ON CHARACTERISTIC CLASSES OF DETERMINANTAL CREMONA TRANSFORMATIONS

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1. Introduction

Let F be a birational map of degree n of the projective space $\mathbb{P}^n$ over an algebraically closed field $\mathbb{K}$, given by the maximal minors of a $n \times (n+1)$ matrix with general linear forms as entries. These *determinantal Cremona transformations* may be defined by n correlations in general position and they have been considered in the classic literature on Cremona transformations by several authors, e. g. [2], [1] and [13, Chap. VIII, §4].

On the other hand, the family of base schemes of determinantal maps (not necessary birational) may be identified as an open and connected subset of the Hilbert scheme of the 2-codimensional arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^n$ (see [4]).

In this article we compute the *multidegrees* of such F and the *Segre classes* of its base scheme B_F, by specializing to the *standard Cremona transformation*

$$S_n := (X_1X_2\cdots X_n : \cdots : X_0\cdots \widehat{X_i}\cdots X_n : \cdots : X_0X_1\cdots X_{n-1})$$

and by applying methods of toric geometry. In this way this translates into computing mixed volumes of some special polytopes with integer vertices.

The sequence of multidegrees $(d_0, \ldots, d_k, \ldots, d_n)$, classically called the "type" of a Cremona transformation $T : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, are given by the degrees of the (direct) strict transforms by T of general k-dimensional linear subvarieties H^{n-k} of $\mathbb{P}^n$; for a reference on this "type" via intersection theory see [14]. The multidegrees are closely related (Proposition 5) to the *Segre class* of T, $s(B_T, \mathbb{P}^n)$, defined as the inverse of the Chern class of the normal bundle of the embedding $i : B_T \hookrightarrow \mathbb{P}^n$ of the base scheme B_T of T, if B_T is regulary embedded. The general definitions are given later. This Segre class lives in the Chow group $A_\bullet(B_T)$ of B_T ; we also consider its image $s(B_T)$ in $A_\bullet(\mathbb{P}^n)$ and the *Segre numbers* $s_k = \int s(B_T) \cdot H^k$.

The main results are the following:

Theorem 1. *The determinantal Cremona transformations of $\mathbb{P}^n$ with generically reduced base scheme and the standard Cremona transformation S_n have the same multidegrees and Segre numbers.*

Theorem 2. *Let n be an integer, $n \geq 2$. The multidegrees of the standard Cremona transformation S_n are equal to the binomial coefficients:*

$$d_k = \binom{n}{k},\ 0 \leq k \leq n.$$

Theorem 3. *The image of the Segre class of S_n in the Chow group $A_\bullet(\mathbb{P}^n)$ is*

$$s(B) := i_* s(B, \mathbb{P}^n) = \sum_{k=0}^{n-2} s_k [H^{n-k}]$$

with Segre numbers

$$\begin{aligned} s_k &= (-1)^{n-k-1} \int_{\mathbb{P}^n} \pi_*(E^{n-k}) \cdot H^k \\ &= (-1)^{n-k-1} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} \binom{n}{j} n^{n-k-j} \end{aligned}$$

for $0 \leq k \leq n-2$, where $\pi : \mathbb{B}l_B(\mathbb{P}^n) \to \mathbb{P}^n$ is the blowing-up of $\mathbb{P}^n$ with center the base scheme $B = B_{S_n}$ and where E is the exceptional divisor of π.

Corollary 4. *The Chow group of the base scheme B of S_n is*

$$A_k(B) = \begin{cases} \mathbb{Z}[H^{n-k}]_B & \text{for } 0 \leq k \leq n-3 \\ \bigoplus_{i=1}^{n(n+1)/2} \mathbb{Z}[\alpha_i]_B & \text{for } k = n-2 \end{cases}$$

where the α_i are the irreducible components of the reduced base scheme $|B|$ of S_n, the 2-codimensional skeleton of the arrangement of the fundamental hyperplanes of $\mathbb{P}^n$. With the above notations, the Segre class of S_n is

$$s(B, \mathbb{P}^n) = \sum_{k=0}^{n-3} s_k [H^{n-k}]_B + \big([\alpha_1]_B, \dots, [\alpha_{n(n+1)/2}]_B\big)$$

where the s_k are the Segre numbers of Theorem 3.

