Rotation Vectors for Homeomorphisms of Non-Positively Curved Manifolds

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Abstract

The notion of a rotation vector, as defined for homeomorphisms of the torus that are isotopic to the identity, is generalized to such homeomorphisms of any complete riemannian manifold with non-positive sectional curvature. These generalized rotation vectors are shown to exist for almost every orbit of such a dynamical system, if a certain condition is satisfied by the invariant measure under consideration.

1 Introduction

The concept of rotation vectors, has its origin in the works of Poincaré on the dynamics of homeomorphisms of the circle. Since then, the definition has been generalized to higher dimensional tori and, for example, has been used successfully to study the conjugacy classes (see [Jäg09]), the existence of periodic orbits (see [Fra89]), and entropy (see [LM91]) of homeomorphisms of the two-torus that are isotopic to the identity.

The main ergodic tool used to show the existence of rotation vectors has been Birkhoff's ergodic theorem. This is one of the reasons why there have been few extentions of the concept to manifolds other than the torus (although there are some notable exceptions, such as the homological rotation vectors defined in [Fra96]).

In [Kaĭ87] Kaimanovich proved an ergodic theorem that states that the orbits of the action of a stationary random sequence of elements of a certain type of Lie group on a symmetric space almost-surely remain close to a geodesic. This theorem has been generalized to orbits of cocycles of isometries (or semi-contractions) acting on a certain types of "hyperbolic" metric spaces by Anders Karlsson and Gregory Margulis in [KM99]. And, more recently, by shifting focus from geodesics to Busseman functions, the result has been generalized to actions of such cocycles on any proper metric space (see [KL06]).

One can show quite easily that the existence of a rotation vector for an orbit of a homeomorphism of the d-dimensional torus, is equivalent to the fact that

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the lift of this orbit to the universal covering space $\mathbb{R}^d$ is close to a geodesic in $\mathbb{R}^d$.

The question therefore arises: Can the ergodic tools developed by Kaimanovich, Karlsson, Margulis, Ledrappier, and others provide the framework to define a more general concept of rotation vector? And, can this concept then be applied usefully to the study of dynamical systems on a wide range of manifolds? The main obstacle in answering the first question, is that it seems dificult to obtain the (lifted) orbits of a general homeomorphism as the orbits of a cocyle of semi-contractions or isometries (e.g. the points on a orbit might not have locally isometric neighborhoods).

This article answers the first of the two questions posed above. We define the concept of a rotation vector for an orbit of a homeomorphism that is isotopic to the identity on any non-positively curved complete riemannian manifold and show that the existence of such vectors is generic from the point of view of the homeomorphism's invariant measures. It remains to be seen whether this notion can be put to use to obtain more information about such dynamical systems. For this purpose, it seems desirable to obtain a suitable generalization of the notion of rotation sets (see [MZ89]) and also one would like to clarify the connection between the rotation vectors defined here and homological rotation vectors as defined in [Fra96].

The proof of our main theorem depends basically on three things:

1. Kingman's subadditive ergodic theorem (in particular corollary 4).

2. A necessary condition (provided by lemma 10) for a sequence in a simply connected complete riemannian manifold of non-positive curvature to be "escorted by geodesic".

3. An ergodic theorem (theorem 12) for random sequences in metric spaces, that guarantees that almost all such sequences will satisfy the condition given by lemma 10.

Kingman's subadditive ergodic theorem is, of course, quite classical and in particular corollary 4 has been used in previous work.

Item 2 in the above list is basically contained in [KM99], in particular lemma 3.1 in that work is more general than our lemma 9. However, the corresponding version of our lemma 10 isn't stated explicitly in [KM99] but rather is implicit in the proof of the main theorem. This is part of the reason why we include a full proof.

The main novelty introduced in this work is theorem 12 (corresponding to item 3) which can be viewed as a generalization of theorem 1.2 in [KL06] to random sequences which are not orbits of cocycles of semi-contractions (also, our argument goes through without any ergodicity assumptions).

2 Statement of the Main Theorem

Let us begin by reviewing the definition of a rotation vector for a homeomorphism of the torus.

