ON SINGULAR QUADRATIC COMPLEXES, QUINTIC CURVES AND CREMONA TRANSFORMATIONS

Contents

D. AVRITZER[1], G. GONZALEZ-SPRINBERG[2] and I. PAN[3]

1. Introduction

Let G denote a smooth hyperquadric in $\mathbb{P}^5$ over the field of complex numbers, considered as the Plücker hyperquadric parametrizing lines in $\mathbb{P}^3$. A *quadratic complex* or to be more precise a *quadratic line complex* is by definition a complete intersection $X = G \cap F$, with a hyperquadric $F \subset \mathbb{P}^5$ different from G. To give X is equivalent to give the pencil of hyperquadrics defined by F and G in $\mathbb{P}^5$. Such pencils of hyperquadrics are classified by a so called *Segre symbol* (see 2.1).

Smooth quadratic complexes are classical objects that have been studied extensively since the 19th century. In modern terms such a quadratic complex X is a Fano threefold of Picard number 1 and index 2. So, as one may expect, the family of lines contained in X is related deeply to geometric properties of X. For example, if $L \subset X$ is a line and $\pi : X \dashrightarrow \mathbb{P}^3$ is the restriction to X of a linear projection of center L, then π is birational which shows X is

[1]Partially supported by CNPq, FAPEMIG, cooperação franco-brasileira
[2]Partially supported by Capes-Cofecub, IFUM
[3]Partially supported by Capes-Cofecub, ANII, IFUM

rational. One may prove that the base locus scheme of π^{-1} is a smooth quintic curve of genus 2. Furthermore, this quintic curve, say C_L, may be identified with the set B_L of lines in X meeting L, and its Jacobian parametrizes the entire family of lines in X (see [GH, Chap. 6]).

Now we assume that X is a singular irreducible and reduced quadratic complex; let $\mathrm{Sing}(X)$ denote the singular set of X. Let $L \subset X$ be a line and consider a (general) projection $\pi : X \dashrightarrow \mathbb{P}^3$ with center L. The purpose of this work is to classify quintic curves arising as the base locus scheme C_L of π^{-1} in the case where π is birational, and the Cremona transformations obtained by composing π^{-1} with another projection of the same type.

More precisely, we first prove that π is birational if and only if L is not contained in $\mathrm{Sing}(X)$; in this case C_L is an arithmetical Cohen-Macaulay quintic curve of arithmetic genus 2 (Proposition 3.3).

If L contains a singular point $o \in \mathrm{Sing}(X)$ then L belongs to the ruling of the cone $\mathrm{CL}_o(X)$ with vertex o which consists of lines in X going through o. We describe a directrix for such a cone, depending on the Segre symbol of X and on the singular point o (Propositions 4.4 and 4.6). In this singular case X may contain planes (see Corollary 4.3 and Proposition 4.14). Moreover B_L may contain a dimension 2 component and dimension 1 components of B_L are sent to components of C_L. In this situation B_L and C_L are different in general. For the most part and for simplicity, in this work we consider the cases where the pencil defining X admits a unique cone of multiplicity ≥ 2 (see § 2.2 and Table 1).

Using the description of the cone above together with (Singular [DGPS]) formal computations we obtain the ideal defining C_L for every such line L, and we describe the irreducible components and the non reduced structures of these quintic curves. The classification of these quintic curves associated to quadratic complexes are given in Theorems 5.2 for quadratic complexes with isolated singularities, 5.4 and 5.5 for quadratic complexes singular along a dimension 1 subvariety, and 5.7 for quadratic complexes singular along a plane. For a quick visualization these results are summarized in tables with figures, following the theorems on dimension 0 and 1 cases.

Every such quintic curve is non reduced or non irreducible, contrary to the case where the quadratic complex is smooth. Furthermore the types of directrices of $\mathrm{CL}_o(X)$ and quintic curves in fact characterize each Segre symbol.

In the last section we describe all Cremona transformations of $\mathbb{P}^3$ which factorize *via* a singular quadratic complex X as a product of two projections whose centers are lines meeting $\mathrm{Sing}(X)$ (Theorems 6.4 and 6.10). As in [AGP] we obtain Cremona transformations of bidegree (3,3) which are either determinantal or de Jonquières, according to whether, the lines, centers of projection, are disjoint or intersecting in a smooth point of X, respectively. However, in the case of a non normal quadratic complex (singular along a plane) all these transformations are *ruled*; these do not appear when X is normal.

On the other hand, all the cases of Cremona transformations of bidegree (2,2) are obtained when the two lines, centers of projection, meet in a singular point of X. These Cremona transformations do not appear when the quadratic complex is smooth (see [AGP]). We also give numerous examples.