The multidegrees of the standard Cremona transformation S_n are well known for small dimensions as the "types" of S_n. We didn't find a classical proof for higher dimensions. We thank I. Dolgachev for his advice on this subject. Finally we thank F. Russo for useful conversations on conservation of numbers for flat deformations.

2. Segre classes and multidegrees

Let $F = (F_0 : \cdots F_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map. Denote by $B = B_F$ its base scheme, the subscheme of $\mathbb{P}^n$ defined by the homogeneous ideal $(F_0, \dots, F_n)$, of codimension at least two. Let $\pi : \mathbb{B}l_B(\mathbb{P}^n) \to \mathbb{P}^n$ be the blowing-up of $\mathbb{P}^n$ with center B, and $E := \pi^{-1}(B)$ its exceptional divisor.

This blowing-up resolves the indeterminacies of F and we get a commutative diagram

$$\begin{array}{ccccc} E & \hookrightarrow & \mathbb{B}l(\mathbb{P}^n) & & \\ \downarrow{\scriptstyle \pi|_E} & & \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \varphi} & \\ B & \overset{i}{\hookrightarrow} & \mathbb{P}^n & \overset{F}{\dashrightarrow} & \mathbb{P}^n \end{array}$$

The Segre class of B in $\mathbb{P}^n$ is defined as the Segre class in the Chow group $A_{\bullet}(B)$ of the normal cone $C_B(\mathbb{P}^n)$; it is equal to

$$s(B,\mathbb{P}^n) := \sum_{i\geq 1}(-1)^{i-1}(\pi|_E)_*(E^i)$$

(see [6, Cor. 4.2.2]). We consider its image in $A_{\bullet}(\mathbb{P}^n)$,

$$s(B) := i_* s(B,\mathbb{P}^n) = \sum s_k[H^{n-k}]$$

where the $s_k = \int_{\mathbb{P}^n} s(B)\cdot H^k$ are integers called the *Segre numbers* of F. The Segre classes and numbers are invariant under birational morphisms ([6, Prop. 4.2]).

The *multidegrees* d_k of F, $0 \leq k \leq n$, defined as the degrees of the (direct) strict transforms by F of general linear subvarieties H^{n-k} of dimension k of $\mathbb{P}^n$, may be computed via a resolution of indeterminacies of F.

Let H_1 (resp. H_2) be the (total) pullback of a hyperplane H by π (resp. by φ). Then

$$d_k = d_k(F) = \int_{\mathbb{B}l_B(\mathbb{P}^n)} H_1^{n-k}\cdot H_2^k.$$

Let $\partial = \deg(F) := \deg(F_i), \forall\, i$; then $\varphi^*\mathcal{O}(1) = \pi^*\mathcal{O}(\partial)\otimes\mathcal{O}(-E)$. It follows $H_2 \sim \partial H_1 - E$.

Proposition 5. *The Segre numbers s_k and the multidegrees d_k of a rational map F are related by:*

(i) $d_0 = 1$, $d_1 = \partial$, and $d_k = \partial^k - \sum_{0\leq \ell\leq k}\binom{k}{\ell}\partial^{k-\ell}s_{n-\ell}$ for $2 \leq k \leq n$.

(ii) $s_k = (-1)^{n-k-1}\sum_{0\leq\ell\leq n-k}(-1)^{\ell}\binom{n-h}{\ell}\partial^{n-k-\ell}d_\ell$.

Proof. (i) The equality follows from

$$d_k = \int_{\mathbb{B}l_B(\mathbb{P}^n)} H_1^{n-k}\cdot H_2^k = \int_{\mathbb{B}l_B(\mathbb{P}^n)} H_1^{n-k}\cdot(\partial H_1 - E)^k,$$

$$s_{n-\ell} = (-1)^{\ell-1}\int_{\mathbb{P}^n} H_1^{n-\ell}\cdot\pi_*(E^\ell)\ ,\quad \int_{\mathbb{B}l_B(\mathbb{P}^n)} H_1^n = 1$$

and the projection formula.

