3. is essentially the same as for solvable Lie groups (cf. [22]). Remark also that the Poisson boundary is always trivial for compactly supported symmetric measures absolutely continious with respect to the Haar measure on all amenable connected Lie groups (in particular, on solvable Lie groups) [13].

Since polycyclic groups can be characterized as solvable groups with finitely generated subgroups, we obtain the following result:

Theorem 4.4. *If the Poisson boundary* (G, μ) *is non-trivial for a certain symmetric probability measure* μ *with a finite first moment on a finitely generated solvable group* G*, then* G *contains an infinitely generated subgroup.*

5. POISSON BOUNDARY FOR RANDOM WALKS ON AFFINE GROUP OF DIADIC-RATIONAL LINE

Consider the group of matrices

$$(x,f)=\begin{pmatrix}2^x & f\\ 0 & 1\end{pmatrix}, \qquad f=\frac{m}{2^n} \qquad (x,m,n\in\mathbb{Z}),$$

i.e. the *affine group* $\mathrm{Aff}(\mathbb{Z}[\frac12])$ *of the diadic-rational line* $\mathbb{Z}[\frac12]$. The group $\mathrm{Aff}(\mathbb{Z}[\frac12])$ is isomorphic to the semi-direct product of the multiplicative group

$$\mathbb{Z}\cong\left\{\begin{pmatrix}2^x & 0\\ 0 & 1\end{pmatrix}\right\}$$

of the ring $\mathbb{Z}[\frac12]$ by its additive group, and it acts on $\mathbb{Z}[\frac12]$ by transformations

$$(x,f)\,t=2^x t+f\,.$$

The group operation in $\mathrm{Aff}(\mathbb{Z}[\frac12])$ can be written in the coordinates $(x,f)\,, x\in\mathbb{Z}, f\in\mathbb{Z}[\frac12]$ as

$$(x_1,f_1)(x_2,f_2)=(x_1+x_2,f_1+2^{x_1}f_2)\,. \tag{1}$$

If (x_i,f_i) is a sequence of elements from $\mathrm{Aff}(\mathbb{Z}[\frac12])$, then their product is

$$(x_1,f_1)\,(x_2,f_2)\cdot\ldots\cdot(x_n,f_n)=(x_1+x_2+\cdots+x_n,f_1+2^{y_1}f_2+\cdots+2^{y_{n-1}}f_n)\,, \tag{2}$$

as it follows from the formula (1).

Denote by $e=(0,0)$ the identity of $\mathrm{Aff}(\mathbb{Z}[\frac12])$. The group $\mathrm{Aff}(\mathbb{Z}[\frac12])$ is solvable with degree 2, has exponential growth and can be presented using the generators

$$a=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}=(1,0)\,, \qquad b=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=(0,1)$$

and the relation

$$b^2a=ab\,.$$

The group $\mathrm{Aff}(\mathbb{Z}[\frac12])$ is the homomorphic image of the wreath product $\widetilde{G}=\mathbb{Z}\leftthreetimes\mathrm{fun}(\mathbb{Z},\mathbb{Z})$ (see above Section 3) under the homomorphism

$$\pi:(x,f)\mapsto\big(x,\sum_k 2^k f(k)\big)\,.$$

Hence for a probability measure μ on G its Poisson boundary could be described as the space of ergodic components of the Poisson boundary $\Gamma(\widetilde{G},\widetilde{\mu})$ with respect to the action of

$$\ker\pi=\{(x,f)\in G_1: x=0\,,\sum_k 2^k f(k)=0\}\,,$$

where $\widetilde{\mu}$ is a lift of the measure μ from G to $\widetilde{G}$ (see [20]). Nonetheless it is easier to describe the Poisson boundary in a more direct way (cf. the discussion in Section 3.2).

For a number $f \in \mathbb{Z}[\frac{1}{2}] \setminus \{0\}$ consider the (uniquely determined) binary decomposition of its absolute value

$$|f| = \sum \varepsilon_i \, 2^i \,, \qquad \varepsilon_i = 0, 1 \,.$$

Let

$$d_-(f) = \min\{i : \varepsilon_i = 1\}$$

and

$$d_+(f) = \max\{i : \varepsilon_i = 1\} \,.$$

One can easily see that

$$d_+(f) \le \log |f| < d_+(f) + 1$$

and

$$d_-(f) = -\log |f|_2 \,,$$

where $|f|$ and $|f|_2$ are the ordinary absolute value of f and its 2-adic absolute value, respectively (all logarithms in this Section are taken with base 2).