2. Quadratic Complexes

Let $G \subset \mathbb{P}^5$ be a smooth quadric which we fix in the sequel. We may consider G as the Plücker quadric parametrizing lines in $\mathbb{P}^3$. Let F denote a quadric in $\mathbb{P}^5$, different from G. The complete intersection

$$X = F \cap G,$$

parametrizes a set of lines in $\mathbb{P}^3$, which is classically called a quadratic line complex. We consider only the cases where X is *reduced* and *irreducible*.

A quadratic line complex determines a pencil of quadrics in $\mathbb{P}^5$ which we call the pencil *associated* to the quadratic complex. Pencils of quadrics are classified according to their Segre symbol. The *Segre symbol of the quadratic complex* is by definition the Segre symbol of the associated pencil of quadrics.

2.1. **Segre symbol of a pencil of quadrics.** Let us recall the definition of the Segre symbol of a pencil of quadrics. Let $(x_1, \ldots, x_{n+1})$ be coordinates for $\mathbb{C}^{n+1}$ and let

$$F = \sum_{i,j=1,\ldots,n+1} a_{ij} x_i x_j, \quad G = \sum_{i,j=1,\ldots,n+1} b_{ij} x_i x_j$$

be quadratic polynomials where $A = (a_{ij})$ and $B = (b_{ij})$ are symmetric matrices. By a slight abuse of notation we will identify such polynomials with the hyperquadrics in $\mathbb{P}^n$ defined by them, so we also write $F \subset \mathbb{P}^n, G \subset \mathbb{P}^n$.

Consider the pencil

$$\mathcal{P} = \{Q_{(\lambda:\mu)} = \lambda F + \mu G \mid (\lambda : \mu) \in \mathbb{P}^1\}$$

of hyperquadrics in $\mathbb{P}^n$. The *discriminant of the pencil* $\mathcal{P}$ is by definition the binary $(n+1)$-form

$$\Delta = \Delta(\lambda, \mu) := \det(\lambda A + \mu B).$$

We assume in what follows that $\Delta(\lambda, \mu)$ is not identically zero. So it has $n+1$ roots counting multiplicities. The general element of the pencil is then a nonsingular quadric and to distinct roots there correspond distinct cones in the pencil.

Suppose $(\bar{\lambda} : \bar{\mu})$ is a root of Δ. It may also happen that all the subdeterminants of $\bar{\lambda}A + \bar{\mu}B$ of a certain order vanish. Suppose that all subdeterminants of order $n+1-d$ vanish for some $d \geq 0$, but not all subdeterminants of order $n-d$. This means that the quadric $Q_{(\bar{\lambda}:\bar{\mu})}$ is a d-cone with vertex a linear space of dimension d and directrix a smooth quadric in a linear subspace of dimension $n-d-1$ in $\mathbb{P}^n$.

Let l_i denote the minimum multiplicity of the root $(\bar{\lambda} : \bar{\mu})$ in the subdeterminants of order $n+1-i$, for $i = 0, 1, \ldots, d$. Then $l_i > l_{i+1}$ for all i so that $e_i := l_i - l_{i+1} > 0$, and we have :

$$\Delta(\lambda, \mu) = (\lambda\bar{\mu} - \bar{\lambda}\mu)^{e_0} \ldots (\lambda\bar{\mu} - \bar{\lambda}\mu)^{e_d} \Delta_1(\lambda, \mu),$$

with $\Delta_1(\bar{\lambda}, \bar{\mu}) \neq 0$. The numbers e_i are called the *characteristic numbers* of the root $(\bar{\lambda} : \bar{\mu})$ and the factors $(\lambda\bar{\mu} - \bar{\lambda}\mu)^{e_i}$ are called the *elementary divisors* of the pencil $\mathcal{P}$.

If $(\lambda_i : \mu_i)$ for $i = 1, \ldots, r$ are the roots of Δ and $e_0^j, \ldots, e_{d_j}^j$ the characteristic numbers associated to the root $(\lambda_j : \mu_j)$ and $d_1 \geq d_2 \geq \cdots \geq d_r$, then

$$\sigma(X) = \sigma(\mathcal{P}) = [(e_0^1 \ldots e_{d_1}^1)(e_0^2 \ldots e_{d_2}^2) \ldots (e_0^r \ldots e_{d_r}^r)]$$

is called the *Segre symbol* of the pencil of quadrics $\mathcal{P}$. The parentheses are omitted if $d_i = 1$. In order to make it unique, we assume that the expressions $(e_0^i, \ldots, e_{d_i}^i)$ are ordered lexicographically if $d_i = d_j$. We call these expressions the *brackets* of the Segre symbol of the pencil $\mathcal{P}$ (even if the parentheses are omitted, i.e. $d_i = 0$).

