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ON THE UNTWISTING OF GENERAL DE JONQUIÈRES AND CUBO-CUBIC CREMONA TRANSFORMATIONS OF $\mathbb{P}^3$

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ABSTRACT. We consider Cremona transformations of $\mathbb{P}^3$ which either stabilize the set of planes passing through a point or they are defined by the maximal minors of a 4×3 matrix of linear forms. For a general transformation satisfying one of these two conditions we give an explicit decomposition as a product of elementary links; in other words, we perform the so-called Sarkisov program for (general) transformations belonging to these two classes. Conversely, we show that a Cremona transformation whose decomposition coincides with one of those we have obtained belongs to one of these classes.

Keywords: Cremona Transformations, Sarkisov program

1. Introduction

Let $\mathbb{P}^n$ be the dimension n projective space over the complex number field $\mathbb{C}$. A Cremona transformation of $\mathbb{P}^n$ is a birational map

$$\phi = (f_0 : \cdots : f_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n,$$

where $f_0, \ldots, f_n \in \mathbb{C}[x_0, \ldots, x_n]$ are relatively prime homogeneous polynomials of the same degree; the common degree of the $f_i's$ is the *degree* of ϕ. The group $\mathrm{Cr}(\mathbb{P}^n)$ of such transformations is the so-called *Cremona Group* of $\mathbb{P}^n$.

One of the most famous theorems of Max Noether states that a Cremona transformation $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ may be decomposed as a product of quadratic transformations (see [Al, Chap. 8]). More precisely, $\mathrm{Cr}(\mathbb{P}^2)$ is generated by the linear automorphisms and the *standard quadratic transformation*

$$(x : y : z) \mapsto (yz : xz : xy).$$

It is an easy exercise to show that the standard quadratic transformation may be decomposed by blowing up a point, then performing two elementary transformations of Hirzebruch surfaces, and finally by blowing down a rational curve. A Hirzebruch surface together with its

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fiber structure is a special case of what today we call a Mori fiber space, which appears as a fundamental piece in the Minimal Model Program of Mori; the blow-ups, blow-downs and elementary transformations are special types of what nowdays we call elementary links; in fact, there are four types of elementary links, the so-called links of types I, II, III and IV.

In [Sa89] V. G. Sarkisov proposed a way to decompose a birational map between Mori fiber spaces as a product of elementary links, that was resumed and clarified by Miles Reid in [Re91]; unfortunately we could not find that Sarkisov's preprint. Finally, Alessio Corti proves in [Co95] that the so-called *Sarkisov Program* works in dimension three. Roughly speaking, the idea of the program consists of define a degree for such a map and construct an elementary link which "untwists" it in such a way that one obtain another map with lesser degree.

It is not difficult, although not completely trivial, to prove Noether's Theorem via the Sarkisov program. The main fact is that, in dimension two, elementary links have a very simple structure (see [An-Me03, Part 5], [KoSmCo, §2.5] or [Is, §1.5]).

On the other hand, in a certain sense, there is no equivalent of Noether's Theorem for dimension three. More precisely, if G is a set of generators for $\mathrm{Cr}(\mathbb{P}^3)$, one may prove G contains uncountable many transformations which are not isomorphisms, i.e., which have degree > 1; a raison for it is we may find families of transformations whose base locus scheme contains a component with birational type varying continuously (see [Hu, Chap. XVI, §32] and [Pan99]). Thus, though Corti's Theorem can be thought as a structure result (or generalization of Noehter's Theorem) for $\mathrm{Cr}(\mathbb{P}^3)$, that is far from satisfactory in this regard.

The aim of this paper is to describe the untwisting procedure for members of some families of Cremona transformations which characterize, as we will see, special blocks of elementary links in the Sarkisov program; moreover, these blocks realize the shortest way to connect $\mathbb{P}^3$ to itself via a sequence of elementary links. This characterization leads us to believe that those transformations play a special role in the understanding of the structure of the Cremona group of $\mathbb{P}^3$.

Let $\phi \in \mathrm{Cr}(\mathbb{P}^3)$. We say ϕ is a *de Jonquières transformation* if there exists a point $o \in \mathbb{P}^3$ such that ϕ acts birationally on the set of hyperplanes passing through o. We say ϕ is a *cubo-cubic transformation* if this map and its inverse are defined by cubic polynomials, i.e. both maps have degree 3; when the base locus scheme of such a transformation is smooth we say it is *special*.

We consider the Sarkisov program in dimension three for two celebrated types of Cremona transformations: "general" de Jonquières transformations and special cubo-cubic transformations (for a more precise definition see §3 and §4, respectively). The main results may be summarized as follows:

• A general de Jonquières transformation of degree 2 characterizes the product of a link of type II connecting $\mathbb{P}^3$ to a smooth hyperquadric in $\mathbb{P}^4$ and a link of type II isomorphic to the inverse of that (Theorem 9).

• A general de Jonquières transformation of degree > 2 characterizes the product of a link of type I, a link of type II connecting the blow-up of a point in $\mathbb{P}^3$ to itself, and a link of type III which is isomorphic to the inverse of the first one (Theorem 11).

• A special Cremona transformation characterizes the shortest link of type II connecting $\mathbb{P}^3$ to itself (Theorem 16).

Moreover, in the way to obtain the main results, we characterize the possible "shortest" (we mean without flips) links of type II connecting $\mathbb{P}^3$ either to itself or to a smooth hyperquadric in $\mathbb{P}^4$ (Lemma 3), and the links of type I connecting $\mathbb{P}^3$ to a Mori fiber space of the form $h : Z \to \mathbb{P}^2$ (Lemma 6).

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2. Generalities

In this paragraph we establish some properties of links of type I and II in the special case where they come from $\mathbb{P}^3$. Before to start with, let us fix some notations and introduce some terminology.

2.1. **Basic notions and notations.** Let X be a dimension 3 projective normal variety which is $\mathbb{Q}$-factorial and has, at most, terminal singularities. Denote by K_X a (Weil) canonical divisor for it. We denote by $N_1(X) \simeq \mathbb{R}^{\rho(X)}$ the real vector space generated by the set of numerical equivalence classes of curves on X, where $\rho(X)$ is the *Picard number* of X, and by $\overline{\mathrm{NE}}(X) \subset N_1(X)$ the *closed cone of curves* of X.