Definition 1 (Rotation Vector for Torus Homeomorphisms). *Let $T^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus, and let $f : T^d \to T^d$ be a homeomorphism that is isotopic to the identity. Also, fix a lift $F : \mathbb{R}^d \to \mathbb{R}^d$ of f.*

The rotation vector $v_F(x)$ of a point $x \in T^d$ is the following limit in case it exists:

$$v_F(x) = \lim_{n\to+\infty} \frac{F^n(\tilde{x}) - \tilde{x}}{n} \text{ where } \tilde{x} \in \mathbb{R}^d \text{ is any lift of } x$$

The following is a well known consequence of Birkhoff's Ergodic Theorem.

Proposition 1. *If $f : T^d \to T^d$ is isotopic to the identity, $F : \mathbb{R}^d \to \mathbb{R}^d$ is a lift of f, and μ is an f-invariant Borel probability measure. Then the limit $v_F(x)$ exists for μ-almost every $x \in T^d$.*

Note that a point $x \in T^d$ has rotation vector $v_F(x)$ for a certain lift F of a homeomorphism $f : T^d \to T^d$, if and only if for any lift $\tilde{x}$ of x it holds that $\|F^n\tilde{x} - (\tilde{x} + nv_F(x))\| = o(n)$ when $n \to +\infty$. If one considers the usual flat metric on T^d then the fact that the curve $t \mapsto x + tv_F(x)$ is a geodesic shows that the following definition generalizes Definition 1. Note that, we will always assume that the universal covering space of a riemannian manifold is endowed with the riemannian metric and distance d such that the covering map is a local isometry.

Definition 2 (Rotation Vector). *Let M be a complete riemannian manifold, $\tilde{M}$ it's universal covering space, $f : M \to M$ a homeomorphism, and $F : \tilde{M} \to \tilde{M}$ a lift of f.*

A rotation vector $v_F(x)$ of a point $x \in M$ is a vector in the tangent space T_xM such that for any lift $\tilde{x}$ of x the following holds:

$$d(\tilde{\alpha}(n), F^n\tilde{x}) = o(n) \text{ when } n \to +\infty$$

where $\tilde{\alpha}$ is the lift starting at $\tilde{x}$ of the geodesic $\alpha : [0, +\infty) \to M$ defined by $\alpha(t) = \exp_x(tv_F(x))$.

If the lift F in the above definition commutes with all covering transformations then the existence of a rotation vector $v_F(x)$ is a statement about the F-orbit of a single lift $\tilde{x}$ of x. Also, notice that in this case the expression $\rho(x) = d(\tilde{x}, F\tilde{x})$ where $\tilde{x}$ is a lift of x is a well defined function from M to $[0, +\infty)$ (i.e. it's value doesn't depend on the choice of $\tilde{x}$). Finally, notice that if $f : M \to M$ is isotopic to the identity then one can always obtain a lift $F : \tilde{M} \to \tilde{M}$ of f that commutes with all covering transformations (e.g. this can be done by lifting the isotopy between the identity map on M and f).

We can now state our main theorem as follows.

Theorem 2 (Main Theorem)**.** *Let M be a complete riemannian manifold with non-positive sectional curvature and let $\tilde{M}$ be it's universal covering space. For each $x \in M$ let $\tilde{x} \in \tilde{M}$ denote an arbitrary lift of x.*

Suppose that $f : M \to M$ is a homeomorphism that is isotopic to the identity, $F : \tilde{M} \to \tilde{M}$ is a lift of f that commutes with all covering transformations, and μ is an f-invariant Borel probability measure satisfying the condition:

$$\int_M d(\tilde{x}, F\tilde{x})\mathrm{d}\mu(x) < +\infty$$

Then for μ-almost every $x \in M$ there exists a rotation vector $v_F(x) \in T_xM$.

