ON BIRATIONAL PROPERTIES OF SMOOTH CODIMENSION TWO DETERMINANTAL VARIETIES

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Abstract. We show that a smooth arithmetically Cohen-Macaulay variety X, of codimension 2 in $\mathbb{P}^n$, $3 \le n \le 5$, general if $n > 3$, admits a morphism onto a hypersurface of degree $(n+1)$ in $\mathbb{P}^{n-1}$ with, at worst, double points; moreover, this morphism comes from a (global) Cremona transformation which induces, by restriction to X, an isomorphism in codimension 1. We deduce that two such varieties are birationally equivalent *via* a Cremona transformation if and only if they are isomorphic.

Mathematical Subject Classification (2000): 14E07, 14E05, 13C40

1. Introduction

Arithmetically Cohen-Macaulay (ACM for short) codimension 2 subschemes of $\mathbb{P}^n$ are geometric objects whose cohomology satisfy very restrictive properties. In fact, the ideal sheaf of such a subscheme admits a *determinantal resolution* of length 2 whose Betti numbers determine a certain invariant, the so-called *type* (following the terminology of G. Ellingsrud in [Ell]) for the subscheme; in particular, most of their algebro-geometric properties are completely determined by this resolution. As shown in [Ell], the ACM codimension 2 subschemes of $\mathbb{P}^n$ of a fixed type may be parametrized by an open, smooth and connected subset of a Hilbert scheme. Among these ACM subschemes, a still more special subfamily is that consisting of subschemes $X \subseteq \mathbb{P}^n$ whose ideal sheaf $\mathcal{J}_X$ has a determinantal minimal resolution of the form

$$(1) \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n-1)^n \xrightarrow{\mu} \mathcal{O}_{\mathbb{P}^n}(-n)^{n+1} \longrightarrow \mathcal{J}_X \longrightarrow 0\ .$$

This family we will denote by $\mathcal{U}_n$.

From geometers of the so-called Italian's School of mathematicians, we know that such a general object is the base locus scheme of a Cremona transformation of $\mathbb{P}^n$ (see either [Cr], [Ca], [Hu, Chap. XIV, §11] or Proposition 1); therefore it is not a complete intersection, following [PR, Prop. 2.1] (see also [CK, Prop. 1]). Moreover, in this case, X is

[1]Partially supported by CNPq-Brazil. Research grant: 307833/2006-2

smooth if and only if $n \in \{3,4,5\}$ ([ESB, Thm.3.2]), thus constituting a rare, and maybe therefore interesting object, as pointed out by Sample and Tyrrel in [ST]. One other reason for paying attention to this kind of variety and consider its study as an interesting subject, is the special place these varieties have with respect to the Harshorne's Conjecture on complete intersections: in fact, by a theorem due to Peskine and Szpiro ([PS, Thm. 5.1]), an ACM smooth codimension 2 subvariety of $\mathbb{P}^n$ is a complete intersection when $n > 5$; hence, when $n > 5$, varieties as in our setup yield examples which show that the smoothness hypothesis in Peskine and Szpiro's Theorem is necessary.

In this work we describe some birational properties of such smooth codimension 2 subvarieties. More precisely, first we show that for a smooth $X \in \mathcal{U}_n$ (then $n = 3,4,5$), general for $n > 3$, there exists a *Cremona transformation* $\phi_X : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ that induces, by restriction, a birational morphism $\eta : X \to Y$, where Y is a hypersurface of degree $n+1$ in $\mathbb{P}^{n-1}$. Then we show that η is an isomorphism in codimension 1, which is an isomorphism for $n = 3,4$ and, at worst, a crepant resolution of a finite number of double points for $n = 5$; from this we will conclude that a birational map $\varphi : X \dashrightarrow X$ is actually an automorphism; see Theorems 8 and 9. On the other hand, we show that two such smooth subvarieties of $\mathbb{P}^n$ are isomorphic if and only if they are birational equivalents via a global birational map of $\mathbb{P}^n$: this is, essentially, the statement of Theorem 10.

The starting point to establish our main results is the existence of ϕ_X. The idea to construct this Cremona transformation is very natural; we conclude this introduction describing it in a few words. We take general elements $V \in \mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n)), S \in \mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n+1))$. The theory of linkage ([PS]) shows that X is linked to another $X' \in \mathcal{U}_n$ by the complete intersection $V \cap S$; hence $V \in \mathbb{P}\mathrm{H}^0(\mathcal{J}_{X'}(n))$. Thus ϕ_X is a Cremona transformation, of degree n, whose base locus scheme is X'. We need to prove that the base locus scheme of ϕ_X^{-1} is smooth and also belongs to $\mathcal{U}_n$. Then ϕ_X maps V to a hyperplane and one expects that it maps S to an element of $\mathbb{P}\mathrm{H}^0(\mathcal{J}_{X'}(n+1))$, from which it follows that $Y = \phi_X(V) \cap \phi_X(S)$.