Let $R \subset \overline{\mathrm{NE}}(X)$ be an extremal ray, i.e. a dimension 1 face, which is negative, in the sense

$$K_X \cdot z < 0, \forall\, z \in R - \{0\},$$

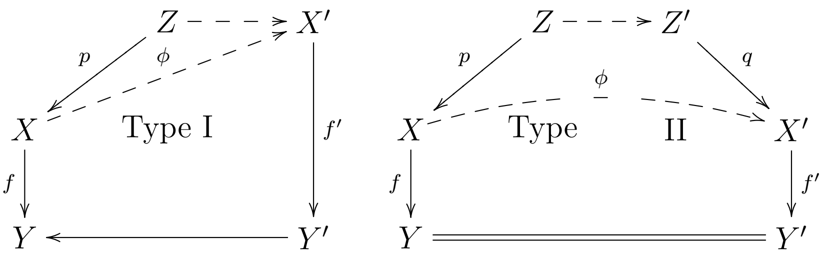
and $f = \mathrm{cont}_R : X \to Y$ the extremal contraction associated to R. We recall that f is said to be a

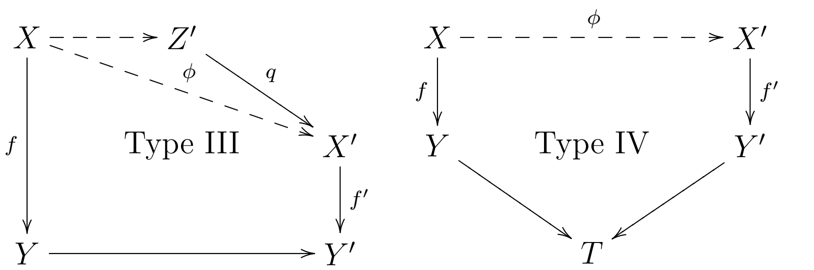
i) divisorial contraction or extraction, if it contracts a divisor;
ii) Mori fiber space (MFS for short) if $\dim X > \dim Y$;
iii) small contraction otherwise (this kind of contraction will not be used in this work). In this case there exists a unique contraction $X^+ \to Y$, associated to a ray R^+ such that $K_{X^+} \cdot z > 0, \forall\, z \in R^+ - \{0\}$: one say the birational map $X \dashrightarrow X^+$ is a flip.

To our purpose (that is, apply the Sarkisov program) we will use the basic results of the so-called *logarithmic* Minimal Model Program (MMP for short) and *relative* MMP; in particular, we will also deal, then, with extremal contractions in these two contexts. The former one consists, roughly speaking, of replace the canonical divisor K_X with $K_X + b\mathcal{H}_X$, where $\mathcal{H}$ varies in a linear system and b is a non-negative rational number. Analogously, in the second one, the closed cone of curves $\overline{\mathrm{NE}}(X)$ is replaced with the *relative* cone of curves $\overline{\mathrm{NE}}(X \to W) = \overline{\mathrm{NE}}(X/W)$ generated by (classes of) curves contracted by a morphism $X \to W$. We refer the reader to [KoMo] and/or [Ma] for further explanations.

Let $\phi : X \dashrightarrow X'$ be a birational map between dimension 3 projective varieties; take and fix an ample divisor $A_{X'}$ on X'. Suppose these varieties admit MFS structures $f : X \to Y$ and $f' : X' \to Y'$, respectively. Choose and also fix a positive rational number μ' such that $\mu' K_{X'} + A_{X'}$ is the pullback of an ample divisor on Y'; in particular $\mu' K_{X'} + A_{X'}$ is f'-trivial, i.e., it intersects non negatively all curves contracted by f'. For example, if f' is $\mathbb{P}^3 \to \{pt.\}$ (respectively, $Q \to \{pt.\}$, where $Q \subset \mathbb{P}^4$ is a smooth hyperquadric), then we may take $A_{X'}$ to be a plane (respectively, a hyperplane section) and $\mu' = 1/4$ (respectively, $\mu' = 1/3$).

There are four types of birational transformations as above which constitute the so-called *elementary links* or *Sarkisov links*, and are said to be of type I, II, III and IV. These are defined by the following diagrams





where all horizontal dashed arrows are composition of a finite number of (logarithmic) flips and p, q are extremal divisorial contractions.

Now we consider a birational morphism $\sigma : V \to X$, obtained via a sequence of blow-ups of smooth centers, which is a resolution of both, the indeterminancy of $\phi : X \dashrightarrow X$, that is $\tau := \phi \circ \sigma$ is a morphism, and the singularities of X; denote by K_V a canonical divisor on V. If $\mathcal{H}_{X'} \in |A_{X'}|$ we define the *homaloidal transform* $\mathcal{H}_X$ of it as being the divisor $\phi_*^{-1}(\mathcal{H}_{X'}) := (\sigma_* \circ \tau^*)(\mathcal{H}_{X'})$. We also denote $\phi_*^{-1}(|A_{X'}|)$, the linear system of homaloidal transforms of the complete linear system $|A_{X'}| = |\mathcal{H}_{X'}|$.

Now we recall the definition of the *Sarkisov degree* of a birational map ϕ as above (we follow [Ma, Chap. 13]):

Quasi-effective threshold: it is the positive rational number $\mu = \mu_\phi$ such that for all irreducible curve C contained into a fiber of $X \to Y$ we have

$$(\mu K_X + \mathcal{H}_X) \cdot C = 0,$$

where $\mathcal{H}_X \in \phi_*^{-1}(|A_{X'}|)$. One knows $\mu \geq \mu'$.

Maximal multiplicity: For $\mathcal{H}_X \in \phi_*^{-1}(|A_{X'}|)$ we have

$$K_V = \sigma^* K_X + \sum_{k=1}^{n} a_k E_k, \quad \mathcal{H}_V := \sigma^* \mathcal{H}_X - \sum_{k=1}^{n} m_k E_k,$$

for some positive rational numbers a_k and some non-negative integer numbers m_k, where E_k is σ-exceptional, for all $k = 1, \dots, n$. The maximal multiplicity of ϕ is defined to be 0 when $\phi_*^{-1}(|A_{X'}|)$ is base points free; otherwise it is defined to be the positive rational number

$$\lambda = \lambda_\phi = \max\{\frac{m_k}{a_k}; k = 1, \dots, n\};$$

we will say λ is *concentrated* at an exceptional divisor E_j when $\lambda = \frac{m_j}{a_j}$.

Number of crepant exceptional divisors: Suppose $\lambda \neq 0$. Keeping the notation above, it is the number $\epsilon = \epsilon_\phi$ of exceptional components E_j such that $\lambda = m_j/a_j$.

The Sarkisov degree of ϕ is, by definition, the triple $(\mu_\phi, \lambda_\phi, \epsilon_\phi)$.

2.2. **Sarkisov program.** It consists of an algorithm to decompose a birational map $\phi : X \to X'$ between two MFS's as a product of elementary links. To perform the algorithm one first calculate the Sarkisov degree of ϕ and then construct an elementary link $\phi' : X \dashrightarrow X_1$ in such a way that $\phi_1 := \phi \circ (\phi')^{-1} : X_1 \dashrightarrow X'$ is another birational map between MFS's with lesser (with respect to the lexicographic order) Sarkisov degree; one say ϕ' untwists ϕ. Then we restart the procedure replacing ϕ with ϕ_1. After a finite number of untwists we finally obtain an automorphism and the program is over.