3 Rate of Escape

Definition 3 (Rate of Escape)**.** *The rate of escape of a sequence $\{x_n\}_{n\geq 0} \subset X$ in a metric space X is the value of the following limit if it exists:*

$$R = \lim_{n\to+\infty} \frac{d(x_0, x_n)}{n}$$

Observe that if $M, \tilde{M}, f$ and F are as in the statement of Theorem 2 and there exists a rotation vector $v_F(x)$ for a certain $x \in M$ then for any lift $\tilde{x}$ of x the sequence $\{F^n\tilde{x}\}_{n\geq 0}$ has rate of escape $\|v_F(x)\|$ (for details see Lemma 5). In view of this our first task is to show that for almost every $x \in M$ the sequence $\{F^n\tilde{x}\}_{n\geq 0}$ has a well defined finite rate of escape.

The purpose of this section is to show a somewhat more general fact, i.e. that a certain general class of random sequences in a metric space almost surely have a well defined finite rate of escape. This is a consequence of Kingman's Subadditive Ergodic Theorem which we restate in a convenient fasion.

Theorem 3 (Kingman's Subadditive Ergodic Theorem [Kin68])**.** *If $(M, \mathcal{B}, \mu)$ is a probability space, $f : M \to M$ is a measure preserving measurable function, and $\{a(m,n)\}_{0\leq m<n}$ (where $m, n \in \mathbb{Z}$) is a family of measurable functions from M to $\mathbb{R}$ satisfying:*

1. $a_x(l,n) \leq a_x(l,m) + a_x(m,n)$ *for all* $x \in M$ *and all* $0 \leq l < m < n$
2. $a_x(m+1,n+1) = a_{f(x)}(m,n)$ *for all* $x \in M$ *and all* $0 \leq m < n$
3. $\int_M |a_x(0,1)|\mathrm{d}\mu(x) < +\infty$
4. $\lim_{n\to+\infty} \int_M \frac{a_x(0,n)}{n}\mathrm{d}\mu(x) > -\infty$

Then there exists an invariant and integrable function $R : M \to \mathbb{R}$ such that

$$\int_A R(x)\mathrm{d}\mu(x) = \lim_{n\to+\infty} \int_A \frac{a_x(0,n)}{n}\mathrm{d}\mu(x)$$

for every invariant measurable set $A \subset M$. And for μ almost every $x \in M$ the following holds:

$$R(x) = \lim_{n \to +\infty} \frac{a_x(0,n)}{n}$$

A family of measurable functions satisfying condition 1 is called a subadditive process. Condition 2 guarantees that this process is stationary. Conditions 3 and 4 imply in particular that each function in the process is integrable (this is a consequence of subadditivity). Processes satisfying condition 4 are said to have "finite time constant", one can prove Kingman's Theorem without this condition but the pointwise limit R will no longer be integrable (e.g. it might be equal to $-\infty$ on a set of positive probability). Since it's first proof in [Kin68], the theorem has recieved many alternative proofs. See [Kre85] for a general reference.

Kingman's theorem has the following immediate corollary.

Corollary 4 (Rate of Escape for Random Sequences)**.** *Suppose $(M, \mathcal{B}, \mu)$ is a probability space and $f : M \to M$ is a measurable and measure preserving transformation.*

If X is a metric space and $\phi : M \to X^{\mathbb{N}}$ is a measurable function such that the family of functions $\{a(m,n)\}_{n>m\geq 0}$ that is defined by

$$a(m,n) : M \to [0, +\infty)$$

$$a_x(m,n) = d(\phi(x)_m, \phi(x)_n)$$

satisfies the hypothesis of Theorem 3. Then there exists an invariant and integrable function $R : M \to [0, +\infty)$ such that

$$\int_A R(x)\mathrm{d}\mu(x) = \lim_{n \to +\infty} \int_A \frac{a_x(0,n)}{n} \mathrm{d}\mu(x)$$

for every invariant measurable set $A \subset M$. And for almost every $x \in M$ the following holds:

$$R(x) = \lim_{n \to +\infty} \frac{a_x(0,n)}{n}$$

In particular, for almost every $x \in M$ the sequence $\phi(x)$ has a finite rate of escape.

4 Geodesic Escorts

The main theorems of [Kaĭ87] and [KM99] can be restated in terms of the following definition. We will prove in this section that it is also connected to rotation vectors.