(ii) From (i) we have

$$(d_0,\ldots,d_n)^t = M(-1,0,s_{n-2},s_{n-3},\ldots,s_0)^t,$$

where $M = (a_{k\ell})$ is the lower triangular matrix with entries $a_{k\ell} = -\binom{k}{\ell}\partial^{k-\ell}$ if $0 \leq \ell \leq k \leq n$. It follows that the inverse matrix $M^{-1} = (b_{k\ell})$ is given by

$b_{k\ell} = (-1)^{1+k+\ell}\binom{k}{\ell}\partial^{k-\ell}$ if $0 \leq \ell \leq k \leq n$ and 0 otherwise. It is enough to check that $\chi(n,k) := \sum_{k\leq\ell\leq n}(-1)^\ell\binom{n}{\ell}\binom{\ell}{k} = 0$ for $k < n$; this follows from $\chi(n,k) = -\chi(n-1,k-1)$ by Stieffel formula and by induction on k since $\chi(n,0) = 0$ for $n > 0$. □

3. General determinantal and standard Cremona transformations

If F is a determinantal Cremona transformation, according to [4] B_F may be considered as a point of an open connected and smooth set of the Hilbert scheme parameterizing the arithmetically Cohen-Macaulay schemes of codimension 2 whose sheaf ideals admit a minimal resolution as the following (see [4, Thm. 2(ii)]). Recall that for the base scheme $B = B_F$ under consideration there exists a minimal resolution

$$0 \longrightarrow \mathcal{O}^n_{\mathbb{P}^n}(-n-1) \xrightarrow{M_F} \mathcal{O}^{n+1}_{\mathbb{P}^n}(-n) \longrightarrow \mathcal{I}_B \longrightarrow 0 \qquad (1)$$

where $\mathcal{I}_B$ denotes the ideal sheaf of B and M_F is the $(n+1)\times n$ matrix of linear forms attached to F.

The standard Cremona transformation is determinantal: it may be defined by the maximal minors of the matrix $M_{S_n} = (m_{ij})$ with entries given by

$$m_{ij} = \begin{cases} \delta_{ij}X_{i-1} & \text{if } j < n+1 \\ -X_n & \text{if } j = n+1, i = 1,\dots,n \end{cases}$$

By the Peskine-Szpiro deformation theorem [12, Thm. 6.2] there exists a dense open set S of a projective space over $\mathbb{K}$ and a S-scheme X of codimension 2 in $\mathbb{P}^n_S = \mathbb{P}^n \times S$, flat over S, such that

(a) the ideal sheaf $\mathcal{I}_X$ of X has a minimal resolution

$$0 \longrightarrow \mathcal{O}^n_{\mathbb{P}^n_S}(-n-1) \longrightarrow \mathcal{O}^{n+1}_{\mathbb{P}^n_S}(-n) \longrightarrow \mathcal{I}_X \longrightarrow 0 \qquad (2)$$

(b) for each codimensional 2 scheme B having a minimal resolution as in (1) there is a unique point $s \in S$ with $X(s) = B \times \{s\}$ such that the minimal resolution of the associated ideal sheaf is obtained from (2) by tensorizing with $\mathbb{K}(s)$ over $\mathcal{O}_S$.

Proof of **Theorem 1** (stated in the introduction):

Suppose that F is a general determinantal Cremona transformation in the sense of the statement ; with the above notations let f and s_n in S be the points corresponding to the base schemes of F and S_n respectively in the Hilbert scheme. Let T be the intersection of S with the line joining f and s_n.

For $s \in S$ let $B_s \subset \mathbb{P}^n_s = \mathbb{P}^n \times \{s\}$ the corresponding arithmetically Cohen-Macaulay scheme of codimension 2.

The set of $s \in T$ such that B_s is generically reduced contains a dense open set, since B_f and B_{s_n} are generically reduced.