For the convenience of the reader we briefly recall how to construct a link in order to untwist a map as above ([Co95] and [Ma, Chap. 13]); a nice *résumé* for it may be found in [La]. There are two cases depending on either $\lambda = \lambda_\phi$ is greater than $\mu = \mu_\phi$ or not:

2.2.1. *Untwisting procedure 1:* $\lambda > \mu$. First we construct a maximal extraction $\pi : Z \to X$, i.e., a K_X-negative extremal contraction whose exceptional set $E = \mathrm{Exc}(\pi)$ corresponds via $\pi^{-1} \circ \sigma : V \dashrightarrow Z$ to an exceptional component of σ at which the maximal multiplicity λ is concentrated. Then we perform a (logarithmic) MMP with respect to the divisor $K_Z + \frac{1}{\lambda}\mathcal{H}_Z$ starting at Z and relative to $f \circ \pi : Z \to Y$, i.e, by contraction of extremal rays of the relative cone $\overline{\mathrm{NE}}(Z/Y)$. This procedure terminates by giving a MFS $f_1 : X_1 \to Y_1$ and an elementary link $\phi' : X \dashrightarrow X_1$; the link ϕ' is of type I or II, the last case occurring if and only if that MMP finishes by a divisorial contraction.

Moreover, denote by $(\mu_1, \lambda_1, \epsilon_1)$ the Sarkisov degree of $\phi_1 := \phi \circ (\phi')^{-1}$. If either $Y = \{pt.\}$ or $Y = C$ and $\pi(E) = C'$ are curves with $f(C') = C$, then $\mu_1 < \mu$. Otherwise $\mu_1 = \mu$ and $\lambda_1 \leq \lambda$; if $\lambda_1 = \lambda$, then $\epsilon_1 < \epsilon$.

2.2.2. *Untwisting procedure 2:* $\lambda \leq \mu$. Denote by R the extremal ray in $\overline{\mathrm{NE}}(X)$ corresponding to the MFS $f : X \to Y$ which belongs to the hyperplane on $N_1(X)$ defined by $\mu K_X + \mathcal{H}_X = 0$. If that divisor is nef we know ϕ is an isomorphism by the Noehter-Fano-Iskovskikh criterion ([Ma, Prop. 13-1-3]), and we have nothing to do in this case. On the other hand, suppose there exists an extremal ray P in the semi-space $\mu K_X + \mathcal{H}_X < 0$; we can choose P in order to $\langle R, P\rangle$ defines a face of $\overline{\mathrm{NE}}(X)$; we denote by $X \to T$ the contraction of this face. Now we perform a $(\mu K_X + \mathcal{H}_X)-$MMP relative to T. If we get a MFS $X_1 \to Y_1$ with respect to $\mu K_{X_1} + \mathcal{H}_{X_1}$, it is also a MFS with respect to K_{X_1} (a usual MFS) and one can see it produces a link of type III or IV. In this case $\mu_1 < \mu$.

If we get a minimal model X_1, with respect to $\mu K_{X_1} + \mathcal{H}_{X_1}$ and relative to T, i.e., that divisor intersects non negatively all elements in $\overline{\mathrm{NE}}(X_1)$ coming from elements in $\overline{\mathrm{NE}}(X/T)$, one can see $X \dashrightarrow X_1$ necessarily is a link of type III. Moreover, in this case $\mu_1 = \mu$, $\lambda_1 \leq \lambda$ and $\rho(X) > \rho(X_1)$.

2.3. **Preliminary results.** The canonical class of $\mathbb{P}^3$ will be denoted by K. As we know, the real vector space $N_1(\mathbb{P}^3)$ of classes of 1-cycles is generated by the class ℓ of a line.

By an *extraction* we mean a Mori (extremal) divisorial contraction $p : X \to Y$ between normal, projective and $\mathbb{Q}$-factorial varieties whose singularities are, at most, terminal. If $E \subset X$ is the exceptional divisor, we call $\mathrm{Center}(p) := p(E)$ the *center* of p.

Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation of degree $d \geq 1$. The quasi-effective threshold of ϕ is $\mu = d/4$. The following lemma is a special case of the Noether-Fano-Iskovskikh criterion.

Lemma 1. *If $d > 1$, then $\lambda_\phi > \mu_\phi$.*

In other words, for the nontrivial Cremona transformation ϕ we are in the the situation of §2.2.1, and one say there exists a *base component of high multiplicity*. Therefore, the first step of the Sarkisov program for ϕ consists of construct a maximal extraction $p : Z \to \mathbb{P}^3$.

Remarks 2. (a) Suppose Cener(p) $= \{pt.\}$ and Z is smooth. According to [Mo82, Thm. 3.3 and Cor. 3.4] the extraction p is the blow-up of a point $P \in \mathbb{P}^3$. Therefore, the two faces of $\overline{\mathrm{NE}}(Z)$ are extremal contractions: that associated to p and the other one to a MFS $Z \to \mathbb{P}^2$ whose fibers are strict transform of lines in $\mathbb{P}^3$ passing through P.

(b) When Z is not smooth (but normal, $\mathbb{Q}$ factorial and with terminal singularities), according to [Kaw01] the extraction p is a weighted blow-up of a point $P \in \mathbb{P}^3$, with weight $wt(P) = (1, a, b)$; here a, b are coprime positive integers with $ab > 1$ or $a = b = 1$. On the other hand, $\mathbb{P}^3$ may be thought as a toric variety in such a way that the point $P \in \mathbb{P}^3$ is fixed by the torus action and we can produce toric wighted blow-ups with weights as above. We deduce there are exactly one of the following situations (for a classification of toric links see [Sh05]):

• $wt(P) = (1, 1, 1)$, and $p : Z \to \mathbb{P}^3$ is the usual blow-up of P and $q : Z \to \mathbb{P}^2$ is a fibering contraction.

• $wt(P) = (1, a, b)$, with a, b coprime positive integers, $ab > 1$, and $q : Z \to X$ is either a small contraction, or a divisorial contraction onto a singular variety X. Moreover, it is divisorial when $1 = a < b$ and small otherwise.

More generally, suppose $p : Z \to X$ is an extraction whose center $\Gamma := \mathrm{Cener}(p)$ is an (irreducible) curve. If Z is smooth, then Γ and X also are smooth and p is the blow-up of γ ([Mo82, Thm. 3.3 and Cor. 3.4]). In the general case p coincides with the blow-up of Γ along the smooth points but terminality on X impose strong conditions on singularities in Γ; the general description of such type of extremal contractions is given in [KoMo92, Thm. 4.9].

Now we give necessary conditions to the existence of some special kind of links of type II which start with $\mathbb{P}^3$.

Lemma 3. *Consider a nontrivial link of type II as follows:*

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{p} & & \searrow^{q} & \\ \mathbb{P}^3 & & & & X \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & & = \!= \!= & & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Where $X = \mathbb{P}^3$ or X is a smooth hyperquadric $Q \subset \mathbb{P}^4$. Suppose that p is a maximal extraction whose center is an irreducible curve Γ of degree d. Then $q \circ p^{-1} : \mathbb{P}^3 \dashrightarrow X$ is defined by a linear system of surfaces of degree n which are generically smooth along Γ. Moreover, we have:

(a) If $X = \mathbb{P}^3$, then $n = 3$ and $d = 6$.

(b) If $X = Q$, then either $n = d = 2$ or $n = 3$ and $d = 5$.