Acknowledgements:

1) I thank Igor Dolgachev for having asked me about the existence of a Cremona transformation of $\mathbb{P}^3$ transforming an ACM curve of degree 6 and genus 3 onto another one; that question origined this paper.

2) I thank Francesco Russo, Ciro Ciliberto and Fernando Cuckierman for useful conversations.

2. Some remarks on codimension 2 determinantal varieties

Let $\mathbb{P}^n$ be the n-dimensional projective space over the field $\mathbb{C}$ of complex numbers. A Cremona transformation on $\mathbb{P}^n$ is a birational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. We denote the *base locus scheme* of ϕ by $\mathrm{Base}(\phi)$. We can write

$$\phi = (f_0 : \ldots : f_n),$$

where $f_0, \ldots, f_n \in \mathbb{C}[X_0, \ldots, X_n]$ are forms without common factors, of same degree, denoted by $\deg \phi$; the integer number $\deg \phi$ is the *degree* of ϕ. If $Z \subseteq \mathbb{P}^n$ is an irreducible variety not contained in $\mathrm{Base}(\phi)$, we denote by $\widetilde{\phi}(Z)$ the closure of $\phi(Z \backslash \mathrm{Base}(\phi))$ and call it the *strict transform* of Z by ϕ.

A linear system on $\mathbb{P}^n$ is the projective space associated to a vector space of forms on $\mathbb{C}^{n+1}$ of a certain fixed degree. We say that a linear system Λ is *homaloidal* when it has dimension n and the associated rational map

$$\phi_\Lambda : \mathbb{P}^n \dashrightarrow \Lambda^\vee$$

is a birational map. Clearly, every Cremona transformation is associated to a homaloidal linear system.

Let $X \subset \mathbb{P}^n$ be an ACM codimension 2 subscheme. There is a minimal resolution of the form

$$0 \longrightarrow \sum_{j=1}^{n} \mathcal{O}_{\mathbb{P}^n}(-n_j) \xrightarrow{\mu} \sum_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(-d_i) \longrightarrow \mathcal{J}_X \longrightarrow 0\ , \tag{2}$$

where μ correspond to a matrix M_X whose entries are (zero or) forms of degree $n_j - d_i > 0$.

When X is the base locus scheme of a Cremona transformation and $n \in \{3, 4, 5\}$ we know ([PR, Thm. 1.8]) that $d_i = n, n_j = n + 1$, for all i, j. Conversely, suppose that we have a minimal resolution as above with $d_i = n, n_j = n + 1$, and n arbitrary. The ideal sheaf $\mathcal{J}_X$ is generated by its global sections in degree n; more precisely ([PS, §3]), the set of maximal minors $\Delta_0, \ldots, \Delta_n$ of M_X is a minimal set of generators of it. Let $\mathbb{H}\mathrm{ilb}$ denote the Hilbert scheme attached to the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^n}/\mathcal{J}_X$. The set of subschemes of $\mathbb{P}^n$ which have a resolution as in equation (1) form an open and connected subset of $\mathbb{H}\mathrm{ilb}$ of dimension $n^3 - n$: see [Ell, Thm.1 and Thm. 2] or [Pa, Cor. 2.1]. We denote this open set by $\mathcal{U}_n$; we also write simply $X \in \mathcal{U}_n$.

Therefore we now have a codimension 2 flat family $\mathcal{X} \subset \mathbb{P}^n_{\mathcal{U}_n} = \mathbb{P}^n \times \mathcal{U}_n$ and a resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_{\mathcal{U}_n}}(-n-1)^n \xrightarrow{\mu} \mathcal{O}_{\mathbb{P}^n_{\mathcal{U}_n}}(-n)^{n+1} \longrightarrow \mathcal{J}_{\mathcal{X}} \longrightarrow 0\ . \tag{3}$$

For the following result we adapt the deformation trick found in [PR, Thm.1.8]. Moreover, the same idea was used in [GSP, Thm1] to show that the so-called multidegrees of a Cremona transformation ϕ, whose base locus scheme is generically reduced and belongs to $\mathcal{U}_n$, are exactly the binomial coefficients

$$\binom{n}{k}, k = 1, \ldots, n;$$

in particular, ϕ and its inverse ϕ^{-1} have degree n in this case (we will use this in the proof below). We also observe that an easy (and known) exercise shows that if ϕ may be defined using the maximal minors of an $(n+1) \times n-$matrix of linear forms, perhaps not relatively primes, then its inverse map ϕ^{-1} has the same property.