Put for $f \in \mathbb{Z}[\frac{1}{2}]$

$$\|f\| = \begin{cases} 1 + \max\{|d_-(f)|, |d_+(f)|\} \,, & f \neq 0 \\ 0 \,, & f = 0 \,. \end{cases}$$

Denote by $|x|$ the ordinary absolute value of a number $x \in \mathbb{Z}$, and by $|(x,f)|$ the length of an element $(x,f) \in G \,, x \in \mathbb{Z} \,, f \in \mathbb{Z}[\frac{1}{2}]$ with respect to the generating set $\{a, a^{-1}, b, b^{-1}\}$.

Lemma 5.1. *There exist positive constants C_1, C_2 such that*

$$C_1(|x| + \|f\|) \le |(x,f)| \le C_2(|x| + \|f\|)$$

for all $(x,f) \in \mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$.

Proof. Let $(x,f) \in \mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$. If $f = 0$, then clearly $|(x,0)| = |x|$, so that we have to consider only the case when $f \neq 0$. To simplify the exposition suppose that f is positive (for negative f one has to change signs in formulas below). Then f can be desomposed as

$$f = \sum \varepsilon(i) \, 2^i \,, \qquad \varepsilon_i = 0, 1 \,.$$

It is clear from the formula (2) that the element (x,f) can be presented as the product

$$(x,f) = a^{d_-} \, g_0 \, a \, g_1 \, \cdot \ldots \cdot a \, g_k \, a^{x - d_+} \,,$$

where $d_- = d_-(f)$ and $d_+ = d_+(f)$ are defined as above, $k = d_+ - d_-$, and

$$g_i = \begin{cases} b \,, & \varepsilon(d_- + i) = 1 \,, \\ e \,, & \varepsilon(d_- + i) = 0 \,. \end{cases}$$

Thus

$$\begin{aligned} |(x,f)| &\le |d_-| + 2(d_+ - d_- + 1) + |x - d_+| \\ &\le 2 + |x| + 3|d_-| + 3|d_+| \\ &\le |x| + 6 \, \|f\| \\ &\le 6 \, (|x| + \|f\|) \,. \end{aligned}$$

Conversely, since the image of the generator $a = (1,0)$ of $\mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$ under the homomorphism $(x,f) \mapsto x$ from $\mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$ to $\mathbb{Z}$ is 1, we have

$$|x| \le |(x,f)| .$$

It easily follows from the formula (2) that

$$|d_-(f)| \le |(x,f)|$$

and

$$|f| \le 2^{|(x,f)|} ,$$

hence

$$|d_+(f)| \le \log|f| \le |(x,f)| .$$

Finally,

$$|x| + \|f\| \le 1 + 2|(x,f)| \le 3|(x,f)|$$

(here $|(x,f)| \ge 1$, since $f \ne 0$). □

Theorem 5.1. *Let μ be a probability measure on the group $G = \mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$ with a finite first moment and such that the group $\mathrm{gr}\,\mu$ generated by $\mathrm{supp}\,\mu$ is non-abelian. Denote by $\mu_{\mathbb{Z}}$ the image of the measure μ on the group $\mathbb{Z}$ under the homomorphism $(x,f) \mapsto x$ from $\mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$ to $\mathbb{Z}$, and by $\alpha = \sum x\,\mu_{\mathbb{Z}}(x)$ the mean of the mesaure $\mu_{\mathbb{Z}}$.*

(i) *If $\alpha < 0$, then the Poisson boundary $\Gamma(G,\mu)$ is non-trivial. For almost all paths (y_n, φ_n) of the random walk (G,μ) there exists the limit*

$$\lim_{n\to\infty} \varphi_n = f_\infty \in \mathbb{R},$$

and the Poisson boundary $\Gamma(G,\mu)$ is isomorphic to the real line $\mathbb{R}$ with the arising harmonic measure (i.e. the distribution of φ_∞) on it;

(ii) *If $\alpha = 0$, then the Poisson boundary $\Gamma(G,\mu)$ is trivial;*

(iii) *If $\alpha > 0$, then the Poisson boundary $\Gamma(G,\mu)$ is non-trivial. For almost all paths (y_n, φ_n) of the random walk (G,μ) there exists the limit*

$$\lim_{n\to\infty} \varphi_n = f_\infty \in \mathbb{Q}_2 ,$$

in the 2-adic topology, and the Poisson boundary $\Gamma(G,\mu)$ is isomorphic to the 2-adic line $\mathbb{Q}_2$ with the arising harmonic measure on it.