Proof. By definition Z is normal and admits, at most, $\mathbb{Q}$-factorial and terminal singularities.

Denote by H the class of an element in the complete linear system $|p^*\mathcal{O}^3_{\mathbb{P}}(1)|$. Then $K_Z = -4H + E$ and $|\mathcal{H}_Z| = |nH - mE|$, where E denotes the exceptional locus of the extraction p and $m, n \geq 1$ are positive integers.

We adapt the proof given in [Ka87, Prop. 2.1].

We have

$$K_Z + \frac{1}{\lambda}\mathcal{H}_Z = p^*(K + \frac{1}{\lambda}\mathcal{H}_{\mathbb{P}^3}) + (1 - \frac{m}{\lambda})E,$$

where λ is the corresponding maximal multiplicity associated to $q \circ p^{-1}$. By definition, we obtain $\lambda = m$.

Notice that the quasi-effective threshold of $q \circ p^{-1}$ is $n/4$ in the both cases considered for X.

Since E is a component of high multiplicity, thus

$$\mu = \frac{n}{4} < m,$$

from which $n < 4m$.

The projection formula yields

$$H^3 = 1, H^2 \cdot E = 0, H \cdot E^2 = -d.$$

On the other hand, since $q : Z \to X$ is birational and it is defined by the linear system $|\mathcal{H}_Z|$, we get

$$(nH - mE)^3 = n^3 - 3nm^2d - m^3E^3 = d_0, \tag{1}$$

where $d_0 = 1$ or 2 following that X is $\mathbb{P}^3$ or Q, respectively. Then m^2 divides $n^3 - d_0$.

Now, by a slight abuse of notation, we write $q : Z \to \mathbb{P}^N$, where $N = 3$ or $q(Z) = Q \subset \mathbb{P}^4 = \mathbb{P}^N$. Since $q : Z \to \mathbb{P}^N$ is a morphism, by applying Bertini's Theorem, first on $Z - \mathrm{Sing}(Z)$ (remember that Z has terminal singularities, therefore isolated) and then on a general member of the linear system $|\mathcal{H}_Z|$, we see that the strict transform of a general codimension 2 linear space, by q, is a smooth complete intersection of two general members of $|nH - mE|$, which are smooth. The adjunction formula yields

$$\begin{aligned}
-2 &= (nH - mE)^2 \cdot [2(nH - mE) + p^*(K) + E] \\
&= (n^2H^2 - 2nmH \cdot E + m^2E^2) \cdot [(2n-4)H - (2m-1)E] \\
&= 2\left[n^2(n-2) - nmd(2m-1) - m^2(n-2)d\right] - \\
&\quad\; m^2(2m-1)E^3
\end{aligned} \tag{2}$$

and it follows that m^2E^3 is even. Therefore $2|mE^3$ and then $m|n^2(n-2)+1$.

If $d_0 = 1$, we conclude that $m|n(n^3 - 2n^2 + 1) - (n-2)(n^3 - 1) = 2(n-1)$, hence $2(n-1) = km, k < 8$. Similary, if $d_0 = 2$ we obtain $m|n(n^3-2n^2+1)-(n-2)(n^3-2) = 3n-4$, hence $3n-4 = km, k < 12$.

Finally, eliminating $(2m-1)m^3E^3$ between Equations (2) and (1), we obtain the relation

$$(n^2 - m^2d)(4m - n) = 2m(d_0 + 1) - d_0. \tag{3}$$

First suppose $d_0 = 1$. Then

$$(n^2 - m^2d)(8m - 2n) = 8m - 2,\ 2n = 2 + km, k < 8;$$

it follows that $(8-k)m-2$ divides $8m-2$. The values for n and d in the statement (a) of the Lemma and $m=1$ constitute the unique system of solutions for this equation.

Now suppose $d_0 = 2$. Hence

$$(n^2 - m^2 d)(12m - 3n) = 18m - 6,\ 3n = 4 + km, k < 12,$$

which implies that $(12-k)m-4$ divides $18m-6$. The solutions (n, m, d) are $(5, 1, 29)$ for $k = 11$, $(3, 1, 5)$ for $k = 5$ and $(2, 1, 2)$ for $k = 2$. Since $n^2 > d$, we may take off the first solution, which completes the proof. □

The following example gives a link of type II connecting $\mathbb{P}^3$ to a singular hyperquadric in $\mathbb{P}^4$, hence excluded in the statement of the Lemma 3.

Example 4. Let $\Gamma \subset \mathbb{P}^3$ be a smooth elliptic curve of degree 5; denote by $\mathcal{J}_\Gamma$ its ideal sheaf. It is a curve such that $\mathcal{O}_\Gamma(1)$ is non-special ([Har, Chap. IV, Exa. 6.4.2]). Then $\mathrm{h}^0(\mathcal{O}_\Gamma(3)) = 15$. It follows that $r := \mathrm{h}^0(\mathcal{J}_\Gamma(3)) \geq 5$; denote by $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r-1}$ the rational map defined by the cubic forms vanishing on Γ. If $p : Z \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ along Γ, by restricting ϕ to planes, we observe that $q = \phi \circ p$ is a morphism whose image, say X, is a 3-fold. Take a general divisor $H \in |p^*\mathcal{O}_{\mathbb{P}^3}(1)|$ and let us denote $E := p^{-1}(\Gamma)$. By construction, q is defined by the complete linear system $|3H - E|$.

On the other hand, we have

$$\begin{aligned} p_*(E - E^2 + E^3) &= \Gamma + p_*(E^3). \\ &= \Gamma + 2 - 2g(\Gamma) + K_{\mathbb{P}^3} \cdot \Gamma \end{aligned} \tag{4}$$

The projection formula implies

$$H^3 = 1, H^2 \cdot E = 0, H \cdot E^2 = -5, E^3 = -20;$$

hence $(3H - E)^3 = 2$. Since $q(Z)$ is nondegenerate by construction, we deduce q is birational onto its image which is, in fact, a dimension 3 variety of degree 2, that is, $r-1 = 4$ and $q(Z) \subset \mathbb{P}^4$ is an hyperquadric.

Projecting from a point of Γ to $\mathbb{P}^2$, we see that through such a point, there passes two 3-secant lines of Γ; then the union of these 3-secant lines is a surface. Note that q contracts (strict transforms of) 3-secant lines of Γ. Moreover, if $L \subset Z$ is the pullback of such a 3-secant line, then

$$\begin{aligned} K_Z \cdot L &= (-4H + E) \cdot L \\ &= -1 \end{aligned}$$

which implies q is a divisorial contraction, then $q(Z)$ is smooth and q is the blow-up a smooth curve.

To finish this paragraph, we consider links of type I connecting $\mathbb{P}^3$ to a Mori fiber space $Z \to \mathbb{P}^2$. Let us begin with some examples.

Examples 5.