Proposition 1. *Let $\mathcal{B}_n \subset \mathcal{U}$ be the open set consisting of the subschemes $X \in \mathcal{U}_n$ whose associated linear system $\Lambda_X := \mathbb{P}(\mathrm{H}^0(\mathcal{J}_X(n)))$ defines a dominant rational map. Then there is a family of Cremona transformation*

$$\begin{tikzcd} \mathbb{P}^n_{\mathcal{B}_n} \arrow[rr, dashed, "\Phi"] \arrow[dr] & & \mathbb{P}^n_{\mathcal{B}_n} \arrow[dl] \\ & \mathcal{B}_n & \end{tikzcd}$$

where we identify the dual of $\mathbb{P}\mathrm{H}^0(\mathcal{J}_\mathcal{X}(n))$ with $\mathbb{P}^n_{\mathcal{B}_n}$ (i.e. the map $\Phi_u : \mathbb{P}^n \times \{u\} \dashrightarrow \mathbb{P}^n \times \{u\}$ is birational if $u \in \mathcal{B}_n$). Moreover, if $u \in \mathcal{B}_n$ corresponds to a generically reduced scheme then $\mathrm{Base}(\Phi_u^{-1}) \in \mathcal{B}_n$.

Proof. The existence of a rational map $\Phi : \mathbb{P}^n_{\mathcal{B}_n} \dashrightarrow \mathbb{P}^n_{\mathcal{B}_n}$ is clear. For the first assertion it suffices then to prove that, for each $X \in \mathcal{B}_n$, the linear system Λ_X is homaloidal.

Denote by M_0 the matrix of linear forms whose maximal minors define the *Standard Cremona Transformation*

$$S_n = (\frac{1}{X_0} : \cdots : \frac{1}{X_n}).$$

The maximal minors of the parametric matrix $tM_0 + (1-t)M_X,\ t \in \mathbb{C}$, define a curve

$$T := (t \mapsto u_t)$$

when t varies in $\mathbb{C}$, after eliminating, if necessary, a finite number of values $t_1, \ldots, t_\ell$.

Let $x, s_n \in \mathcal{U}_n$ be the points of $\mathcal{U}_n$ associated to X and $\mathrm{Base}(S_n)$. Let $\mathcal{F}_T \subset \mathbb{P}^n_T := \mathbb{P}^n \times T$ be the flat family of the u_t's parameterized by $t \in T$; notice that x, s_n are regular points of T because this curve can be realized as a line in the projective space whose homogeneous coordinates are the coefficients of the entries of the $(n+1)\times n$-matrix coming from the sequence in (3) (see [PS, §6]). Consider the blowing-up $\pi : \widetilde{\mathbb{P}^n_T} \to \mathbb{P}^n_T$ of $\mathbb{P}^n_T$ with center $\mathcal{F}_T$ and let E be the exceptional divisor; denote by $\pi_t : \widetilde{\mathbb{P}^n_t} \to \mathbb{P}^n_t$ the corresponding blowing-up in level $t \in T$. Choose a general section $\mathcal{H} \in \pi^*\mathcal{O}_{\mathbb{P}^n_T}(n) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n_T}}(-E)$.

The family $(\mathcal{H})_t$ is flat over T and its members are the fibers of a dominant morphism, therefore the intersection number $\mathcal{H}^n$ is well defined. By the "conservation of number" (see [Fu, Cor. 10.2.1]) one has

$$(\mathcal{H})^n_x = (\mathcal{H})^n_{s_n}.$$

Now, the birationality of S_n is equivalent to $(\mathcal{H})^n_{s_n} = 1$, from which the birationality of Φ_x follows.

Finally, as we have remarked before, we know that $\deg(\Phi_u^{-1}) = \deg(\Phi_u) = n$, thus $\mathrm{Base}(\Phi^{-1}) \in \mathcal{B}_n$, completing the proof. $\square$

Corollary 2. *There exists a birational map* $\mathcal{U}_n \dashrightarrow \mathcal{U}_n$.

Proof. The family of inverse maps $\Psi := \Phi^{-1}$ defines a new family of Cremona transformations on $\mathbb{P}^n_{\mathcal{B}_n}$, such that

$$\Psi_{\mathrm{Base}(\Phi_u^{-1})} = \Phi_u^{-1},$$

for a generic $u \in \mathcal{B}_n$. The equation $\Phi \circ \Psi = Id$ defines a relation on $\mathbb{P}^n_{\mathcal{B}_n} \times \mathbb{P}^n_{\mathcal{B}_n}$; this induces a generically one to one correspondence on $\mathcal{B}_n \times \mathcal{B}_n$. $\square$

Lemma 3. *We have*

$$\dim \mathrm{H}^0(\mathcal{J}_X(k)) = \begin{cases} 0 & \text{if } k < n, \\ n+1 & \text{if } k = n, \\ n^2+n+1 & \text{if } k = n+1. \end{cases}$$

Proof. By [PR, Prop. 1.2] we only need to prove case $k = n+1$. From equation (2) we obtain an exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathcal{O}^n) \longrightarrow \mathrm{H}^0(\mathcal{O}^{n+1}(1)) \longrightarrow \mathrm{H}^0(\mathcal{J}_X(n+1)) \longrightarrow 0\ ,$$

from which the assertion follows. $\square$

Now observe that a generic $h \in \mathrm{H}^0(\mathcal{J}_X(n+1))$ is irreducible. In fact, $\mathrm{H}^0(\mathcal{J}_X(k)) = 0$ for $k < n$ and the set of degree $n+1$ forms, which are the product of a form of degree n and a linear one, is contained in a proper subvariety of $\mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n+1))$.