Proof. Let

$$\{(y_n,\varphi_n)\} = (x_1,f_1)(x_2,f_2)\cdots(x_n,f_n) = (x_1+x_2+\dots x_n, f_1+2^{y_1}f_2+\dots+2^{y_{n-1}}f_n)$$

be a path of the random walk (G,μ) (here (x_i,f_i) are independent μ-distributed random variables – increments of the random walk (G,μ)).

First remark that if $\alpha = 0$, then the random walk $\{y_n\}$ on $\mathbb{Z}$ is recurrent, hence the Poisson boundary $\Gamma(G,\mu)$ is trivial (cf. Theorem 3.1).

For $\alpha \neq 0$ we have to show first that the series

$$\sum_{n\geq 0} 2^{y_n} f_{n+1} \tag{3}$$

converges in the ordinary (for $\alpha < 0$) or in the 2-adic (for $\alpha > 0$) topology. From the finiteness of the first moment of μ and from Lemma 5.1 follows that a.e.

$$\log(1 + |f_n|) = o(n)$$

and

$$\log(1 + |f_n|_2) = o(n)\,.$$

Since $y_n/n \to \alpha$ almost everythere , the series (3) converges in the ordinary ($\alpha < 0$) or in the 2-adic ($\alpha > 0$) topology. Non-triviality of the Poisson boundary, i.e. the fact that the sum of the series (3) is not the same for almost all paths can be proved by repeating the argument used in Theorem 3.3 in a similar situation.

Now we have to show that the hitting points φ_∞ on the real line ($\alpha < 0$) or on the 2-adic line ($\alpha > 0$) generate the whole stationary σ-algebra of the random walk (G, μ). In order to do it we shall use Theorem 1.3, and show that the paths $\{(y_n, \varphi_n)\}$ admit a good approximation in terms of the limit point φ_∞ only. Consider first the case when $\alpha < 0$. For the sake of simplicity suppose that $\alpha = -1$. For a point $\psi \in \mathbb{R}$ define its *truncations*

$$[\psi]_n = 2^{-n}[2^n\psi]\,,$$

where $[x]$ is the integer part of a number $x \in \mathbb{R}$. In other words, if

$$\psi = \sum \varepsilon_k 2^k\,,$$

then

$$[\psi]_n = \sum_{k\geq -n} \varepsilon_k 2^k\,.$$

Now consider

$$\Delta_n = (-n, [\varphi_\infty]_n)^{-1}(y_n, \varphi_n) = (y_n + n, 2^n\,(\varphi_n - [f_\infty]_n))$$

and show that $|\Delta_n| = o(n)$ for a.e. path $\{(y_n, \varphi_n)\}$. [For an arbitrary $\alpha < 0$ one should take $(-[n\alpha], [\varphi_\infty]_{[n\alpha]})$ instead of $(-n, [\varphi_\infty]_n)$ in the definition of Δ_n.]

Since $y_n/n \to -1$, we have to check only that

$$\|2^n(\varphi_n - [f_\infty]_n)\| = o(n)$$

(cf. Lemma 5.1). Indeed,

$$\begin{aligned} 2^n\,(\varphi_n - [f_\infty]_n) &= 2^n\,\varphi_n - [2^n\,\varphi_\infty] \\ &= \left(\sum_{k\leq n} 2^{n+y_{k-1}}\,f_k - \left[\sum_k 2^{n+y_{k-1}}\,f_k\right]\right)\,, \end{aligned}$$

whence

$$d_-\big(2^n\,(\varphi_n - [f_\infty]_n)\big) \geq \min\big(0, d_-(\sum_{k\leq n} 2^{n+y_{k-1}}\, f_k)\big)$$
$$\geq \min_{k\leq n}\big(0, d_-(2^{n+y_{k-1}}\, f_k)\big)$$
$$= \min_{k\leq n}\big(0, n + y_{k-1} + d_-(f_k)\big)\,.$$