(a) (To compare with [Sh05, Rem. 3]). Let $C \subset \mathbb{P}^3$ be an irreducible conic. Take two degree n irreducible surfaces, say $S_1 := (g_1 = 0)$, $S_2 := (g_2 = 0)$, containing C; denote by $\Gamma = \Gamma_n$ the residual curve in $S_1 \cap S_2$ with respect of C; it has degree $n^2 - 2$. By *liaison* Theory ([PS74, §3]), Γ is arithmetically Cohen-Macaulay, with arithmetic genus $n^3 - 2n^2 - 4n + 8$. Its ideal sheaf is generated by

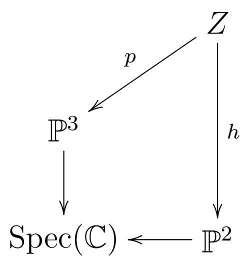
$$g_1, g_2, g$$

where g is a homogeneous polynomial of degree $2n-3$. When $2n-3 = n$, that is, $n = 3$, we obtain a rational fibration $\rho : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$, defined by $\rho = (g_1 : g_2 : g)$, whose general fiber is a conic and $\mathrm{Base}(\rho) = \Gamma$. Since $\Gamma \cup C$ has arithmetic genus 10, the fibers of ρ intersect Γ at 6 points, counting multiplicities.

Since C is smooth, by taking S_1 and S_2 general enough we may suppose that Γ is smooth ([PS74, Prop.4.1]). Then, by blowing up a smooth Γ and contracting the strict transforms of its 6-secant conics, we obtain a link of type I. Note that Γ has degree 7 and genus 5.

(b) If $C \subset \mathbb{P}^3$ is a line, a computation as in (a) shows that we must have $n = 2$, and then Γ is a twisted cubic (ref. Lemma 14).

Lemma 6. *Consider a link of type I as follows:*



where p *is a maximal extraction with center* $\mathrm{Center}(p)$ *and* Z *is smooth. Then the map* $h \circ p^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ *is a rational fibration defined by a linear system of surfaces of degree* n *which are generically smooth along* $\mathrm{Center}(p)$*. Moreover, in this case we have one of the following situations:*

(a) $n = 1$ *and* $\mathrm{Center}(p) = \{pt.\} \subset \mathbb{P}^3$.

(b) $n = 2$ *and* $\mathrm{Center}(p)$ *is a curve of degree 3.*

(c) $n = 3$ *and* $\mathrm{Center}(p)$ *is a curve of degree 7.*

Proof. As we know Center(p) is either a point or an irreducible curve. In the first case we are in situation (a), by Remarks 2(a).

Suppose that $\Gamma :=$ Center(p) is a curve of degree $d \geq 1$; we keep all notations from the proof of Lemma 3. The Mori fiber space $h : Z \to \mathbb{P}^2$ is defined by a dimension 2 linear system of the form $|nH - mE|$, for some $m \geq 1$.

In this case, the equation (2) remains valid while the equations (1) and (3) become, respectively

$$n^3 - 3nm^2d - m^3E^3 = 0, \tag{5}$$

and

$$(n^2 - m^2d)(4m - n) = 2m. \tag{6}$$

From equation (5) we infer $m^2|n^3$, hence $m|n^2$. Since the equation (2) implies $m|n^2(n-2)+1$ we conclude $m = 1$. Equation (6) then becomes

$$(n^2 - d)(4 - n) = 2.$$

Taking into account that $n^2 - d > 0$ we conclude that the only solutions (n, d) for this equation are $(2, 3)$ and $(3, 7)$, which completes the proof. □

3. The de Jonquières case

We say that $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is a *de Jonquières transformation* of degree d if, up to a linear change of coordinates in the domain and the target, we may write

$$\phi = (g : kx : ky : kz),$$

for an irreducible g, in such a way that g and k vanish at $o = (1 : 0 : 0 : 0)$ with multiplicities $d-1$ and $\geq d-2$ respectively; note that the *base locus scheme* Base(ϕ) of ϕ is supported on the curve $\Gamma = \{g = k = 0\}$ unless $d = 2$, in which case Base(ϕ) contains an isolated point; to show that definition of de Jonquières transformation is equivalent to the one introduced in §1 see, for example, [Pa00].

Since ϕ preserves the set of general lines through o and the set of general planes through o, the strict transform of a general line by ϕ is a rational curve of degree d with a $(d-1)$-uple point at o; in particular, $\deg \phi^{-1} = d$. We will say that the de Jonquières transformation is general when g and k are general satisfying these conditions.

Remark 7. The surface $(k = 0)$ and the joint (cone) Join(o, Γ) are contracted by ϕ. More precisely, the first one is contracted onto a point and the second one onto a curve.

We obtain easily the following result.

Lemma 8. *Let* $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ *be a general de Jonquières transformation. A resolution of the indeterminancy of* ϕ *is obtained by blowing up, first the point* o*, and then a smooth curve birational to* Γ*. Furthermore,*

$$K_V = \sigma^* K + 2E + F, \quad \mathcal{H}_V = \sigma^* \mathcal{H} - (d-1)E - F,$$

where $\sigma(E) = \{o\}$ *and* $\sigma(F) = \Gamma$.

A de Jonquières transformation of degree $d = 2$ is a *quadro-quadratic* transformation which we will consider in a seperate setup. From the lemma above, it follows that the Sarkisov degree of ϕ is

$$\left(\frac{d}{4}, \lambda, 1\right),$$

where the maximal multiplicity is $\lambda = 1$, concentrated at F, when $d = 2$, and it is $\lambda = (d-1)/2$ and concentrated at E, when $d > 2$; in the case $d = 3$, one may consider also λ as being concentrated at F.

3.1. **The quadro-quadratic case.** Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a general quadro-quadratic transformation; we know the reduced structure on Base(ϕ) consists of a conic C and a point o outside its plane. By Lemma 8,

$$K_V = \sigma^* K + 2E + F, \quad \mathcal{H}_V = \sigma^* \mathcal{H} - E - F,$$

where $\sigma(E)$ is the point and $\sigma(F)$ is the conic, both constituting the irreducible components of Base(ϕ). The Sarkisov degree (μ, λ, ϵ) of ϕ is

$$\left(\frac{1}{2}, 1, 1\right),$$

with maximal multiplicity λ concentrated at F. In this case, the conic $C \subsetneq \text{Base}(\phi)$ is a base component of high multiplicity and the blow-up of $\mathbb{P}^3$ along C coincides with the maximal extraction $p : Z \to \mathbb{P}^3$ with respect to $K_Z + \mathcal{H}_Z$. Note that the vector space $N_1(Z)$ is generated over $\mathbb{R}$ by $\ell_Z = p^*(\ell)$ and the class f of a fiber $p^{-1}(x) \subset F$ for $x \in C$.