It is a classical fact (see e.g. [HP], p.278) that 2 pencils of quadrics $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{P}^n$, whose discriminants have roots exactly at $(\lambda_i^1 : \mu_i^1)$ and $(\lambda_i^2 : \mu_i^2)$, are isomorphic, that is, projectively equivalent in $\mathbb{P}^n$, if and only if they have the same Segre symbol and there is an automorphism of $\mathbb{P}^1$ taking $(\lambda_i^1 : \mu_i^1)$ to $(\lambda_i^2 : \mu_i^2)$ for all i, where the brackets corresponding to $(\lambda_i^1 : \mu_i^1)$ and

$(\lambda_i^2 : \mu_i^2)$ are of the same type. This can be used to define a normal form for those pencils $\mathcal{P}$, whose discriminant is not identically zero (see [HP], p.280):

For every e^i_j occurring in the Segre symbol of X suppose $\mu_i \neq 0$ and consider the $(e^i_j \times e^i_j)$-matrices

$$A_{ij} = \begin{pmatrix} 0 & 0 & \dots & 1 & \frac{\lambda_i}{\mu_i} \\ 0 & \dots & 1 & \frac{\lambda_i}{\mu_i} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \frac{\lambda_i}{\mu_i} & 0 & \dots & 0 \\ \frac{\lambda_i}{\mu_i} & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad B_{ij} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The coordinates of $\mathbb{P}^n$ can be chosen in such a way that A and B are given as block diagonal matrices as follows

$$A = \operatorname{diag}(A_{10}, \cdots, A_{rd_r}) \quad \text{and} \quad B = \operatorname{diag}(B_{10}, \cdots, B_{rd_r}). \tag{1}$$

We call these coordinates *Segre coordinates* of the pencil of quadrics $\mathcal{P}$. Note that Segre coordinates are not uniquely determined.

In what follows, we will be interested in quadratic line complexes, i.e, the intersection X of the Plücker quadric G parametrizing lines in $\mathbb{P}^3$ with a second quadric $F \subset \mathbb{P}^5$. The quadratic line complex X is by definition the base locus of the pencil $\mathcal{P}$. Thus X is the intersection of any two different quadrics of the pencil. We will use here the Segre symbol of the pencil $\lambda F + \mu G$ as defined above which is by definition the Segre symbol of X.

The brackets in the Segre Symbol correspond 1-1 to the cones in the pencil. We call the cone $Q_{(\lambda_i:\mu_i)}$ corresponding to the bracket $(e^i_0, \dots, e^i_{d_i})$ a cone *of type* $(e^i_0, \dots, e^i_{d_i})$. The quadric $Q_{(\lambda_i:\mu_i)}$ is then a d_i-cone and the corresponding root in the discriminant Δ is a root of multiplicity $e^i := \sum_{j=0}^{d_i} e^i_j$. By a slight abuse of notation we call $Q_{(\lambda_i:\mu_i)}$ a *d-cone of multiplicity* e^i.

2.2. **Vertices and singularities.** In the following, unless explicit mention to the contrary is made, we consider quadratic complexes whose Segre symbols have only one bracket with a cone of multiplicity ≥ 2 in the pencil. Let X be such a quadratic complex. We denote by V the vertex of the cone of multiplicity ≥ 2 and let $\operatorname{Vert}(X) := V \cap X$.

Notice that a point $x \in X$ is singular if and only if $x \in X$ and x is a singular point of a cone in the pencil, i.e. it belongs to its vertex. On the other hand, if there is another cone K in the pencil, its multiplicity is 1 and it is a 0-cone. By using the Segre coordinates (1) it follows that the vertex of K is not in $X = K \cap G$. We deduce that $\operatorname{Vert}(X)$ is supported on the singular set $\operatorname{Sing}(X)$ of X.

Recall that X is assumed reduced and irreducible. Then the Segre symbol of X can not contain a d-cone with $d \geq 3$, i.e. a bracket of *length* ≥ 4 (see [AL, Lemma 4.1]). Table 1 gives a list of results on the brackets occurring in this paper. Let us see how to prove some of the last lines in this table. The rest is similar.

Consider first a pencil of quadrics $\mathcal{P}_1 := \lambda F_1 + \mu G_1 \subset \mathbb{P}^5$ with Segre symbol $[(111)111]$. Considering the normal form we can assume the generators to be given by:

$F_1 = a(x_1^2 + x_2^2 + x_3^2) + a_4x_4^2 + a_5x_5^2 + a_6x_6^2$ and $G_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$. It follows that there is a 2-cone C_1 in the pencil given by $C_1 = F_1 - aG_1 = (a_4 - a)x_4^2 + (a_5 - a)x_5^2 + (a_6 - a)x_6^2$ whose vertex is the plane α given by $x_4 = x_5 = x_6 = 0$. The base locus X is singular along the non-singular conic in α given by $x_1^2 + x_2^2 + x_3^2 = 0$.