Since $y_n/n \to -1$, and $d_-(f_k) = o(k)$, we get that

$$d_-\big(2^n(\varphi_n - [f_\infty]_n)\big) \geq o(n)\,.$$

On the other hand,

$$|2^n(\varphi_n - [f_\infty]_n)| = \left|\sum_{k\leq n} 2^{n+y_{k-1}}\, f_k - \left[\sum_k 2^{n+y_{k-1}}\, f_k\right]\right|$$
$$\leq 1 + \left|\sum_{k\leq n} 2^{n+y_{k-1}}\, f_k - \left[\sum_{k\leq n} 2^{n+y_{k-1}}\, f_k\right] - \left[\sum_{k>n} 2^{n+y_{k-1}}\, f_k\right]\right|$$
$$\leq 2 + \left|\left[\sum_{k>n} 2^{n+y_{k-1}}\, f_k\right]\right|$$
$$\leq 3 + \left|\sum_{k>n} 2^{n+y_{k-1}}\, f_k\right|$$
$$\leq 3 + \sum_{k>n} 2^{n+y_{k-1}}\, |f_k|\,.$$

Since $y_n/n \to -1$, and

$$\log(1 + |f_k|) = o(k)\,,$$

it implies that

$$|2^n(\varphi_n - [f_\infty]_n)| \leq 2^{o(n)}\,.$$

Hence

$$d_+(2^n(\varphi_n - [f_\infty]_n)) \leq o(n)\,,$$

and finally

$$\|2^n(\varphi_n - [f_\infty]_n)\| = o(n)\,.$$

Now Theorem 1.3 implies that the hitting points $\varphi_\infty \in \mathbb{R}$ generate the whole Poisson boundary.

Consider now the case when $\alpha > 0$. The proof here goes along the same lines as in the case $\alpha < 0$ with some obvious modifications. Once again for the sake of simplicity assume that $\alpha = 1$. For a point $\psi \in \mathbb{Q}_2$ define its truncations

$$[\psi]_n = 2^n\{2^{-n}\psi\}\,,$$

where $\{x\}$ is the fractional part of a number $x \in \mathbb{Q}_2$. In other words, if

$$\psi = \sum \varepsilon_k\, 2^k\,,$$

then

$$[\psi]_n = \sum_{k<n} \varepsilon_k \, 2^k \, .$$

Now consider

$$\Delta_n = (n, [\varphi_\infty]_n)^{-1}(y_n, \varphi_n) = (y_n - n, 2^{-n}(\varphi_n - [\varphi_\infty]_n)) \, ,$$

and show that $|\Delta_n| = o(n)$ for a.e. path $\{(y_n, \varphi_n)\}$.

Once again we have to check only that

$$\|2^{-n}(\varphi_n - [\varphi_\infty]_n)\| = o(n) \, .$$

Indeed,

$$\begin{aligned} 2^{-n}(\varphi_n - [\varphi_\infty]_n) =& 2^{-n}\varphi_n - \{2^{-n}\varphi_\infty\} \\ =& \sum_{k\le n} 2^{-n+y_{k-1}} f_k - \left\{ \sum_k 2^{-n+y_{k-1}} f_k \right\} \, , \end{aligned}$$

whence

$$|2^{-n}(\varphi_n - [\varphi_\infty]_n)| \le 1 + \sum_{k\le n} 2^{-n+y_{k-1}} |f_k| \, .$$

Since $y_n/n \to 1$ and $|(f_k)| \le 2^{o(k)}$, we get that

$$|2^{-n}(\varphi_n - [\varphi_\infty]_n)| \le 2^{o(n)} \, ,$$

and

$$d_+(2^{-n}(\varphi_n - [\varphi_\infty]_n)) \le o(n) \, .$$

On the other hand,

$$\begin{aligned} d_-(2^{-n}(\varphi_n - [\varphi_\infty]_n)) &= d_- \left(\sum_{k\le n} 2^{-n+y_{k-1}} f_k - \left\{ \sum_k 2^{-n+y_{k-1}} f_k \right\} \right) \\ &\ge \min \left(0, d_- \left(\sum_{k>n} 2^{-n+y_{k-1}} f_k \right) \right) \\ &\ge \min_{k>n} (0, d_-(2^{-n+y_{k-1}} f_k)) \\ &= \min_{k>n} (0, -n + y_{k-1} + d_-(f_k)) \, . \end{aligned}$$

Since $y_n/n \to -1$ and $d_-(f_k) = o(k)$, we have again that

$$d_-(2^{-n}(\varphi_n - [\varphi_\infty]_n)) \ge o(n)$$

and

$$\|2^n(\varphi_n - [\varphi_\infty]_n)\| = o(n) \, .$$

□

Remarks.