Definition 4 (Geodesic Escort)**.** *Let (X, d) be a metric space. A sequence $\{x_n\}_{n\geq 0} \subset X$ is said to be escorted by a geodesic, if there exists a function $\alpha : [0, +\infty) \to X$ that is either constant or a local isometry onto it's image and satisfies:*

- $\alpha(0) = x_0$
- $d(x_n, \alpha(d(x_0, x_n))) = o(n)$ *when* $n \to +\infty$

Notice that any sequence with rate of escape equal to zero is escorted by a (constant) geodesic.

Before giving a proof of a necessary and sufficient condition for the existence of a rotation vector we would like to recall the following facts:

- If M is a complete riemannian manifold with non-positive sectional curvature then it's universal covering space $\tilde{M}$ is diffeomorphic to $\mathbb{R}^{\dim(M)}$ (see for example [Lan99] Theorem 3.8 on page 252).
- Any two points in $\tilde{M}$ belong to a unique geodesic (up to reparametrizations) and therefore any arclength parametrization of this geodesic is a (global) isometry onto it's image (see [Lan99] p.353 Corolarry 3.11).

Lemma 5. *Let M be a complete riemannian manifold with non-positive sectional curvature and let $\tilde{M}$ be it's universal covering space.*

Suppose that $f : M \to M$ is a homeomorphism that is isotopic to the identity, and that $F : \tilde{M} \to \tilde{M}$ is a lift of f that commutes with all covering transformations.

A rotation vector $v_F(x)$ exists for a point $x \in M$ if and only if for every lift $\tilde{x}$ of x the sequence $\{F^n\tilde{x}\}_{n\geq 0}$ has a finite rate of escape and is escorted by a geodesic.

Proof. First suppose $v = v_F(x) \in T_xM$ is a rotation vector for x. For any lift $\tilde{x}$ of x the geodesic $\tilde{\alpha} : [0, +\infty) \to \tilde{M}$ given by Definition 2 satisfies:

$$d(\tilde{\alpha}(n), F^n\tilde{x}) = o(n) \text{ when } n \to +\infty$$

Since each pair of points in $\tilde{M}$ belongs to a unique geodesic we have that:

$$d(\tilde{\alpha}(0), \tilde{\alpha}(n)) = d(\tilde{x}, \tilde{\alpha}(n)) = n\|\tilde{\alpha}'(0)\| = n\|v\|$$

The triangle inequality now implies:

$$d(\tilde{x}, \tilde{\alpha}(n)) - d(\tilde{\alpha}(n), F^n\tilde{x}) \leq d(\tilde{x}, F^n\tilde{x}) \leq d(\tilde{x}, \tilde{\alpha}(n)) + d(\tilde{\alpha}(n), F^n\tilde{x})$$

And this in turn implies:

$$n\|v\| - o(n) \leq d(\tilde{x}, F^n\tilde{x}) \leq n\|v\| + o(n)$$

Which shows that $\{F^n\tilde{x}\}_{n\geq 0}$ has rate of escape $\|v\|$.

Let $\beta : [0, +\infty) \to \tilde{M}$ be the geodesic starting at $\tilde{x}$ with $\beta'(0) = \tilde{\alpha}'(0)/\|\tilde{\alpha}'(0)\|$. Since

$$\beta(d(\tilde{x}, F^n\tilde{x})) = \beta(n\|v\| + o(n)) = \tilde{\alpha}(n + o(n))$$

one has that:

$$d(F^n\tilde{x}, \beta(d(\tilde{x}, F^n\tilde{x})) \leq d(F^n\tilde{x}, \tilde{\alpha}(n)) + d(\tilde{\alpha}(n), \tilde{\alpha}(n + o(n))) = o(n)$$

and therefore β escorts the sequence $\{F^n\tilde{x}\}_{n\geq 0}$.

Suupose now that $\{F^n\tilde{x}\}_{n\geq 0}$ has rate of escape R and is escorted by a geodesic β. Let $\gamma : \tilde{M} \to \tilde{M}$ be a covering transformation. Since F commutes with γ and γ is an isometry one has that $\gamma \circ \beta$ is a geodesic escort for the sequence $\{F^n(\gamma\tilde{x})\}_{n\geq 0}$ and this sequence has rate of escape R. In particular the rate of escape is independendent of the chosen lift $\tilde{x}$, and therefore if $R = 0$ then $v = 0 \in T_xM$ is a rotation vector for x.