bracket	dim of V	Vert(X)
1	0	$\emptyset$
2	0	1 point
3	0	1 point
4	0	1 point
5	0	1 point
6	0	1 point
(11)	1	2 distinct points
(21)	1	1 double point
(31)	1	1 double point
(41)	1	1 double point
(51)	1	1 double point

bracket	dim of V	Vert(X)
(22)	1	1 line
(32)	1	1 line
(42)	1	1 line
(33)	1	1 line
(111)	2	smooth conic
(211)	2	rank 2 conic
(311)	2	rank 2 conic
(411)	2	rank 2 conic
(221)	2	rank 1 conic
(321)	2	rank 1 conic
(222)	2	1 plane

TABLE 1. Segre symbols and vertices

Let $\mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be pencils of quadrics $\subset \mathbb{P}^5$ with generators $F_i, G_i, i = 2, 3, 4$ and Segre symbols $[(211)11], [(221)1]$ and $[(222)]$ respectively. Here is a list of generators for $\mathcal{P}_i, i = 2, 3, 4$:

$$F_2 = 2ax_1x_2 + ax_3^2 + x_1^2 + ax_4^2 + a_5x_5^2 + a_6x_6^2 \text{ and } G_2 = 2x_1x_2 + x_3^2 + x_4^2 + x_5^2 + x_6^2.$$

$$F_3 = 2ax_1x_2 + 2ax_3x_4 + x_1^2 + x_3^2 + ax_5^2 + a_6x_6^2 \text{ and } G_3 = 2x_1x_2 + 2x_3x_4 + x_5^2 + x_6^2.$$

$$F_4 = 2ax_1x_2 + 2ax_3x_4 + 2ax_5x_6 + x_1^2 + x_3^2 + x_5^2 \text{ and } G_4 = 2x_1x_2 + 2x_3x_4 + 2x_5x_6.$$

The corresponding 2-cones are:

$$C_2 = x_1^2 + (a_5 - a)x_5^2 + (a_6 - a)x_6^2; \;\; C_3 = x_1^2 + x_3^2 + (a_6 - a)x_6^2; \;\; C_4 = x_1^2 + x_3^2 + x_5^2.$$

Then $\mathrm{Vert}(F_i \cap G_i), i = 2, 3, 4$ are a singular conic, a double line and a whole plane respectively.

Remark 2.1. If we consider a more general singular quadratic complex X whose Segre symbol may have several brackets of multiplicity ≥ 2 we have the following cases:
(a) dim Sing$(X) = 0$ and $\#\mathrm{Sing}(X) \leq 6$.
(b) dim Sing$(X) = 1$ and Sing(X) is supported on a line, a conic (possibly reducible) or two smooth skew conics.
(c) dim Sing$(X) = 2$, Sing(X) is a plane and X has Segre symbol $[(222)]$.
(d) Sing(X) is supported on a line or a conic plus either one or two points in general position.

3. The birational geometry of quadratic complexes

In this section we consider an arbitrary quadratic complex $X = F \cap G$ reduced and irreducible, which may be singular. We denote by T_pF the projective tangent hyperplane of F at $p \in F$.

Lemma 3.1. *Assume F to be smooth and let $L \subset F$ be a line. The following assertions hold:*
(a) *the morphism $L = \mathbb{P}^1 \to (\mathbb{P}^5)^\vee, p \mapsto T_pF$, is non constant and $\cup_{p \in L} T_pF = \mathbb{P}^5$.*
(b) *the linear space $V_F^3(L) := \cap_{p \in L} T_pF$ has dimension 3.*
(c) *If L is also contained in G and $L \cap \mathrm{Sing}(X)$ contains at least two points, then $V_F^3(L) = V_G^3(L)$.*

Proof. For the first assertion we restrict to a general dimension 3 linear space containing L. It follows that we are left with quadrics in a $\mathbb{P}^3$ and the result follows.

To prove (b) note that the map $p \mapsto T_pF \in (\mathbb{P}^5)^\vee$ is a linear immersion of L. If $p, q \in L, p \neq q$, then $V_F^3(L) = T_pF \cap T_qF$, since every point in L is a linear combination of p and q. And now (c) follows immediately.

□

Lemma 3.2. *Let $Y \subset \mathbb{P}^4$ be a normal irreducible surface and let $y \in Y$ be a smooth point. Suppose Y is a complete intersection of degree 4. Then, the projection $S \subset \mathbb{P}^3$ of Y from y is a cubic normal surface. Moreover, if Y is smooth, then S has at most two double points.*

Proof. Since y is smooth, S is a cubic surface. Suppose it is not normal. Hence there is a line $L_0 \subset S$ consisting of singular points. On the other hand, the intersection of Y with the plane $\langle y, L_0 \rangle$ contains an irreducible curve D that projects onto L_0.

The fact that S is singular along L_0 implies $\overline{ys}$ intersects $Y \setminus \{y\}$ at two (possibly coincident) points of D, for general $s \in L_0$. Notice that Y being normal a general point in D is smooth in Y. Since Y is an intersection of hyperquadrics, then $\overline{ys} \subset Y$ for all $s \in L_0$. Then $\langle y, L_0 \rangle \subset Y$ which is not possible, since Y does not contain planes. This contradiction shows that S is normal.