On the other hand, take $H \in |p^* \mathcal{O}_{\mathbb{P}^3}(1)|$ and consider the complete linear system $|2H - F|$, consisting of pullbacks, by p, of quadrics containing C. It is base points free, has dimension 4 and $(2H - F)^3 = 2$, hence it defines a birational morphism $q : Z \to Q$, where $Q \subset \mathbb{P}^4$ is a hyperquadric. Since $(2H - F) \cdot (\ell_Z - 2f) = 0$, then q is the divisorial contraction associated to the K_Z-negative extremal ray generated by $\ell_Z - 2f$, whose exceptional divisor is the strict transform, in Z, of the plane on which C lies; in particular

$$\overline{\text{NE}}(Z) = \langle f, \ell_Z - 2f \rangle.$$

Since the normal bundle of this divisor is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)$ we deduce Q is smooth and q is the blow-up of a point. We obtain an (elementary) link of type II as follows:

$$\begin{array}{ccccc} & & Z & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\ \mathbb{P}^3 & & & & Q \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & & = & & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Notice that $\phi_1 := \phi \circ (p \circ q^{-1}) : Q \dashrightarrow \mathbb{P}^3$ has a unique base point at $o_1 := q(p^{-1}(o))$.

By construction of the untwisting link (see §2.2.1), we know $\mu_1 < \mu = 1/2$. Therefore, to continue the Sarkisov program's algorithm we need to blow up the point o_1, since $\lambda_1 = 1 > \mu_1$; denote by $p_1 : Z' \to Q$ this blow-up and by E_1 the exceptional locus of p_1. If e_1 is the class contracted by p_1 with $E_1 \cdot e_1 = -1$, we deduce

$$\overline{\mathrm{NE}}(Z') = \langle e_1, \ell_{Z'} - e_1 \rangle :$$

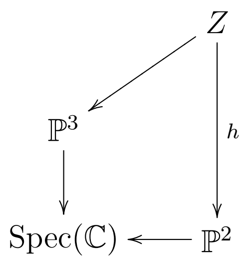
in fact, besides the conic's plane, ϕ contracts the ruling of the cone $\mathrm{Joint}(o, C)$, whose strict transform in Z' defines the class $\ell_{Z'} - e_1$, which is $(K_{Z'} + \mathcal{H}_{Z'})$-negative. This ray defines a divisorial contraction $q_1 : Z' \to X'$. Hence $q_1 \circ p_1^{-1}$ is a link of type II which untwists ϕ_1 to give $\phi_2 := \phi_1 \circ (p_1 \circ q_1^{-1}) : X' \dashrightarrow \mathbb{P}^3$. Since $\lambda_2 = \lambda_{\phi_2} = 0$, i.e. ϕ_2 is a morphism, and $\rho(X') = 1$, the only possible end for the algorithm is to have $X' \simeq \mathbb{P}^3$, and thus the output of the Sarkisov program for ϕ is a flowchart of the form:

$$\begin{array}{ccccccccc} & & Z & & & & Z' & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & & \overset{p_1}{\swarrow} & & \overset{q_1}{\searrow} & \\ \mathbb{P}^3 & & & & Q & & & & \mathbb{P}^3 \\ \downarrow & & & & \downarrow & & & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & & = & & \mathrm{Spec}(\mathbb{C}) & & = & & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Theorem 9. *Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation which is not a morphism. Then ϕ is a quadro-quadratic transformation, if and only if, the Sarkisov program for ϕ leads a product of two links of type II where the Mori fiber spaces are as in the flowchart above with* $\mathrm{Center}(q) \neq \mathrm{Center}(p_1)$ *and neither of these centers is a quintic curve.*

Proof. Since Z and Z' are smooth Remarks 2(a) implies that the center of p is an irreducible curve. By Lemma 3 this curve is a conic C, and the rational map $q \circ p^{-1}$ is defined by a linear system of quadrics which (generically) are smooth along C. Therefore q is necessarily the contraction of the conic's plane to a point: that is the blow-up of a point in Q. By symmetry, the same holds for p_1 and q_1. Since ϕ is not a morphism we conclude it is a general quadro-quadric transformation. □

3.2. **Case** $d > 2$**.** The point $o = (1 : 0 : 0 : 0)$ corresponds to a component of high multiplicity. The maximal extraction $p : Z \to \mathbb{P}^3$ is nothing but the ordinary blow-up of $\mathbb{P}^3$ at o. By using Remarks 2(a), we conclude that the first untwisting step is provided by a link of type I of the form:



We know that $\mu_1 < \mu$ in this case (see §2.2.1). More precisely, since $h : Z \to \mathbb{P}^2$ contracts the strict transform of lines passing through o, we obtain μ_1 from the equation

$$(\mu_1 K_Z + \mathcal{H}_Z) \cdot (\ell_Z - e) = 0.$$

Keeping notations from Lemma 8, we have

$$\begin{aligned} \mu_1 K_Z + \mathcal{H}_Z &= \mu_1(-4H + 2E) + dH - (d-1)E \\ &= (-4\mu_1 + d)H + (2\mu_1 - d + 1)E, \end{aligned}$$

from which we conclude $\mu_1 = 1/2$.

Since $\lambda_1 = 1 > \mu_1$ and λ_1 is concentrated at F, to get a maximal extraction, we may blow up the strict transform $\widetilde{\Gamma}$ of Γ by p, say $p_1 : Z' = V \to Z$, with $F = p_1^{-1}(\widetilde{\Gamma})$. Now we need to run a $(K_{Z'} + \mathcal{H}_{Z'})$-MMP over $\mathbb{P}^2$.

Note that, by genericity on the de Jonquières map ϕ, the fibers of h which intersect $\widetilde{\Gamma}$ do it transversely and at a unique point; then the strict transform, by p_1, of such fibers are numerically equivalent.

We deduce that the only possible output for the (logarithmic) relative MMP above is to contract the strict transform of generetrices of the cone Joint(o, Γ) (compare with Remark 7).

Then we have constructed a divisorial contraction $q_1 : Z' \to X$, whose center is an irreducible curve which is birationally equivalent to Γ. At this point, $\mathcal{H}_X$ is base points free and the only thing to do is to contract the strict transform, say S, of the rational surface ($k = 0$) to a point (Remark 7), by obtaining a link of type III which necessarily is of the form

$$\begin{array}{ccc} X & \searrow & \\ \downarrow & & \mathbb{P}^3 \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Notice that Remarks 2(a) implies the contraction $X \to \mathbb{P}^3$ is the blow-up of a point.

Thus, the Sarkisov program for ϕ has reached the end. We obtain a composition of three links of types I, II and III. Moreover, the link of type I is the unique one obtained from the blow-up of a point of $\mathbb{P}^3$, and that of type III is (isomorphic to) its inverse.

In the sequel, we show that an untwisting flowchart, as above, is necessarily the output, for the Sarkisov program, of a de Jonquières transformation of degree $d \geq 3$. We need some preliminary considerations.