1. Obviously, Theorem 5.1 can be reformulated for the affine group $\mathrm{Aff}(\mathbb{Z}[\frac{1}{p}])$ of the ring $\mathbb{Z}[\frac{1}{p}]$ corresponding to an arbitrary base p.

2. The description of the Poisson boundary for the group $\mathrm{Aff}(\mathbb{Z}[\frac{1}{2}])$ obtained in the case $\alpha < 0$ coincides with the description of the Poisson boundary for random walks on the Lie group $\mathrm{Aff}(\mathbb{R})$ [9]. Non-triviality of the Poisson boundary for $\alpha > 0$ is a phenomenon specific for discrete affine groups. It would be interesting to investigate the Poisson boundary for higher-dimensional solvable groups over diadics, for example, for the group of triangular matrices. Probably, for these groups the Poisson boundary will be mixed – consisting of both real and 2-adic components. This problem is also closely related with finding out a description of the Poisson boundary for random walks on Lie groups over p-adic fields.

6. EXCHANGEABLE σ-ALGEBRA OF RANDOM WALKS ON $\mathbb{Z}$

In this Section we apply the techique developed in this paper to obtain a description of the exchangeable σ-algebra of random walks on $\mathbb{Z}$ with a finite first moment.

Let $\{y_n\}_{n=0}^{\infty}$ be a homogeneous Markov chain on a countable state space X with transition probabilities $p(x,y)$, $x, y \in X$. As usually, by $\mathbf{P}_x$ denote the probability measure in the path space $X^\infty = \{y = \{y_n\}_{n=0}^{\infty}\}$ corresponding to the initial distribution δ_x concentrated on a point $x \in X$, and by $\mathbf{P}_\theta = \sum \theta(x)\mathbf{P}_x$ the measure in the path space corresponding to an arbitrary initial distribution θ on X.

The group $S(\infty)$ of *finite permutations* of the parameter set $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ of the chain $\{y_n\}$ naturally acts on the path space X^∞. Denote by $\mathfrak{S}$ the *exchangeable* (or: *symmetric*) σ-algebra of the chain $\{y_n\}$ – the (completed) σ-algebra of subsets of the path space invariant with respect to the action of the group $S(\infty)$ on X^∞.

Introduce the *extended chain*

$$\{(y_n, \sum_{k=0}^{n-1} \delta_{y_k})\}$$

on the state space $X \times \mathrm{fun}(X, \mathbb{Z})$, where $\mathrm{fun}(X, \mathbb{Z})$ is the additive group of finitely supported $\mathbb{Z}$ valued configurations on X. In other words, we add to the states y_n of the original chain the *occupation functions*

$$\varphi_n = \sum_{k=0}^{n-1} \delta_{y_k}$$

saying how many times each of the points of the state space X was visited by the path $\{y_n\}$ up to the time n. The transition probabilities of the extended chain have the form

$$\widetilde{p}\left((x, f), (y, f + \delta_x)\right) = p\,(x, y)\,.$$

Remark that for a fixed initial distribution θ on X one can naturally identify the path spaces of the original chain on X and of the extended chain on $X \times \mathrm{fun}(X, \mathbb{Z})$.

In a particular case, when X is a group and $\{y_n\}$ is a random walk determined by a probability measure μ on X, the extended chain is the random walk on the wreath product $X \leftthreetimes \mathrm{fun}(X,\mathbb{Z})$ corresponding to the measure

$$\widetilde{\mu}(x,\delta_e) = \mu(x)\,, \qquad x \in X \tag{1}$$

(here e is the identity of X, and δ_x is the unit mass at x).