For the last statement, we note that if Y is smooth, then it is a del Pezzo surface of degree 4. Hence, through y there passes at most two lines which are (-1)-curves in Y. By projecting from y these lines contract to double points: indeed, denote by $\nu : \mathbb{B}l_y(Y) \to Y$ the blow-up of Y at y. Then ν resolves the indeterminacy of the projection $\pi : Y \setminus \{y\} \to S$, that is, $\pi \circ \nu$ is a morphism. Since the (inverse) strict transform by ν of those two lines are (-2)-curves, we deduce $\pi \circ \nu$ contracts them to double points. These are the only possible singular points of S since this surface is normal and $\mathbb{B}l_y(Y)$ is a desingularization.

□

Let L be a line contained in X. Fix a general 3-space M in $\mathbb{P}^5$, and define a projection $\pi : \mathbb{P}^5 \dashrightarrow M = \mathbb{P}^3$, with center L; we assume $M \cap L = \emptyset$.

Proposition 3.3. *The restriction of π to X induces a birational map $X \dashrightarrow \mathbb{P}^3$, which we denote also by π, if and only if $L \not\subset \mathrm{Sing}(X)$. Moreover, the inverse π^{-1} is given by a linear system Λ of cubic surfaces whose base locus scheme is an arithmetically Cohen-Macaulay quintic curve $C \subset \mathbb{P}^3$ of arithmetic genus 2, such that a general element of Λ is:*

(a) a normal cubic surface if $\dim \mathrm{Sing}(X) \leq 1$, *with at most two double points if* $\dim \mathrm{Sing}(X) = 0$.

(b) a non normal cubic surface if $\dim \mathrm{Sing}(X) = 2$.

Proof. Take a general point $y \in \mathbb{P}^3$. Consider the plane $H_y := \langle L, y \rangle$ generated by L and y. Then

$$H_y \cap F = L \cup L_F, H_y \cap G = L \cup L_G,$$

for lines $L_F, L_G \subset H_y$; occasionally $L_F = L$ or $L_G = L$.

Suppose $y \in T_pF = T_pG$ for a $p \in L$. Then H_y is tangent to F and G at p from which we deduce that the lines L_F and L_G pass through p. Hence either $H_y \cap X = L$ or $H_y \cap X = L \cup L'$ with $L' = L_F = L_G \neq L$.

On the other hand, the birationality of $\pi : X \dashrightarrow \mathbb{P}^3$ is equivalent to the fact that for a general $y \in \mathbb{P}^3$ the lines L_F and L_G intersect at a point of $X \setminus L$. We deduce π is birational if and only if for a general $y \in \mathbb{P}^3$ we have

$$p \in L, T_pF = T_pG \;\Rightarrow\; y \notin T_pF.$$

In other words, since $\cup_{p\in L} T_pF = \mathbb{P}^5$ (Lemma 3.1), π is birational if and only if $\dim T_pF \cap T_pG = 3$ for a general $p \in L$, which means $L \not\subset \mathrm{Sing}(X)$ and proves the first part of the Proposition.

Now we assume $L \not\subset \mathrm{Sing}(X)$. A general hyperplane section $Y = \mathcal{H} \cap X$ is then an irreducible degree 4 surface such that $\mathrm{Sing}(Y) \cap L = \emptyset$. Hence, by projecting Y form L we obtain a cubic surface. The general elements of the linear system Λ defining π^{-1} are, by construction, these cubic surfaces.

Note that $\mathrm{Sing}(Y)$ is either a line or a discrete set, depending on whether $\dim \mathrm{Sing}(X) = 2$ or less. Moreover, the surface $Y = \mathcal{H} \cap X$ is a complete intersection of degree 4 in $\mathcal{H} = \mathbb{P}^4$. Therefore a general hyperplane section $\mathcal{H}' \cap Y$ of the surface Y is then an irreducible quartic curve C of arithmetical genus 1, with at most a unique double point, and this curve is disjoint from L. Since all chords of C are contained in $\mathcal{H}' \cap \mathcal{H} = \mathbb{P}^3$, we deduce $D := \pi(C)$ is a (complete intersection) quartic curve in $\mathbb{P}^3$ isomorphic to C.

Finally, take general irreducible cubic surfaces $S_1, S_2 \in \Lambda$, such that $S_1 \cap S_2 \supset D$.

Since $\deg S_i = 3$, $i = 1, 2$, then $S_1 \cap S_2 = \Gamma \cup D$ where Γ is a curve of degree $9-4 = 5$ containing the base locus scheme of π^{-1}. Furthermore, since D is arithmetically Cohen-Macaulay, by *liason* we get that Γ is also arithmetically Cohen-Macaulay of arithmetic genus 2 and $h^0(\mathcal{I}_\Gamma(3)) = 6$ (see [PS, §3]). Since the ideal sheaves of both Γ and $\mathrm{Base}(\pi^{-1})$ are generated by six cubic global sections we conclude $\Gamma = \mathrm{Base}(\pi^{-1})$.