The de Jonquières transformations of $\mathbb{P}^3$ are a special kind of Cremona transformations with the following property: there are two points $o, o' \in \mathbb{P}^3$ such that a general line through o is sent to a line through o'. As showed in [Pa00, §2], up to a linear change of coordinates, on the domain and the target, one may suppose that $o = o' = (1 : 0 : 0 : 0)$ and

$$\phi = (g : kt_1 : kt_2 : kt_3), \tag{7}$$

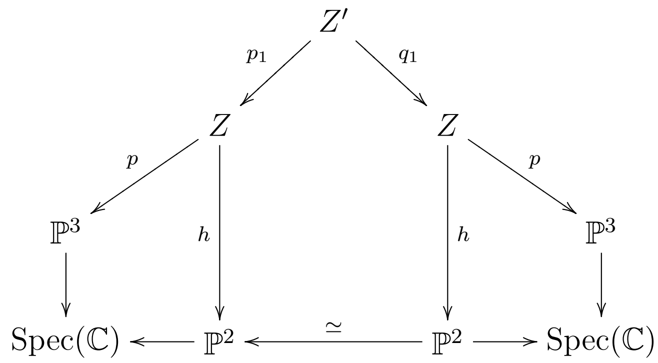
with g, k and t_i vanishing at o with orders $d-1$, not lower than $d-1$, and $d - \deg(g)$, respectively; we call o the *center* of ϕ. In this case, $\tau = (t_1 : t_2 : t_3) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a plane Cremona transformation. In [Hu], a Cremona transformation as above is called a monoidal transformation; we prefer to call it a *star-shaped transformation*, since it preserves the *star of lines* through o (and "monoidal" sometimes means "blow-up").

Remark 10. A star-shaped transformation is a de Jonquières transformation if, and only if, τ is a linear automorphism of $\mathbb{P}^2$.

Thus we obtain the following result.

Theorem 11. *Let* $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ *be a Cremona transformation.*

(a) If ϕ is a general de Jonquières transformation of degree $d \geq 3$ and center o, then the Sarkisov program for ϕ yields a flowchart of the following type:



where p is the blow-up of $o \in \mathbb{P}^3$.

(b) If the Sarkisov program for ϕ yields a flowchart as in (a), where Center(p) *is not a curve, then ϕ is a de Jonquières transformation of degree $d \geq 3$.*

Proof. We only need to prove (b). After Lemma 6 we know that p is the blow-up of a point in $\mathbb{P}^3$. The assumption implies that ϕ sends a line in $\mathbb{P}^3$, which is general among those passing through o, onto a line. Then ϕ is a star shaped transformation which induces an isomorphism of $\mathbb{P}^2$. The result follows from Remark 10. □

In Lemma 14 we give a Cremona transformation which is not a de Jonquières transformation, having an untwisting which looks as the one in Theorem above, but where the extraction $p : Z \to \mathbb{P}^3$ is the blow-up of $\mathbb{P}^3$ along a twisted cubic curve (to compare with Lemma 6(b)).

4. Cubo-cubic transformations

A Cremona transformation $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is said to be *cubo-cubic* is both ϕ and ϕ^{-1} are defined by polynomials of degree 3. There are, essentially, three kind of cubo-cubic transformations. Indeed, if ϕ is cubo-cubic, we have the following (exclusive only generically) possibilities for ϕ (see for example [Pa97]):

(i) its base locus scheme Base(ϕ) is an arithmetically Cohen-Macaulay curve of degree 6 and arithmetical genus 3; in this case we say ϕ is *determinantal.*
(ii) ϕ is a de Jonquières transformation.
(iii) there is a line along which all polynomials defining ϕ vanish at order ≥ 2 along a common line; in this case we say ϕ is *ruled.*

Note 12. The names given to the cubo-cubic transformations in (i) and (iii), which are classic (see [Hu]), are justified by way of (a) and (b) above, respectively:

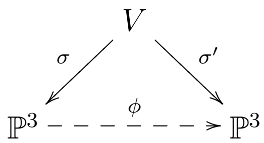
(a) The definition ideal of an arithmetical Cohen Macaulay curve of degree 6 and arithmetic genus 3 is defined by the maximal minors of a 4×3 matrix of linear forms ([Ell75, Example 2, pag. 430].

(b) The linear system associated to a ruled cubo-cubic transformation consists of cubic surfaces containing a common line of double points. All these surfaces are ruled.

On the other hand, let $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformations which is not an isomorphism. If Base(φ) is smooth, then it is a cubo-cubic determinantal Cremona transformation: see [Ka87] and [Hu, Chap. HIV, §11]. Moreover, in the first reference S. Katz proves that it suffices to assume the indeterminacy of φ is resolved by way of the blow-up of an smooth irreducible curve, to obtain the same conclusion.

In a general setup, a Cremona transformation $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is said to be *special* is its base locus scheme Base(ϕ) is connected and smooth (see [ESB89] and [CK89]). Therefore, the result above may be rephrased by saying that the unique special Cremona transformations are the cubo-cubic determinantal ones.

Now fix a special Cremona transformation $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. One has a resolution as follows:



where

$$K_V = \sigma^* K + E, \quad \mathcal{H}_V = \sigma^* \mathcal{H} - E,$$

$E = \sigma^{-1}(\Gamma)$; in particular, the Sarkisov degree is $(3/4, 1, 1)$. Moreover, σ' is a divisorial contraction onto a curve isomorphic to Γ whose exceptional locus is the strict transform by σ of the variety of 3-secant lines to Γ (see [Ka87], [SR, Chap. VIII, §4] or [Hu, Chap. HIV, §11]). From this description we obtain the following lemma:

Lemma 13. *Let* $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ *be a special transformation with* Base(ϕ) $= \Gamma$. *The Sarkisov program for* ϕ *consists of a unique link* *of type II, as follows:*

$$\begin{array}{ccccc} & & V & & \\ & \swarrow & & \searrow & \\ \mathbb{P}^3 & & & & \mathbb{P}^3 \\ \downarrow & & & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & & =\!=\!=\!=\!= & & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Proof. There is a component of high multiplicity whose corresponding maximal extraction is the blow-up of $\mathbb{P}^3$ along Γ. Then $\overline{\mathrm{NE}}(V) = \langle e, \ell_V - 3e\rangle$. Running a MMP with respect to $K_V + (1/\lambda)\mathcal{H}_V$ (over Spec($\mathbb{C}$)) we deal with the contraction associated to the negative extremal ray $\mathbb{R}^+(\ell_V - 3e)$, which is the class of the strict transform of a 3-secant line to Γ. It leads to a divisorial contraction $V \to X$ onto a Fano variety $X \to \mathrm{Spec}(\mathbb{C})$, completing a link of type II. Since $\mathcal{H}_X$ has no base points, then $\mu_1 \geq \lambda_1 = 0$ and $\mu_1 K_X + \mathcal{H}_X$ is nef. The Noether-Fano-Iskovkikh criterion completes the proof. $\square$

The smooth sextic Γ may degenerate to a Fossum-Ferrand double structure in a twisted cubic γ. The degree 8 surface of 3-secant lines to Γ then degenerates to a quartic surface considered twice: in fact it is the developable of γ, that is, the surface of tangent lines to γ (see [SR, Chap. VIII, S 4] and [PR05, §2] for further explanations). To obtain a resolution of the indeterminancy of the corresponding Cremona transformation ϕ_γ we start by blowing up $\mathbb{P}^3$ along γ; this leads to an exceptional divisor $E := \mathbb{P}(N_\gamma \mathbb{P}^3) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5))$. It suffices to blow up the section, say γ_0, of the bundle of projective lines $E \to \Gamma$ associated to the double structure on γ; it leads to an exceptional divisor F. We obtain a resolution $\sigma : V \to \mathbb{P}^3$. Thus,

$$K_V = \sigma^* K + E + 2F, \quad \mathcal{H}_V = \sigma^* \mathcal{H} - E - 2F.$$