On the other hand if $R > 0$, then β is a non-constant geodesic and the image v of the vector $R\beta'(0)$ under the diferential of the covering map $\pi : \tilde{M} \to M$ is independent of the choosen lift $\tilde{x}$. To show that v is a rotation vector for x all that is needed is to prove that $d(\beta(Rn), F^n\tilde{x}) = o(n)$. We can obtain this directly from $d(\tilde{x}, F^n\tilde{x}) = Rn + o(n)$ and the fact that β is a geodesic escort, as follows:

$$d(\beta(Rn), F^n\tilde{x}) \leq d(\beta(Rn), \beta(d(\tilde{x}, F^n\tilde{x}))) + d(\beta(d(\tilde{x}, F^n\tilde{x})), F^n\tilde{x}) = o(n)$$

□

5 Aligned Sequences

Definition 5 (Linear Escape to Infinity)**.** *A sequence $\{x_n\}_{n\geq 0} \subset X$ in a metric space X is said to escape linearly to infinity if it has a positive and finite rate of escape.*

In this section we will give a condition under which a sequence that escapes linearly to infinity will be escorted by a geodesic.

The simplest such condition known to the author is the following (which we will state without proof), valid for d-dimensional hyperbolic space:

Proposition 6. *Let $\mathbb{H}^d$ denote d-dimensional hyperbolic space. Any sequence $\{x_n\}_{n\geq 0} \subset \mathbb{H}^d$ that escapes linearly to infinity and satisfies*

$$d(x_n, x_{n+1}) = o(n) \text{ when } n \to +\infty$$

is escorted by a unique geodesic.

The condition $d(x_n, x_{n+1}) = o(n)$ is almost always satisfied by random sequences, provided that the variables $d(x_n, x_{n+1})$ are identicaly distributed and have finite expectation. Using this fact one can obtain a proof of a special case of our main theorem.

However the sequence in $\mathbb{C}$ defined by $x_n = ne^{i\log(n)}$ satisfies $d(x_n, x_{n+1}) = o(n)$ (where $d(x, y) = |x - y|$ for every $x, y \in \mathbb{C}$) but isn't escorted by a geodesic. This shows that the proposition is false for general non-positively curved spaces.

Since our objective is to prove the existence of geodesic escorts for sequences in complete riemannian manifolds with non-positive sectional curvature we will need a stronger hypothesis then $d(x_n, x_{n+1}) = o(n)$. The following definitions will allow us to formulate such a hypothesis.

Definition 6 (ϵ-Cone)**.** *Let (X, d) be a metric space. If $\epsilon \in [0, +\infty]$ and $x, y \in X$ the ϵ-cone from x to y is defined as the following set:*

$$[x, y]_\epsilon = \{z \in X : e^{-\epsilon} d(x, z) + d(z, y) \leq d(x, y)\}$$

The 0-cone between two points in $\mathbb{C}$ is a segment. The cone $[x, y]_{+\infty}$ is a closed disk centered at y and containing x (in particular note that the definition is not symmetric in x and y).

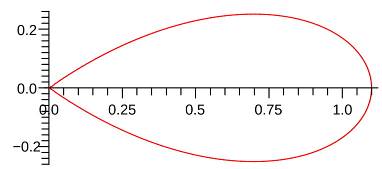

Figure 1: The boundary of $[0, 1]_{0.2}$ in $\mathbb{C}$.

We will now show that, in certain manifolds, the ϵ-cone between two points is close to a geodesic segment for small ϵ.

We will rely on two properties of complete simply connected riemannian manifolds with non-positive curvature which we state below without proof. Lemma 7 is contained in theorems 4.3 and 4.4 of chapter IX in [Lan99]. While the semi-parallelogram law (lemma 8) is proved to hold locally in any non-positively curved manifold in section 3 of chapter XI of the same reference. In the context of simply connected non-positively curved manifolds the same proof gives the (global) statement below.