Lemma 6.1. *For an arbitrary initial distribution θ on X the tail σ-algebra $\widetilde{\mathcal{A}}^\infty$ of the extended chain and the exchangeable σ-algebra $\mathfrak{S}$ of the chain $\{y_n\}$ coincide $\mathbf{P}_\theta$-mod 0.*

Proof. Recall that the tail σ-algebra $\widetilde{\mathcal{A}}^\infty$ of the extended chain can be described as the σ-algebra generated by the *tail equivalence relation*:

$$\{(y_n,\varphi_n)\} \sim \{(y'_n,f'_n)\} \qquad \Longleftrightarrow \qquad \exists\, N \ge 0 : y_n = y'_n\,, \varphi_n = \varphi'_n \ \forall\, n \ge N\,.$$

The exchangeable σ-algebra $\mathfrak{S}$ is generated by the equivalence relation

$$\{y_n\} \approx \{y'_n\} \qquad \Longleftrightarrow \qquad \exists\, g \in S(\infty) : y_n = y'_{g(n)} \ \forall\, n \ge 0\,.$$

Since the occupation functions φ_n for the extended chain has the form $\varphi_n = \sum_{k=0}^{n-1} \delta_{y_k}$, and $S(\infty)$ is the group of *finite* permutations, we immediately get that the equivalence relations $\sim$ and $\approx$ on the path space X^∞ coincide. □

Theorem 6.1 (cf. [3])**.** *If $x \in X$ is a recurrent state for the chain $\{y_n\}$, then the exchangeable σ-algebra $\mathfrak{S}$ of the chain $\{y_n\}$ is trivial $\mathbf{P}_x$-mod 0.*

Proof. Tail and stationary σ-algebras of the extended chain coincide $\mathbf{P}_x$-mod 0, since the sets attainable from the point $(x,0) \in X \times \mathrm{fun}(X,\mathbb{Z})$ in different numbers of steps are pairwise disjoint. The point x being recurrent means that the set $\{x\} \times \mathrm{fun}(X,\mathbb{Z})$ is recurrent for the extended chain. Thus the stationary σ-algebra of the extended chain coincides with the stationary σ-algebra of a certain random walk on the abelian group $\mathrm{fun}(X,\mathbb{Z})$ (cf. Lemma 2.2) , the latter being trivial by the Blackwell – Choquet – Deny theorem. □

This Theorem shows that the exchangeable σ-algebra is essentailly trivial for all recurrent chains (see [3] for its complete description in combinatorial terms for an arbitrary initial distribution). If the chain $\{y_n\}$ is transient, then it visits (almost surely) all points of the state space X only a finite number of times, hence the occupation functions φ_n converge a.e. to a *final occupation function* φ_∞ (depending on the path $\{y_n\}$), such that $\varphi_\infty(x)$ is the number of times when the point x was visited by the trajectory $\{y_n\}$. Clearly, the final occupation function φ_∞ is measurable with respect to the exchangeable σ-algebra of the chain $\{y_n\}$.

Theorem 6.2. *Let μ be a probability measure with a finite first moment on the group of integers $\mathbb{Z}$. Denote by $\mathbf{P}$ the probability measure in the path space of the random walk $(\mathbb{Z},\mu)$ corresponding to the initial distribution δ_0 on $\mathbb{Z}$. If the mean $\bar{\mu}$ of the measure μ is zero, then the exchangeable σ-algebra $\mathfrak{S}$ of the random walk $(\mathbb{Z},\mu)$ is* *trivial* **P**-mod 0. *If* $\bar{\mu} \neq 0$, *then the* σ*-algebra* $\mathfrak{S}$ *is generated by final occupation functions* φ_∞ *of the random walk* $(\mathbb{Z}, \mu)$.

Proof. We shall use coincidence of the exchangeable σ-algebra and the stationary σ-algebra of the random walk on the group $G = \mathbb{Z} \leftthreetimes \mathrm{fun}(\mathbb{Z}, \mathbb{Z})$ determined by the measure $\widetilde{\mu}$ (1). The proof goes along the same lines as in Theorem 5.1.

If the mean $\bar{\mu}$ is zero, then the random walk $(\mathbb{Z}, \mu)$ is recurrent and the exchangeable σ-algebra is trivial **P**-mod 0 by Theorem 3.1.