Fix an irreducible $g \in \mathrm{H}^0(\mathcal{J}_X(n+1))$ and denote by $S = V(g) \subseteq \mathbb{P}^n$ the subscheme defined by g. Take an irreducible element V of the homaloidal linear system Λ_X . Then the scheme-theoretical equality

$$S \cap V = X \cup X'$$

holds, where X' is linked to X and hence it is also a codimension 2 ACM scheme.

Remark 4. By Hartshorne's Connectedness Theorem, $X \cap X'$ is of codimension 3 in $\mathbb{P}^n$.

Lemma 5. *We have $X' \in \mathcal{U}_n$.*

Proof. As we know, $V = V(f)$ for a $f \in \mathrm{H}^0(\mathcal{J}_X(n))$. Then

$$g = \sum_i \lambda_i \Delta_i, \ f = \sum_i \mu_i \Delta_i,$$

where $\lambda_0, \dots, \lambda_n$ are linear forms and $\mu_0, \dots, \mu_n \in \mathbb{C}$.

The ideal sheaf $\mathcal{J}_{X'}$ is generated by the maximal minors of a matrix M' obtained from M_X by adding the $2 \times (n+1)$-matrix

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_n \\ \mu_0 & \mu_1 & \dots & \mu_n \end{pmatrix}.$$

The minors of M' are invariants, up to multiplication by a nonzero constant, by the natural (right) action of $\mathrm{GL}(n+1)$. Hence one may suppose that $\mu_0 = 1$ and $\mu_i = 0$ for $i = 1, \dots, n$. Thus we may replace M' by a matrix obtained from it by taking off the first column and the last row, and the proof is complete. $\square$

Proposition 6. *Let $X \in \mathcal{U}_n$ be. The following statements hold:*

a) $\deg X = n(n+1)/2$.

b) If $X' \in \mathcal{U}_n$ is linked to X then it is done by way of a complete intersection of hypersurfaces of degree n and $n+1$.

c) The set of $X' \in \mathcal{U}_n$ that are linked to X is constructible and irreducible of dimension $n^2 + n$.

d) If X is smooth and X' is general among the schemes as in (c), then $n \in \{3, 4, 5\}$ and X' is also smooth. Moreover, in this case, a general hypersurface $V \in \Lambda_X$ (resp. $S \in \mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n+1))$) is smooth at points of codimension 2 and points of codimension 3 have multiplicity ≤ 2; S is certainly singular if $n = 5$.

Proof. Assertion (a) follows directly from resolution (2). In particular, $\deg(X \cup X') = n(n+1)$; since $\mathrm{H}^0(\mathcal{J}_X(k)) = \mathrm{H}^0(\mathcal{J}_{X'}(k)) = 0$ if $k < n$ (Lemma 3), this prove assertion (b).

For the proof of (c), suppose

$$X \cup X' = S \cap V = S' \cap V',$$

scheme-theoretical, where S, S' have degree $n+1$ and V, V' have degree n; set $S = V(g), S' = V(g'), V = V(f)$, and $V' = V(f')$. Thus $X \cup X' \subseteq V \cap V'$, from which $V = V'$ by (a). We deduce that $g' \in (g, f)$, that is, there exist $\alpha \in \mathbb{C}$ and a linear form λ such that $g' = \alpha g + \lambda f$. Then the set

$$\{S' \in \mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n+1)) : S' \cap V = X \cup X'\}$$

is a projective space of dimension n.

Denote by $\mathbb{P}_1$ and $\mathbb{P}_2$ the projective spaces of forms on $(n+1)$ variables, vanish on X, of degree $n+1$ and n, respectively. The assertion (c) follows from a direct computation of dimensions involving the following commutative diagram, taking account of Lemma 3:

$$\begin{tikzcd}
\mathcal{U}_n \times \mathbb{P}_1 \times \mathbb{P}_2 \arrow[dr, "pr_1"] \arrow[drr, "pr_2"] & & \{(X', S, V) : S \cap V \supseteq X \cup X'\} \arrow[ll, hook'] \arrow[dl, "pr_1|"'] \arrow[d, "pr_2|"] \\
& \mathcal{U}_n & \mathbb{P}_1
\end{tikzcd} \tag{4}$$

Here pr_1, pr_2 denote the canonical projections and $pr_1|$, $pr_2|$ their restrictions to the incidence variety.

Finally, let us consider now the situation of smooth $X \in \mathcal{U}_n$. Since Λ_X is homaloidal, we deduce $n = 3, 4$ or 5 according to [ESB, Thm. 3.2]. Following [PS, Prop. 4.1, parts (4) and (6)] we may choose S and V general enough to have X', $X \cap X'$, $S \backslash (X \cap X')$ and $V \backslash (X \cap X')$ all smooth.