Lemma 7 (Convexity of geodesic distance)**.** *If M is a complete simply connected riemannian manifold with non-positive sectional curvature then for any $x \in M$ and any pair of geodesics $\alpha, \beta : \mathbb{R} \to M$ the following two functions are convex:*

$$t \mapsto d(x, \alpha(t))$$
$$t \mapsto d(\alpha(t), \beta(t))$$

Lemma 8 (Semi-Parallelogram Law)**.** *Let M be a complete simply connected riemannian manifold with non-positive sectional curvature, and let $x, y, z \in M$. Then if m is the midpoint of the geodesic segment $[x, y]_0$ the following inequality holds:*

$$d(x, y)^2 + 4d(m, z)^2 \leq 2d(x, z)^2 + 2d(z, y)^2$$

Lemma 9. *Let M be a complete simply connected riemannian manifold with non positive sectional curvature, and suppose $x, y, z \in M$ and $\epsilon > 0$ are such that $z \in [x, y]_\epsilon$. If $\alpha : [0, +\infty) \to M$ is the unique geodesic parametrized by arclength with $\alpha(0) = x$ and $\alpha(d(x, y)) = y$, and $w = \alpha(d(x, z))$, then the following holds:*

$$d(z, w)^2 \leq 4(1 - e^{-2\epsilon})d(x, z)^2$$

Proof. Let $a = d(x, z) = d(x, w), b = d(x, y) - a$ and $c = d(z, y)$. Notice that $b = d(w, y)$ if $d(x, z) < d(x, y)$ and $b = -d(w, y)$ otherwise. In both cases $|b| \leq c$.

Since $z \in [x, y]_\epsilon$ we have:

$$e^{-\epsilon}a + c \leq a + b$$

which implies

$$a + b - c \geq e^{-\epsilon}a$$

Let m be the midpoint of the segment $[z, w]_0$. Lemma 7 implies that:

$$d(y, m) \leq \max(|b|, c) = c$$

and from this we obtain

$$d(x, m) \geq d(x, y) - d(y, m) \geq a + b - c \geq e^{-\epsilon}a$$

The semi-parallelogram law (lemma 8) now gives:

$$d(w, z)^2 + 4d(x, m)^2 \leq 4a^2$$

Combining these inequalities we obtain:

$$d(w, z)^2 \leq 4(a^2 - d(x, m)^2) \leq 4(1 - e^{-2\epsilon})a^2 = 4(1 - e^{-2\epsilon})d(x, z)^2$$

□

We will now state a condition that will guarantee the existence of a geodesic escort for a sequence that escapes linearly to infinity. One can interpret the condition saying either that, the sequence eventually stays in arbitrarily small ϵ-cones, or that the diferences $d(x_0, x_n) - d(x_k, x_n)$ are close to their largest posible value (i.e. $d(x_0, x_k)$).

Definition 7 (Aligned Sequence)**.** *Let (X, d) be a metric space. A sequence $\{x_n\}_{n\geq 0} \subset X$ is said to be aligned if for each $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that for infinitely many $n \in \mathbb{N}$ the following holds:*

$$x_k \in [x_0, x_n]_\epsilon \text{ for all } K \leq k \leq n$$

Observation 1. Let (X, d) be a metric space and $\{x_n\}_{n\geq 0} \subset X$ a sequence that escapes linearly to infinity with rate of escape R. Then $\{x_n\}_{n\geq 0}$ is aligned if and only if $L \geq R$ where:

$$L = \lim_{K\to+\infty} \limsup_{n\to+\infty} \min_{K\leq k\leq n} \{\frac{d(x_0, x_n) - d(x_n, x_k)}{k}\}$$

Lemma 10. *Let M be a complete simply connected riemannian manifold with non positive sectional curvature. If $\{x_n\}_{n\geq 0} \subset M$ escapes linearly to infinity and is aligned then it is escorted by a geodesic.*

Proof. Let $f : [0, +\infty) \to [0, +\infty)$ be given by $f(\epsilon) = 2\sqrt{1 - e^{-2\epsilon}}$.