Consider the case when $\bar{\mu} \neq 0$ and suppose for the sake of simplicity that $\bar{\mu} = 1$. We have to show that the stationary σ-algebra of the random walk on the wreath product $G = \mathbb{Z} \leftthreetimes \mathrm{fun}(\mathbb{Z}, \mathbb{Z})$ determined by the measure $\widetilde{\mu}$ is generated by the limit configurations φ_∞ on $\mathbb{Z}$. Let $\{(y_n, \varphi_n)\}$ be a path of the random walk $(G, \widetilde{\mu})$, and $\varphi_\infty = \lim_{n\to\infty} \varphi_n$ be the corresponding limit configuration. Define for the points $x \in \mathbb{Z}$ the truncations of the configuration φ_∞:

$$[\varphi_\infty]_x(z) = \begin{cases} \varphi_\infty(z) & , z \leq x\,, \\ 0 & , z > x\,, \end{cases}$$

so that $[\varphi_\infty]_x$ shows how many times the path $\{y_n\}$ visited the points to the left from x. Since $y_n = n + o(n)$, all configurations φ_n are a.e. finitely supported.

Show that the path $\{(y_n, \varphi_n)\}$ in G can be well approximated with the sequence $\{(n, [\varphi_\infty]_n)\}$. Consider

$$\begin{aligned} \Delta_n &= (n, [\varphi_\infty]_n)^{-1} (y_n, \varphi_n) \\ &= (y_n - n, T^{-n} (\varphi_n - [\varphi_\infty]_n)) \\ &= (0, T^{-n} (\varphi_n - [\varphi_\infty]_n)) \cdot (y_n - n, 0)\,, \end{aligned}$$

(here T is the action of $\mathbb{Z}$ on $\mathrm{fun}(\mathbb{Z}, \mathbb{Z})$ by shifts). We have to prove that $|\Delta_n| = o(n)$, where $|\cdot|$ is the length function on G determined by the generators $(1,0)$, $(-1,0)$, $(0,\delta_0)$, $(0,-\delta_0)$. Evidently, we have to estimate only the first factor

$$\Delta_n^1 = (0, T^{-n} (\varphi_n - [\varphi_\infty]_n))$$

in the formula above. The configurations φ_n and $[\varphi_\infty]_n$ differ because either $y_k > n, k \leq n$, or $y_k \leq n\,, k > n$, i.e.,

$$\Delta_n^1 = (0, \psi_1 - \psi_2)$$

with

$$\begin{aligned} \psi_1 &= (T^{-n} \sum_{k\leq n} \delta_{y_k})|_{(n,\infty)} \\ \psi_2 &= (T^{-n} \sum_{k>n} \delta_{y_k})|_{(-\infty,n]} \end{aligned}$$

(here $\varphi|_S$ is the restriction of a configuration φ onto a set $S \subset \mathbb{Z}$).

Since

$$|(0,\psi_1)| = 2\max_{k\le n}\{(y_k-n)_+\} + \mathrm{card}\{k\le n : y_k > n\},$$
$$|(0,\psi_2)| = 2\max_{k> n}\{-(y_k-n)_-\} + \mathrm{card}\{k> n : y_k \le n\},$$

and $y_n/n \to 1$ almost surely, we get that $|\Delta_n| = o(n)$. Now in order to finish the proof we have to apply Theorem 1.3. □

Remarks.

1. The author is not aware of any results about exchangeable σ-algebras for transient Markov chains. It would be very interesting to find a more direct proof of Theorem 6.1, as well as to obtain a description of the exchangeanble σ-algebra for random walks on $\mathbb{Z}$ without a finite first moment, or for the random walks on $\mathbb{Z}_k$ $k>1$. Our method can be probably used in the case when the mean is non-zero, but for transient random walks with zero mean ($k\ge 3$) it seems to be very difficult to find an "approximation" of the conditional random walk in terms of the final occupation function (a random walk with zero mean on $\mathbb{Z}^3$ is transient without any "regular pattern of escaping to infinity").

2. The same argument shows that the Poisson boundary for finitely supported measures on wreath products $\mathbb{Z} \leftthreetimes \mathrm{fun}(\mathbb{Z},A)$ is either trivial or coincides with the space of final configurations $\mathrm{Fun}(\mathbb{Z},A)$ for an arbitrary passive group A. It also works for measures with a finite first moment provided the projection of their support onto $\mathrm{fun}(\mathbb{Z},A)$ is finite. But already for general measures with a finite first moment on the group $\mathbb{Z} \leftthreetimes \mathrm{fun}(\mathbb{Z},\mathbb{Z})$ estimates of the entropy of conditional walks turn out to be much more difficult. The situation becomes worse for the wreath products with the active group $\mathbb{Z}^3$ (see the previous remark).

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