Arguing as above, we may take $V, V' \in \Lambda_X$ general enough to have Y, $X \cap Y$, $V' \backslash (X \cap Y)$ and $V \backslash (X \cap Y)$ all smooth, where $V \cap V' = X \cup Y$.

Fix a closed point $p \in X \cap Y$ (analogously for $X \cap X'$) and take a general 2-plane Π passing through p. The traces of V and V' on Π define curves C and C', respectively, passing through p. Since Π intersects X and Y transversely, we conclude that p is a point of length 2 in $\Pi \cap (X \cup X')$. Thus p can not be a common singular point of V and V' (respectively V and S). In particular:

- V does not have a singular point of codimension 2 since such a point corresponds to an irreducible component of $X \cap Y$ and is also singular for V', by Bertini's theorem; the same argument works for S;
- a point of codimension 3 of V has multiplicity ≤ 2: it corresponds to a subvariety of codimension 3 of V contained in $X \cap Y$.

In the case of S, that is a general determinantal variety, we need to show that singularities must appear in codimension 3 points.

Let $N = (n+1)^2 - 1$ be. Denote by $M_k \subset \mathbb{P}^N$, $k \leq n+1$, the determinantal variety of matrices of rank at most k; here $M_{n+1} = \mathbb{P}^N$. It is well known that M_k is the singular locus of M_{k+1} and that it has codimension $(n-k+1)^2$ in $\mathbb{P}^N$ (see for example [ACGH, Chap. II]). On the other hand, the matrix defining S induces an embedding

$$\mu : S \to M_n.$$

Therefore $\mathrm{Sing}(S) \supseteq \mu^{-1}(M_{n-1})$. We conclude the proof by considering the case $n = 5$.

$\square$

Remark 7. a) We do not know if singularities of S must appear in codimension 3, though the study of this kind of variety is a very classical subject.

b) The indeterminancy of the linear system $\mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n+1))$ is resolved by blowing up $\mathbb{P}^n$ along X: indeed, if $\Lambda_X = \mathbb{P}(\langle f_0, \dots, f_n \rangle)$, this holds for the linear subsystem

$$\mathbb{P}(\langle x_0 f_0, \dots, x_n f_0, \dots, x_0 f_n, \dots, x_n f_n \rangle) \subseteq \mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n+1)),$$

because the homogeneous ideals $(x_0 f_0, \dots, x_n f_0, \dots, x_0 f_n, \dots, x_n f_n)$ and $(f_0, \dots, f_n)$ have the same saturation.

3. The main results

Definition 1. *Let $X_1, X_2 \subseteq \mathbb{P}^n$ be two subschemes. We say that the pairs $(\mathbb{P}^n, X_1)$ and $(\mathbb{P}^n, X_2)$ are birational equivalent if there exists a commutative diagram*

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dashed, "\phi"] & \mathbb{P}^n \\ X_1 \arrow[u, hook] \arrow[r, dashed, "\varphi"] & X_2 \arrow[u, hook] \end{tikzcd}$$

where $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ and $\varphi : X_1 \dashrightarrow X_2$ are birational maps.

We denote by $\mathrm{Bir}(X)$ the group of birational maps from X to X, and by $\mathrm{Aut}(X)$ the subgroup of automorphism of X.

Let W be a normal projective variety. A Weil canonical divisor K_W of W is the closure of a canonical divisor of $W \setminus \mathrm{Sing}(W)$. Suppose there exists a positive $m \in \mathbb{Z}$ such that mK_W is a Cartier divisor, and that for some desingularization $\pi : Z \to W$ there exists a divisor F, with $\pi_*(F) = 0$, such that $mK_Z \sim \pi^*(mK_W) + F$. Under these hypotheses, we say that W has *canonical singularities* if F is effective (see [De, §7.2] or [CCKM, §2.3]); when $F = 0$ the desingularization is said to be *crepant* (see [CCKM, Def. 6.22]).

We now state and prove our main results.

Theorem 8. *Given $n \in \{3,4,5\}$, let $X \in \mathcal{U}_n$ be a smooth codimension 2 subvariety; for $n > 3$ we suppose in addition that X is in general position. Then there exists a Cremona transformation $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that:*

a) the restriction of ϕ to X induces a birational morphism $\eta : X \to Y$, where Y is a hyperplane section of a determinantal hypersurface of degree $n+1$;

b) η is an isomorphism in codimension 1, which is an isomorphism if $n = 3, 4$ and, at worst, a crepant resolution of a finite number of double points if $n = 5$.

Moreover, in the last case, over each double point there is a unique rational curve.

Theorem 9. *Given $n \in \{3,4,5\}$, let $X \in \mathcal{U}_n$ be a smooth codimension 2 subvariety; for $n > 3$ we suppose in addition that X is in general position. Then* $\mathrm{Aut}(X) = \mathrm{Bir}(X)$.