If $\dim \mathrm{Sing}(X) \leq 1$, then Y, the general hyperplane section of X, is normal and it is smooth when $\dim \mathrm{Sing}(X) = 0$. From Lemma 3.2 it follows $\pi(Y) \in \Lambda$ is a normal cubic surface which has at most 2 double points if $\dim \mathrm{Sing}(X) = 0$. This proves (a).

If $\dim \mathrm{Sing}(X) = 2$, then $\mathrm{Sing}(X)$ is a plane. It follows that Y is singular along a line and therefore $\pi(Y) \in \Lambda$ is also singular along a line and is therefore a non normal cubic surface proving (b).

□

It follows from the proposition above that all quadratic complexes X we are considering, may be obtained as the image of a birational morphism $\psi : Z \to X$ where $Z = \mathbb{B}l_C(\mathbb{P}^3)$ is the blow-up of $\mathbb{P}^3$ along an arithmetically Cohen Macaulay quintic curve C of arithmetic genus 2. One constructs ψ by resolving the indeterminacy of the birational map $\pi^{-1} : \mathbb{P}^3 \dashrightarrow X \subset \mathbb{P}^5$ whose base locus scheme is exactly the curve C.

The corollary below is then straightforward.

Corollary 3.4. *Let $X \subset \mathbb{P}^5$ be a quadratic complex containing a line $L \not\subset \mathrm{Sing}(X)$. Then X is obtained from the blow-up of $\mathbb{P}^3$ along an ACM quintic curve of arithmetic genus 2 in such a way that general planes in $\mathbb{P}^3$ correspond to general hyperplane sections of X, containing L.*

4. Lines and Planes in the quadratic complex

4.1. **Lines in the quadratic complex.** For a smooth hyperquadric $F \subset \mathbb{P}^5$ we consider the (extended) Gauss map $\nabla F : \mathbb{P}^5 \to (\mathbb{P}^5)^\vee$ defined by $\nabla F(p) = T_pF$: it is a linear isomorphism. If $L \subset \mathbb{P}^5$ is a line, then $\nabla F(L)$ is a line in $(\mathbb{P}^5)^\vee$. We deduce the following result:

Lemma 4.1. *Let $X = F \cap G$ be a quadratic complex where $F \subset \mathbb{P}^5$ is another smooth hyperquadric. Then $\mathrm{Sing}(X) = \{p \in X; \nabla F(p) = \nabla G(p)\}$. In particular, for a line $L \subset X$, exactly one of the following assertions holds:*

(*a*) $L \subset \mathrm{Sing}(X)$.
(*b*) $L \cap \mathrm{Sing}(X) = \emptyset$.
(*c*) $\# L \cap \mathrm{Sing}(X) = 1$.

(*d*) $\#L \cap \mathrm{Sing}(X) = 2$.

Proof. The first assertion is clear. For the following ones we only note that $\nabla F(p) = \nabla G(p)$ if and only if p is a common zero of the 2×2 minors of the corresponding system on $L = \mathbb{P}^1$: the number of solutions for such a system of quadratic equations is $0, 1, 2$ or ∞. $\square$

Definition 4.2. *Let $o \in \mathrm{Sing}(X)$ be a singular point of X. The cone of lines at o is the union $\mathrm{CL}_o(X)$ of lines in X passing through o.*

Since o is singular in X we have $T_oF = T_oG$. Take a general 4-space $\mathbb{P}^4 \simeq V^4 \subset \mathbb{P}^5$, $o \notin V^4$. Hence $X_{V^4} := X \cap V^4$ is a complete intersection in V^4 of the hyperquadrics $F \cap V^4, G \cap V^4$. If $x \in X_{V^4} \cap T_oG$, then the line $\overline{ox}$ is tangent to F and G at o from which we get $\overline{ox} \subset F \cap G = X$. We deduce that the family of lines in X passing through o is made of the generating lines of the cone with vertex o and directrix $D = X_{V^4} \cap T_oG$. Moreover, we have

$$\mathrm{CL}_o(X) = X \cap T_oG.$$

We also have the following related result:

Corollary 4.3. *Let $X \subset \mathbb{P}^5$ be a quadratic complex and let $\alpha \subset \mathbb{P}^5$ be a plane. Then, $X \supset \alpha$ implies $\mathrm{Sing}(X) \cap \alpha \neq \emptyset$. Moreover, if $o \in \mathrm{Sing}(X)$, then there is a plane in X containing o if and only if $X_{V^4} \cap T_oG$ contains a line, i.e., there is a plane in $\mathrm{CL}_o(X)$ going through o.*

Proof. The second statement is trivial. For the first one notice that $p \in \alpha$ implies $T_pG \supset \alpha$ and $T_pF \supset \alpha$. Since the hyperplanes in $\mathbb{P}^5$ which contain α are the points of a plane $\alpha^\vee$ in the dual space $(\mathbb{P}^5)^\vee$, the maps $p \mapsto \nabla F(p)$ and $p \mapsto \nabla G(p)$ induce linear maps $\nabla F|_\alpha : \alpha \to \alpha^\vee, \nabla G|_\alpha : \alpha \to \alpha^\vee$. Lemma 3.1 implies these maps are both isomorphisms. Then, the one to one correspondence $\nabla F(p) \mapsto \nabla G(p)$ establishes an automorphism of $\alpha^\vee$ which has, of course, (at least) a fixed point and the result follows from Lemma 4.1.