For each $n \in \mathbb{N}$, let $\alpha_n : [0, +\infty) \to M$ be the unique geodesic parametrized by arclength such that $\alpha_n(0) = x_0$ and $\alpha_n(d(x_0, x_n)) = x_n$. Also let R be the rate of escape of the sequence $\{x_n\}_{n\geq 0}$.

For each $\epsilon > 0$ there is a natural number K and a subsequence of values of n such that:

$$x_k \in [x_0, x_n]_\epsilon \ \forall K \leq k \leq n$$

In this sequence of values of n, by lemma 9 we have:

$$d(x_k, \alpha_n(d(x_0, x_k))) \leq f(\epsilon) d(x_0, x_k) \ \forall K_\epsilon \leq k \leq n$$

Consider the pointwise limit α_ϵ of a subsequence of the corresponding geodesics α_n (such a limit exists because $\alpha_n(t) = \exp_{x_0}(\alpha_n'(0)t)$ for every $t \in [0, +\infty)$ and $\alpha_n'(0)$ belongs to the unit sphere of $T_{x_0}M$ which is compact).

The curve α_ϵ is a geodesic and the following holds (since $d(x_0, x_k) = Rk + o(k)$ when $k \to +\infty$):

$$d(x_k, \alpha_\epsilon(d(x_0, x_k)) \leq f(\epsilon) Rk + o(k) \text{ when } k \to +\infty$$

By the triangle inequality we have the following for each pair $\epsilon, \epsilon' > 0$:

$$d(\alpha_\epsilon(d(x_0, x_k)), \alpha_{\epsilon'}(d(x_0, x_k)) \leq (f(\epsilon) + f(\epsilon'))Rk + o(k) \text{ when } k \to +\infty$$

and therefore by convexity (lemma 7)

$$d(\alpha_\epsilon(t), \alpha_{\epsilon'}(t)) \leq (f(\epsilon) + f(\epsilon'))t \ \forall t \in [0, +\infty)$$

This shows that there is a geodesic $\alpha : [0, +\infty) \to M$ such that $\alpha(t) = \lim_{\epsilon\to 0} \alpha_\epsilon(t)$ and therefore:

$$d(\alpha_\epsilon(t), \alpha(t)) \leq f(\epsilon)t \ \forall t \in [0, +\infty), \epsilon > 0$$

This implies

$$d(x_k, \alpha(d(x_0, x_k))) = o(k) \text{ when } k \to +\infty$$

□

6 Alignment of Random Sequences

The following lemma will enable us to prove that, in a sense, almost all random sequences that escape linearly to infinity are aligned.

Lemma 11 ([KM99] Lemma 4.1)**.** *Suppose $\{a(m,n)\}_{0\le m<n}$ satifies the hypothesis of Theorem 3 and that:*

$$\lim_{n\to+\infty}\int_M \frac{a_x(0,n)}{n}\mathrm{d}\mu(x) > 0$$

Then the probability of the following set is strictly positive:

$$\{x \in M : \text{ for infinitely many } n \in \mathbb{N}, a(0,n) - a(k,n) > 0 \text{ for all } 1 \le k \le n\}$$

The following theorem completes the prerequisites for the proof of our main theorem.

Theorem 12 (Alignment of Random Sequences)**.** *Suppose $(M, \mathcal{B}, \mu)$ is a probability space and $f : M \to M$ is a measurable and measure preserving transformation.*

If X is a metric space and $\phi : M \to X^{\mathbb{N}}$ is a measurable function such that the family of functions $\{a(m,n)\}_{n>m\ge 0}$ defined by:

$$a(m,n) : M \to [0,+\infty)$$

$$a_x(m,n) = d(\phi(x)_m, \phi(x)_n)$$

satisfies the hypothesis of Theorem 3. Then for almost every $x \in M$ if the sequence $\phi(x)$ escapes linearly to infinity then it is aligned.

Proof. Let $R : M \to [0,+\infty)$ be the function given by Corollary 4. For almost every $x \in M$, the sequence $\phi(x)$ escapes linearly to infinity if and only if $R(x) > 0$.