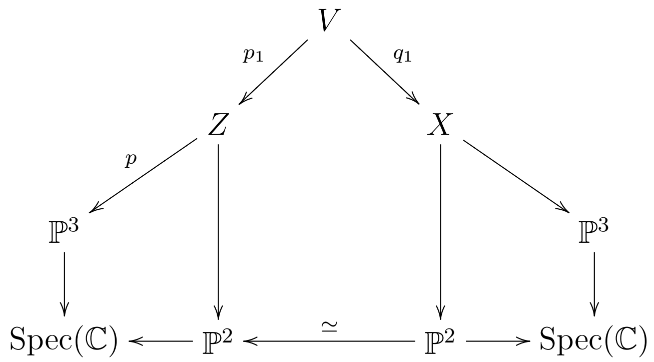
The Sarkisov degree of ϕ_γ is

$$(3/4, 1, 2);$$

A priori, the maximal multiplicity may be concentrated at E or at F. The blow-up $p : Z \to \mathbb{P}^3$ of $\mathbb{P}^3$ along γ is obviously a maximal extraction, hence we may start the program with it.

The strict transforms of quadrics containing γ cut out a base point free linear system on Z. This defines a MFS $h : Z \to \mathbb{P}^2$ which contracts all strict transforms of bisecants of γ. We deduce that $\overline{\mathrm{NE}}(Z) = \langle e, \ell_Z - 2e\rangle$, where the ray $\mathbb{R}^+(\ell_Z - 2e)$, which is $K_Z + \mathcal{H}_Z$ negative, defines

h. Now we blow up γ_0 using $p_1 : V \to Z$. The program forces us to untwist by a link of type I or II by contracting a $(K_V + \mathcal{H}_V)$-negative ray in $\overline{\mathrm{NE}}(V/\mathbb{P}^2)$. Among the 2-secant lines to γ, there are those whose strict transforms in Z intersect γ_0. Then $\overline{\mathrm{NE}}(V/\mathbb{P}^2) = \langle f, \ell_V - 2e_V - f\rangle$, where $f := p_1^{-1}(x)$ for $x \in \gamma_0$. There is no choice: we need to contract the ray $\mathbb{R}^+(\ell_V - 2e_V - f)$ which corresponds to a divisorial contraction $q_1 : V \to X$, whose center is a smooth rational curve, obtaining a link of type II onto a MFS $X \to \mathbb{P}^2$. For this one, the program is now in the case $0 = \lambda_2 < \mu_2 = \mu_1$. We finish it by contracting the strict transform of the developable of γ untwisting by a link of type III to obtain a decomposition as follows:



Moreover, there is no other possibility to obtain the maximal extraction $Z \to \mathbb{P}^3$. Indeed, suppose we may contract the divisor E from V to obtain a maximal extraction $p' : Z \to \mathbb{P}^3$, with exceptional divisor F. Notice that $E = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5))$ may be contracted in a unique form in order to obtain a curve: the same in which it was produced. The strict transform, by the resolution σ, say S, of a general plane in $\mathbb{P}^3$, is obtained from it by blowing up 3 points and then 3 other points over each of them. By contracting E we contract three (-2) curves in S and then it produces a surface with 3 double points; these points vary when S varies. By Bertini's Theorem, Z should be singular along a curve, contradicting terminality.

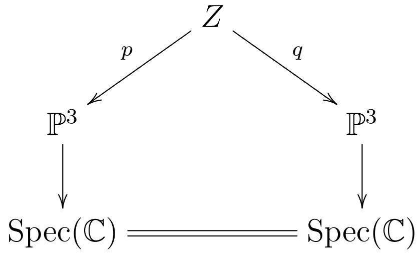
Thus, we have proven the following result.

Lemma 14. *Let $\gamma \subset \mathbb{P}^3$ be a twisted cubic and let $\phi_\gamma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a cubo-cubic transformation defined by a double structure on γ. The Sarkisov program for ϕ leads to a sequence of links of type I, II and III.*

Remark 15. It is known that $\phi_\gamma^{-1} = \phi_{\gamma'}$, where γ' is another twisted cubic. Then Z and X are both the blow-up of $\mathbb{P}^3$ along a twisted curve.

Since γ and γ' are projectively equivalent we conclude that the link of type III above is isomorphic to the inverse of that of type I.

Theorem 16. *Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be an arbitrary Cremona transformation which is not an isomorphism. Suppose that the Sarkisov program for ϕ yields an end consisting of a link of type II as follows:*



where Center(p) *and* Center(q) *are curves. Then ϕ is a (cubo-cubic) determinantal transformation. Moreover, ϕ is special if and only if Z is smooth.*

Proof. Suppose $\Gamma = \text{Center}(p)$ is (an irreducible) curve of degree d (see Remark below). Then p coincides with the blow-up of Γ along smooth points of it. Lemma 3 implies that $d = 6$ and ϕ is defined by a linear system of cubic surfaces which are generically smooth along Γ.

Denote by H the class of an element in the complete linear system $|p^*\mathcal{O}^3_{\mathbb{P}}(1)|$. Then $K_Z = -4H+E$ and $\mathcal{H}_Z = |3H-E|$, where $E = p^{-1}(\Gamma)$ denotes the exceptional locus of p.

Take two general cubic surfaces S, S' in the linear system $\mathcal{H}$; hence

$$S \cap S' = \Gamma \cup D,$$

where D is a rational curve of degree $3 = \deg \phi^{-1}$. We consider two cases:

1. D is singular. The classification of cubo-cubic transformations (see [Pa97, Thm. 1.2]) implies that ϕ is a de Jonquières transformation. We may suppose it is written as

$$\phi = (g : kx : ky : kz),$$

where k and g are homogeneous polynomials of degree 2 and 3 vanishing at $(0 : 0 : 0 : 1)$, and of order 1 and 2, respectively. Since $q \circ p^{-1}$ contracts a unique irreducible surface and ϕ contracts $k = 0$ onto a point, we conclude Center(q) is a point, which is not possible and proves that D can not be singular.

2. D is smooth. Hence, it follows that D is a twisted cubic. Since the base locus scheme of a ruled cubo-cubic transformation contains a line,

we conclude ϕ is determinantal, that is, Γ is a genus 3 arithmetically Cohen-Macaulay curve.

The last assertion in the statement of the theorem follows from the first one, Lemma 13 and the fact that Z smooth together with Remarks 2(a) imply p is the blow-up of a smooth curve.

□

Remark 17. The hypothesis on Center(p) and Center(q) in the Theorem is not necessary if we admit Remarks 2(b); we preferred not to use this part of Remarks 2 since we did not prove it in the text.

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