$\square$

Now we describe the quartic curve $D = X_{V^4} \cap T_oG \subset V^4 \cap T_oG \simeq \mathbb{P}^3$. We know it is a complete intersection of quadrics so it is an arithmetic genus 1 (possibly reducible or non reduced) curve. The next result is well known (see [HP, Chap XIII, §11]).

Proposition 4.4. *Let $D \subset \mathbb{P}^3$ be the base locus scheme of a pencil $\mathcal{P}$ of smooth quadrics in $\mathbb{P}^3$. We assume $\dim D = 1$ and denote by $\sigma(\mathcal{P})$ the Segre symbol of $\mathcal{P}$. Then exactly one of the following situations occur:*

(*a*) *D is an elliptic curve C_4 and the Segre Symbol $\sigma(\mathcal{P})$ is* [1111].

(*b*) *D is a rational curve C_4' with an ordinary double point and the Segre Symbol $\sigma(\mathcal{P})$ is* [211].

(*c*) *D is a rational curve C_4'' with a cusp and the Segre Symbol $\sigma(\mathcal{P})$ is* [31].

(*d*) *D is the union of a twisted cubic and one of its chords $C_3 \cup L$ and the Segre Symbol $\sigma(\mathcal{P})$ is* [22].

(*e*) *D is the union of a twisted cubic and one of its tangent lines $C_3 \cup T$ and the Segre Symbol $\sigma(\mathcal{P})$ is* [4].

(*f*) *D is the union of two smooth conics $C_2 \cup C_2'$ intersecting transversely at two points and the Segre Symbol $\sigma(\mathcal{P})$ is* [(11)11].

(*g*) *D is the union of a smooth conic C_2 and a rank 2 conic $L_1 \cup L_2$ such that $|C_2 \cap L_i| = 1$, $i = 1, 2$: here we have two possibilities depending on whether $C_2 \cap L_1 \cap L_2 = p$ is a point, in which case the plane $\langle L_1, L_2 \rangle$ is tangent to C_2, or it is empty; and the Segre Symbol $\sigma(\mathcal{P})$ is* [(31)] *or* [2(11)]*, respectively.*

(*h*) *D is a non degenerate union of 4 lines* $\cup_{i=1}^{4} L_i$ *such that* $L_i \cap L_j \neq \emptyset$ *if and only if* $|i-j| = 1$ *and the Segre Symbol* $\sigma(\mathcal{P})$ *is* $[(11)(11)]$.

(*i*) *D is an open polygonal of three lines one of which supports a double structure* $2L_0 \cup L_1 \cup L_2$ *and the Segre Symbol* $\sigma(\mathcal{P})$ *is* $[(22)]$.

(*j*) *D is a double rank 2 conic* $2(L_1 \cup L_2)$ *whose double structure lies over a plane and the Segre Symbol* $\sigma(\mathcal{P})$ *is* $[(211)]$.

(*k*) *D is the union of two irreducible conics* $C_2 \cup C_2'$ *intersecting at a (double) point and the Segre Symbol* $\sigma(\mathcal{P})$ *is* $[(21)1]$.

(*l*) *D is a conic* C_1 *with a double structure lying over a plane and the Segre Symbol* $\sigma(\mathcal{P})$ *is* $[(111)1]$.

Remark 4.5. (*a*) The curve D in case (i) is the complete intersection of a smooth quadric $Q \subset \mathbb{P}^3$ and a reduced and reducible quadric $H_1 \cup H_2$ such that $H_1 \cap H_2 \subset Q$.

(*b*) The curve D in case (j) is the complete intersection of a smooth quadric $Q \subset \mathbb{P}^3$ and a double plane $2H$ such that H is tangent to Q at a point.

If $o \in \mathrm{Sing}(X)$ is a singular point as we saw before the ruling of the cone $\mathrm{CL}_o(X)$ is the set of lines in X passing through o. Proposition 4.4 applied to $D = X_{V^4} \cap T_oG$ gives us information about the singularity $o \in X$, and D depends on the quadratic complex type.

Recall that if the pencil associated to X has a unique cone of multiplicity ≥ 2, then the intersection of its vertex with X, $\mathrm{Vert}(X)$ (see § 2.2), is supported on $\mathrm{Sing}(X)$.