Theorem 10. *Given $n \in \{3,4,5\}$, let $X_1, X_2 \in \mathcal{U}_n$ be two smooth codimension 2 subvarieties; for $n > 3$ we suppose in addition that X_1 and X_2 are in general position. The following statements are equivalent:*
a) $(\mathbb{P}^n, X_1)$ and $(\mathbb{P}^n, X_2)$ are birational equivalents.
b) X_1 and X_2 are birational equivalents.
c) X_1 and X_2 are isomorphic.

For the proof of Theorem 10 we need some technical lemmas; along the way we also prove Theorem 8 and then Theorem 9. First we explain our main argument.

Main construction. Fix $X \in \mathcal{U}_n$ smooth and general, $n \in \{3,4,5\}$. Take general elements, V in the homaloidal linear system Λ_X, and $S \in \mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n+1))$. Then $S \cap V = X \cup X'$ with $X' \in \mathcal{U}_n$ being general too; in particular it is smooth. Therefore there exists a Cremona transformation $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ such that $\mathrm{Base}(\phi) = X'$ and with $X'' := \mathrm{Base}(\phi^{-1}) \in \mathcal{U}_n$ being smooth.

Conversely, let $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a determinantal Cremona transformation of degree n that is general, in the sense that $\mathrm{Base}(\phi)$ is a general element in $\mathcal{U}_n$, and with $\mathrm{Base}(\phi^{-1}) \in \mathcal{U}_n$ smooth. Hence we may *link* ϕ to elements $X \in \mathcal{U}_n$ in general position.

The strategy to prove Theorem 8 is to restrict to X one of its *linked* Cremona transformations, in order to transform it into a hyperplane section of a general determinantal hypersurface of degree $n+1$.

As we saw, $\Lambda_{X'}$ is also homaloidal (Proposition 1); note $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ a Cremona transformation with associated linear system $\Lambda_{X'}$. Hence the strict transform of V by ϕ is a hyperplane $H \subseteq \mathbb{P}^n$.

Lemma 11. *Let $n \in \{3,4,5\}$. The map that associates the strict transform $\widetilde{\phi}(S)$ to a general $S \in \mathbb{P}\mathrm{H}^0(\mathcal{J}_{X'}(n+1))$ induces an isomorphism*

$$\mathbb{P}\mathrm{H}^0(\mathcal{J}_{X'}(n+1)) \simeq \mathbb{P}\mathrm{H}^0(\mathcal{J}_{X''}(n+1))$$

.

Proof. Let $p : Z \to \mathbb{P}^n$ be the blowup of $\mathbb{P}^n$ along X'; denote by E its exceptional divisor. We have a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\phi"] & & \mathbb{P}^n \end{tikzcd} \tag{5}$$

where q is a birational morphism defined by the complete linear system $|p^*\mathcal{O}(n) \otimes \mathcal{O}(-E)|$. Since $X'' = \{y \in \mathbb{P}^n; \dim q^{-1}(y) \geq 1\}$ is smooth, it follows from [ESB, Thm. 1.1 and lem. 2.4] that q is the blowup of $\mathbb{P}^n$ along X'' and $q(E)$ is an irreducible variety of degree n^2-1; this variety is the *jacobian* of ϕ^{-1}, that is, the hypersurface defined by the jacobian determinant associated to the $(n+1)$ polynomials defining ϕ^{-1}.

Let S be a general hypersurface of degree $n+1$ containing X; it is singular at points that have codimension 2 in X (Proposition 6d). If $H \subseteq \mathbb{P}^n$ is a general hyperplane, then p^*S is linearly equivalent to $(n+1)p^*H+E$, from which we conclude that q_*p^*S is linearly equivalent to a hypersurface of degree $(n+1)n-(n^2-1) = n+1$ containing the base locus scheme of ϕ^{-1}. □

Remark 12. We conclude that the strict transform of X by ϕ is a hypersurface of $H = \mathbb{P}^{n-1}$ of degree $n+1$. We denote this variety by $Y = Y_X \subseteq \mathbb{P}^{n-1}$; observe that $(\mathbb{P}^n, X)$ is then birationally equivalent to $(\mathbb{P}^n, Y)$.

Proof of Theorem 8.

The case $n=3$ is an easy consequence of Lemma 11 and Remark 12, because Y is a quartic plane curve of geometric genus 3 (therefore smooth). In what follows, we assume $n > 3$ and keep all notations from Lemma 11 and its proof.

The commutative diagram in equation (5) gives also a resolution of indeterminancies of the inverse map ϕ^{-1} of ϕ; let $F = q^{-1}(X'')$ denote the exceptional divisor of q. As we know, $q : Z \to \mathbb{P}^n$ is the blowup of the smooth variety X''.