Let $\mathcal{X}_T \subset \mathbb{P}^n_T := \mathbb{P}^n \times T$ be the flat family of the B_t's parameterized by $t \in T$. Consider the blowing-up $\pi : \widetilde{\mathbb{P}^n_T} \to \mathbb{P}^n_T$ of $\mathbb{P}^n_T$ with center $\mathcal{X}_T$, let E be the exceptional divisor; denote by $\pi_t : \widetilde{\mathbb{P}^n_t} \to \mathbb{P}^n_t$ the corresponding blowing-up in level $t \in T$. Choose general sections $\mathcal{H}_1 \in \pi^*\mathcal{O}_{\mathbb{P}^n_T}(n)$ and $\mathcal{H}_2 \in \pi^*\mathcal{O}_{\mathbb{P}^n_T}(n) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^n_T}}(-E)$.

The family $(\mathcal{H}_2)_t$ is flat over T (see [6, App. B.6.7]). For general $t \in T$, since the divisor $\pi_t^*((\mathcal{H}_1)_t)$ is regular in codimension 1 then it is normal. It follows that $(\mathcal{H}_1)_t$ is a generically flat family over T.

By the "conservation of number" (see [6, Cor. 10.2.1]) one has

$$(\mathcal{H}_1)^k_f \cdot (\mathcal{H}_2)^{n-k}_f = (\mathcal{H}_1)^k_{s_n} \cdot (\mathcal{H}_2)^{n-k}_{s_n},$$

so we get the equality of multidegrees, and the equality of Segre numbers follows from the preceding Proposition. □

4. Fans and mixed volumes

Consider the projective space $\mathbb{P}^n$ over $\mathbb{K}$ as the toric variety defined by the fan Δ associated to the faces of the simplex with vertices $e_0, e_1, \dots, e_n$ where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$ and $e_0 := -\sum_{i=1}^n e_i$.

The central symmetry $-Id$ in $\mathbb{R}^n$ induces the standard Cremona transformation S_n viewed as a monomial birational map ([9]). A natural way to resolve the indeterminacies of S_n is to factorize it through the toric variety X_Σ associated to the fan given by the minimal common subdivision of Δ and $-\Delta$.

$$\begin{array}{ccc} & X_\Sigma & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X_\Delta & \overset{S_n}{\dashrightarrow} & X_{-\Delta} \end{array}$$

Intersection numbers for divisors in toric varieties may be computed as mixed volumes of Minkowsky sums of polytopes ([11], [6]), and we obtain:

Proof of **Theorem 2** (stated in the introduction):

The multidegrees are independent of the resolution of indeterminacies, so we may use the toric variety X_Σ. Let H_1 (resp. H_2) be the pullback by π_1 (resp. by π_2) of the hyperplane associated to the one-dimensional cone through e_0 (resp. through $-e_0$). It turns out that the polytope associated to H_1 (resp. to H_2) is the simplex $\delta_n = [0, e_1, \dots, e_n]$ (resp. the simplex $-\delta_n$). It follows that the intersection number $d_k = \int_{X_\Sigma} H_1^k \cdot H_2^{n-k}$ equals the mixed volume $V(\delta_n, k; -\delta_n, n-k)$; i.e. the coefficient of the monomial $\nu_1 \cdots \nu_n$ in the homogeneous polynomial $\mathrm{Vol}\left((\nu_1 + \cdots + \nu_k)\delta_n + (\nu_{k+1} + \cdots + \nu_n)(-\delta_n)\right)$.

Lemma 6. *Let a, b be non negative integers. Then*

$$\mathrm{Vol}(a\delta_n + b(-\delta_n)) = \sum_{j=0}^{n} \binom{n}{j} \frac{a^j b^{n-j}}{j!(n-j)!}$$

Proof. This is a variant of the Steiner decomposition formula. It may be proved by intersecting $a\delta_n + b(-\delta_n)$ with the 2^n orthants of $\mathbb{R}^n$. Notice that each intersection is a cartesian product of simplices, with $\binom{n}{j}$ products of type $a\delta_j \times b(\delta_{n-j})$, for each j, $0 \leq j \leq n$. □

Then the value of d_k follows by putting $a = \nu_1 + \cdots + \nu_k$, $b = \nu_{k+1} + \cdots + \nu_n$. The only term with the monomial $\nu_1 \cdots \nu_n$ appears in $a^k b^{n-k}$ with coefficient $k!(n-k)!$, so the lemma gives $d_k = \binom{n}{k}$.