Consider the measurable function $L : M \to [0,+\infty)$ defined by:

$$L(x) = \max(0, \lim_{K\to+\infty} \limsup_{n\to+\infty} \min_{K\le k\le n} \{\frac{d(\phi(x)_0, \phi(x)_n) - d(\phi(x)_n, \phi(x)_k)}{k}\})$$

By Observation 1 it suffices to show that $L \ge R$ on a set of full measure.

First we will show that the function L is invariant on a set of full measure. For this purpose it suffices to show that $L(x) \le L(f(x))$ for almost every x, since the probability of the set $\{x \in M : L(x) \ge a\}$ is equal to that of the set $\{x \in M : L(f(x)) \ge a\}$ for every $a \in \mathbb{R}$ because f is measure preserving.

If $L(x) = 0$ then trivially $L(x) \le L(f(x))$. Suppose $L(x) \ge t > 0$. This implies that for each $\epsilon > 0$ there exists $K \in \mathbb{N}$ and infinitely values of $n \in \mathbb{N}$ such that:

$$d(\phi(x)_0, \phi(x)_n) - d(\phi(x)_n, \phi(x)_k) > te^{-\epsilon}k \text{ for all } K \le k \le n$$

for a slightly larger K the following holds:

$$\begin{aligned} d(\phi(x)_1, \phi(x)_n) - d(\phi(x)_n, \phi(x)_k) &\ge -d(\phi(x)_1, \phi(x)_0) + d(\phi(x)_0, \phi(x)_n) - d(\phi(x)_n, \phi(x)_k) \\ &> -d(\phi(x)_1, \phi(x)_0) + te^{-\epsilon}k > te^{-2\epsilon}(k-1) \text{ for all } K \le k \le n \end{aligned}$$

and therefore $L(f(x)) \geq te^{-2\epsilon}$. Since this holds for every $\epsilon > 0$ we have shown the claim that $L(x) \leq L(f(x))$ almost everywhere and therefore L is invariant on a set of full measure.

To simplify the discussion, modify L on a set of measure 0 so that it is strictly invariant.

We will now show that $L \geq R$ almost everywhere.

This is trivially true in the set where $R = 0$.

Suppose that for some $\epsilon > 0$ the set $A = \{x \in M : L(x) < e^{-\epsilon}R(x)\}$ has positive measure. Let 1_A be the function that takes the value 1 on A and 0 outside of A, and consider the stationary subadditive process $\{b(m,n)\}_{0\leq m<n}$ defined by

$$b_x(m,n) = (d(\phi(x)_m, \phi(x)_n) - (n-m)e^{-\epsilon}R(x))1_A(x)$$

Since this process satisfies

$$\lim_{n\to+\infty}\int_M \frac{b_x(0,n)}{n}\mathrm{d}\mu(x) = (1-e^{-\epsilon})\int_A R(x)\mathrm{d}\mu(x) > 0$$

by Lemma 11 there is a set of positive probability of $x \in M$ such that there exist infinitely many n satisfying the following:

$$b_x(0,n) - b_x(k,n) > 0 \text{ for all } 1 \leq k \leq n$$

However since

$$b_x(0,n) - b_x(k,n) = (d(\phi(x)_0, \phi(x)_n) - d(\phi(x)_n, \phi(x)_k) - ke^{-\epsilon}R(x))1_A(x)$$

this would imply that for some $x \in A$ we have $L(x) \geq e^{-\epsilon}R(x)$ contradicting the definition of A. Therefore we must have $L \geq R$ almost everywhere as claimed. □

7 Proof of the Main Theorem

We will now prove the main theorem (Theorem 2).

Proof. One can choose $\tilde{x}$ so that it is a measurable function of x. By Corollary 4, for almost every $x \in M$ the sequence $\{F^n\tilde{x}\}_{n\geq 0}$ has a finite rate of escape $R(x)$.

If $R(x) = 0$ then $0 \in T_xM$ is a rotation vector for x.

On the other hand, by Theorem 12, for almost every $x \in M$ in the case that $R(x) > 0$ the sequence $\{F^n\tilde{x}\}_{n\geq 0}$ is aligned. In this case the existence of a geodesic escort follows from Lemma 10. This implies the existence of a rotation vector by Lemma 5. □

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