Denote by $\widetilde{X}, \widetilde{S} \subseteq Z$ the strict transforms by p^{-1} of X and S, respectively; we may suppose that $\widetilde{S}$ is smooth, since p resolves the indeterminancy of the linear system $\mathbb{P}\mathrm{H}^0(\mathcal{J}_X(n+1))$ (Remark 7b)).

Since $X \cap X'$ has codimension 1 in X (Remark 4), p induces an isomorphism from $\widetilde{X}$ onto X; then it suffices to show that q restricted to $\widetilde{X}$ gives a birational morphism $\eta : \widetilde{X} \to Y$ satisfying the required properties.

When we restrict q to $\widetilde{X} \backslash (F \cap \widetilde{X})$, we obtain an isomorphism onto $Y \backslash (Y \cap X'')$.

Now recall that q induces, by restriction, a projective line bundle $q : F \to X''$.

Since X is general in $\mathcal{U}_n$, by Proposition 6d and Lemma 11 we may also assume $W := \widetilde{\phi}(S) = q(\widetilde{S})$ to be smooth at codimension 2 points and to have a finite number of double points of codimension 3 (eventually nonempty for $4 \leq n \leq 5$); in particular, the singular set $\mathrm{Sing}(W)$ of W may intersect Y only for $n = 5$ and does it, at worst, in a finite number of (closed) double points: indeed, since X is general, we may suppose S and V to be generals containing X', and then Y is the intersection of W with a general hyperplane. Denote $U'' := X'' - (X'' \cap \mathrm{Sing}(W))$.

On the other hand, the canonical exact sequence of vector bundles

$$0 \longrightarrow T_{U''} \longrightarrow T_W|_{U''} \longrightarrow N_{U''}W \longrightarrow 0$$

associated to the normal line bundle $N_{U''}W$ of U'' in W, induces, after projectivization, a section of the projective line bundle $q : q^{-1}(U'') \to U''$. By construction, the closure of the image of this section is $\widetilde{S} \cap F$. Since $\widetilde{X} = \widetilde{q^{-1}}(Y) \subseteq \widetilde{S}$, we conclude that $q|_{\widetilde{X} \cap F} : \widetilde{X} \cap F \to \widetilde{X''} \cap Y$ is an isomorphism if $n = 4$ and contracts a fiber of F onto each double point of Y if $n = 5$.

Combining this result with that of the first part of the proof we obtain a birational morphism $\eta := q|_{\widetilde{X}} : \widetilde{X} \to Y$ that is essentially the one which we want; it remains to prove that, for $n = 5$, the resolution η is crepant.

First we observe that the Weil canonical divisor K_Y of Y is itself a Cartier divisor: since Y is a hypersurface of degree 6 in $\mathbb{P}^4$ the divisor K_Y is the closure of a hyperplane section H_Y; we can then move H_Y and assume it lies in $Y \backslash \mathrm{Sing}(Y)$.

Finally, as η induces an isomorphism from $\widetilde{X} \backslash \eta^{-1}(Y \backslash \mathrm{Sing}(Y))$ onto $Y \backslash \mathrm{Sing}(Y)$ and $\eta^{-1}(\mathrm{Sing}(Y))$ is a reunion of a finite number of lines, we have $K_{\widetilde{X}} = \eta^* K_Y$, completing the proof.

$\square$

Remark 13. As it is shown in [ESB, Prop. 2.3] the image by p of the fibers $F_{x''}$ of $q|_F : F \to X''$, $x'' \in X''$, are the n-secant lines of X. Then, for $n = 5$, we reach, eventually, a finite number of n-secant lines of X', contained in X, which will be contracted by ϕ to double points of Y. Therefore, the existence of sectional singularities on S depend on the existence of n-secant lines to X', contained in X (c.f. remark 7a))

We have the following lemma:

Lemma 14. *Let $m \geq 3$ be a positive integer. Let $Y_1, Y_2 \subseteq \mathbb{P}^{m-1}$ be irreducible hypersurfaces of the same degree $r \geq m+1$ and $\varphi : Y_1 \dashrightarrow Y_2$ a birational map. Assume that Y_1 and Y_2 are smooth if $m \leq 4$ and have at most isolated canonical singularities if $m \geq 5$. Then φ is an isomorphism. Moreover, if $r = m+1 \leq 5$ or $m \geq 5$, the morphism φ extends to a linear automorphism of $\mathbb{P}^{m-1}$.*

In the case of smooth hypersurfaces, the lemma above follows, as a particular case, from a theorem of Severi ([Se]), which it was generalized by C. Ciliberto in [Ci], for the case of Castelnouovo varieties. As we will see, we will use the lemma in a situation which is essentially covered by the Ciliberto's Theorem [Ci, Thm. 3.1.1]; however, and for the convenience of the reader, we will give a complete proof, which leads with a slightly more general situation, though only for hypersurfaces.

Proof. The case $m = 3$ is clear. Suppose that $m \geq 4$ and observe that the dualizing sheaf $\omega^o_{Y_i}$ of Y_i is $\omega = \mathcal{O}_{Y_i}(\ell) = \mathcal{O}(\ell)$, where $\ell = r - m \geq 1$, for $i = 1, 2$.