□

5. Last Proofs and examples

The proof of **Theorem 3** (stated in the introduction) follows from Proposition 5, Theorem 2 and the fact that $\deg(S_n) = n$.

The reduced base scheme $|B|$ of S_n is the union of the $n(n+1)/2$ codimension 2 linear subspaces obtained by cutting pairwise the fundamental hyperplanes. This is a connected set for $n \geq 3$. By induction using the fundamental exact sequence

$$A_\bullet(X \cap Y) \to A_\bullet(X) \oplus A_\bullet(Y) \to A_\bullet(X \cup Y) \to 0$$

one gets that $A_k(B)$ is an infinite cyclic group, for $0 \leq k \leq n-3$, and $A_{n-2}(B) \simeq \mathbb{Z}^{n(n+1)/2}$ with a natural basis given by the fundamental classes in B of the irreducible components.

If $0 \leq k \leq n-3$ then $i : B \hookrightarrow \mathbb{P}^n$ induces an isomorphism from $A_k(B)$ onto $A_k(\mathbb{P}^n)$. By theorem 3, the Segre number $s_{n-2} = n(n+1)/2$, which is the number of irreducible components of $|B|$. Then the symmetries of B give the $(n-2)$-dimensional part of $s(B, \mathbb{P}^n)$, which proves **Corollary 4**(stated in the introduction). □

Example 1. A Cremona transformation $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is called *special* if its base scheme B_F is smooth and irreducible. It follows from [5] (see also [3]) that if a determinantal Cremona transformation is special then $n = 3, 4, 5$ and in this case $\deg(F) = \deg(F^{-1}) = n$; moreover, if $n \neq 4$ these are the only special Cremona transformations. From [12, thm. 6.2] such a transformation exists. The equality on degrees of F and F^{-1} is a particular case of our theorems 1 and 2.

In [10] the case $n = 3$ is considered (see also [13, chap. VIII, §4]). In particular the image of the Segre class of B_F in $\mathbb{P}^3$ is computed there:

$$s(B_F) = B_F - 28p$$

where B_F is a smooth curve of degree 6 and genus 3 and p is the class of a point. This result confirms the number obtained for S_3.

Example 2.

Example 3. Let $F : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be the Cremona transformation

$F = (X_0(X_1X_2-X_0X_3) : X_1(X_1X_2-X_0X_3) : X_0X_1(X_2-X_3) : X_0X_1(X_0-X_1))$;

its inverse is

$$F^{-1} = (X_0X_3 : X_1X_3 : X_0(X_2 - X_1) : X_1(X_2 - X_0)).$$

This transformation F is defined by the maximal minors of

$$M_F = \begin{pmatrix} 0 & X_1 & 0 \\ X_0 & 0 & 0 \\ -X_1 & -X_1 & X_1 - X_0 \\ X_3 & X_3 & X_2 - X_3 \end{pmatrix}.$$

Remark that $d_2(F) = d_1(F^{-1}) = 2$, but for the standard Cremona transformation of $\mathbb{P}^3$ we have $d_1 = d_2 = 3$. A direct verification shows that B_F is not reduced along its irreducible component $X_0 = X_1 = 0$.

For higher n we obtain analogous examples by induction: complete the matrix M_F with a line of zeroes and then a column of general linear forms to obtain a 4×5 matrix. The maximal minors define a determinantal Cremona transformation $G : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$. The matrix $M_{G^{-1}}$ can be obtained from $M_{F^{-1}}$ in the same way that one obtains M_G from M_F. We obtain $\deg(G) > \deg(G^{-1})$.

Example 4. In $\mathbb{P}^3$ there exists an involutory determinantal Cremona transformation whose base scheme is a sextic of arithmetic genus 3 supported on a twisting cubic γ. Following [13, pag. 180], such transformation may be constructed by associating to a generic point $p \in \mathbb{P}^3$ the intersection of the polar planes of p with respect to three general quadrics containing γ. Theorem 1 does not apply for this type of determinantal transformation, but its Segre numbers are also those of S_3.

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