Case 1. $m = 4$. Then, for $i = 1, 2$, $Y_i \subseteq \mathbb{P}^3$ is a smooth surface of general type. By Kodaira's Vanishing Theorem,

$$\mathrm{H}^1(\omega^\vee) = \mathrm{H}^1(\omega^{\otimes 2}) = 0$$

and it follows that Y_i is minimal for $i = 1, 2$ (see [BPV, Chap. VII, Pro.5.5]). Hence φ is well defined (see [Ba, Thm.10.21]).

Moreover, if $r = 5$, the Riemann-Roch Theorem implies

$$h^0(\omega) - h^1(\mathcal{O}) + h^0(\mathcal{O}) = 5.$$

From the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O} \longrightarrow 0$$

we conclude that $h^1(\mathcal{O}) = 0$, and therefore $h^0(\omega) = h^0(\mathcal{O}(1)) = 4$. This means that an isomorphism $\varphi : Y_1 \to Y_2$ comes from a linear isomorphism of $\mathbb{P}^3$.

Case 2. $m \geq 5$. Then, for $i = 1, 2$, $Y_i \subseteq \mathbb{P}^4$ is a $(m-2)$-fold of degree $r \geq m+1$.

Let H_i be a hyperplane section of Y_i, $i = 1, 2$. By the Grothendieck-Lefschetz Theorem (see, for example, [Ha1, Chap. IV, Cor. 3.2]), $\mathrm{Pic}(Y_i) = \mathbb{Z}[H_i]$, where $[H_i]$ is the hyperplane class of Y_i. Denote by K_i a divisor associated to the dualizing sheaf $\omega^o_{Y_i}$, $i = 1, 2$; then K_i is linearly equivalent to ℓH_i. Observe that, by construction, K_i is the Weil canonical divisor K_{Y_i} of Y_i; then it is a Cartier divisor.

On the other hand, take a resolution of indeterminancies of φ that resolves also the singularities of Y_1, i.e. a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y_1 \arrow[rr, dashed, "\varphi"] & & Y_2 \end{tikzcd}$$

where p and q are birational morphisms and Z is smooth. Take a canonical divisor K_Z of Z and $H_Z = q^*H_2$. Since Y_i has at most canonical singularities, we obtain $K_Z = q^*K_2 + E_q = p^*K_1 + E_p$, where E_q and E_p are effective divisors such that $q_*(E_q) = 0$ and $p_*(E_p) = 0$.

Now, we have $p_*q^*[H_2] = d[H_1]$ for a positive integer d; it is the class of the strict transform H_0 of H_2 by φ^{-1}. In particular, $H_Z = p^*H_0 - F_p$, where F_p is an effective divisor with $p_*(F_p) = 0$.

Hence we have the following numerically equivalent relations:

$$\begin{aligned} K_Z - \ell H_Z &\equiv q^*(K_2 - \ell H_2) + E_q \\ &\equiv E_q \\ &\equiv p^*(K_1 - \ell H_0) + E_p + \ell F_p \\ &\equiv p^*\left((\ell - d\ell)H_1\right) + E_p + \ell F_p \end{aligned}$$

Applying p_* we finally obtain that $(\ell - d\ell)H_1$ is numerically equivalent to an effective divisor, from which $d \leq 1$ and then $d = 1$. Finally, since a general section of Y_i is a smooth complete intersection, it is linearly normal (see [Ha2, Chap. II, Exa. 8.4]); hence it is also true for Y_i. This completes the proof of Lemma 14. □

Proof of Theorem 9. For $n = 3, 4$ the theorem follows directly from Theorem 8 and Lemma 14. For $n = 5$, the same results give a birational map $\varphi : X \dashrightarrow X$ that is an isomorphism in codimension 1, and which extends, by construction, to an automorphism of the crepant resolution $\eta : X \to Y$. □

Proof of theorem 10. The assertions (a) implies (b) and (c) implies (b), are trivial. Moreover, (b) implies (c) follows directly from Theorem

8 and Lemma 14 in the same way as we proved Theorem 9. Therefore it suffices to show that (b) implies (a).

There are Cremona transformations $\phi_1, \phi_2 : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ inducing, via restriction to X_1 and X_2 respectively, birational maps to hypersurfaces $Y_1, Y_2 \subseteq \mathbb{P}^{n-1}$ (Theorem 8); in particular, Y_1 and Y_2 are birationally equivalents.

Now, by Lemma 14, a birational map from Y_1 to Y_2 extends to a linear automorphism of $\mathbb{P}^{n-1}$; hence it extends to a linear automorphism of $\mathbb{P}^n$, which we denote by ρ. Thus, $\phi_2^{-1} \circ \rho \circ \phi_1$ is a Cremona transformation mapping X_1 onto X_2, birationally. This completes the proof of Theorem 10. $\square$

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