Proposition 4.6. *Let* $X = F \cap G$ *be a a singular quadratic complex. For a point* $o \in \mathrm{Sing}(X)$ *we think of* $D = X_{V^4} \cap T_oG$ *as associated to a pencil* $\mathcal{P}$ *of quadrics in* $\mathbb{P}^3$ *induced by the pencil of* X. *The following statements hold:*

(*a*) *Suppose* $\mathrm{Vert}(X)$ *is discrete and* $\sigma(X)$ *contains a unique bracket with a cone of multiplicity* ≥ 2. *Then:*

- (*a*1) *If* $\sigma(X) = [21111]$ *or* $[3111]$, *then* $\sigma(\mathcal{P}) = [1111]$.
- (*a*2) *If* $\sigma(X) = [411], [51]$ *or* $[6]$, *then* $\sigma(\mathcal{P}) = [211], [31]$ *or* $[4]$, *respectively.*
- (*a*3) *If* $\sigma(X) = [(11)1111]$ *or* $[(21)111]$, *then* $\sigma(\mathcal{P}) = [1111]$.
- (*a*4) *If* $\sigma(X) = [(31)11], [(41)1]$ *or* $[(51)]$, *then* $\sigma(\mathcal{P}) = [(11)11], [(21)1]$ *or* $[(31)]$, *respectively.*

(*b*) *Suppose* $\mathrm{Vert}(X) = R$ *is a line and* $\sigma(X)$ *contains a unique bracket with a cone of multiplicity* ≥ 2. *Then:*

- (*b*1) *If* $\sigma(X) = [(22)11]$, *then there exist* $r_1, r_2 \in R$ *such that* $\sigma(\mathcal{P}) = [211]$ *for* $o \notin \{r_1, r_2\}$ *and* $\sigma(\mathcal{P}) = [(11)11]$ *for* $o \in \{r_1, r_2\}$.
- (*b*2) *If* $\sigma(X) = [(32)1]$, *then there exists* $r \in R$ *such that* $\sigma(\mathcal{P}) = [31]$ *for all* $o \neq r$ *and* $\sigma(\mathcal{P}) = [(21)1]$ *for* $o = r$.
- (*b*3) *If* $\sigma(X) = [(42)]$, *then there exist* $r \in R$ *such that* $\sigma(\mathcal{P}) = [4]$ *for all* $o \neq r$ *and* $\sigma(\mathcal{P}) = [(22)]$ *for* $o = r$.
- (*b*4) *If* $\sigma(X) = [(33)]$, *then* $\sigma(\mathcal{P}) = [(31)]$ *for all* $o \in R$.

(*c*) *Suppose* $\mathrm{Vert}(X) = C_2$ *is a (possibly rank 1 or 2) conic and* $\sigma(X)$ *contains a unique bracket with a cone of multiplicity* ≥ 2. *Then:*

- (*c*1) *If* $\sigma(X) = [(111)111]$, *then* $\sigma(\mathcal{P}) = [1111]$ *for all* $o \in C_2$.
- (*c*2) *If* $\sigma(X) = [(211)11]$ (*resp.* $[(411)]$, *resp.* $[(311)1]$), *then there exists* $r \in C_2$ *such that* $\sigma(\mathcal{P}) = [211]$ (*resp.* $[4]$, *resp.* $[(21)1]$) *for* $o \neq r$ *and* $\sigma(\mathcal{P}) = [(11)11]$ (*resp.* $[(211)]$, *resp.* $[(111)1]$) *for* $o = r$.
- (*c*3) *If* $\sigma(X) = [(221)1]$, *then there exist* $r_1, r_2 \in C_2$ *such that* $\sigma(\mathcal{P}) = [(21)1]$ *for* $o \notin \{r_1, r_2\}$ *and* $\sigma(\mathcal{P}) = [(111)1]$ *for* $o \in \{r_1, r_2\}$.

($c4$) *If* $[(321)]$, *then there exists* $r \in C_2$ *such that* $\sigma(\mathcal{P}) = [(31)]$ *for* $o \neq r$ *and* $\sigma(\mathcal{P}) = [(211)]$ *for* $o = r$.

(d) *If* $\mathrm{Vert}(X) = \alpha$ *is a plane, that is* $\sigma(X) = [(222)]$, *then there exists a smooth conic* $C_\alpha \subset \alpha$ *such that* $\sigma(\mathcal{P}) = [(211)]$ *for* $o \in C_\alpha$ *and* $\sigma(\mathcal{P}) = [(22)]$ *otherwise.*

Proof. We first recall that $\mathrm{CL}_o(X) = X \cap T_oG$. We prove case by case and use notations as in [J], where it is written the normal forms F and G associated to the (for example) Segre symbol $[(21)111]$ as if it were $[111(12)]$.

Case ($a1$) We prove case $[3111]$, the other is similar. We may assume

$$G = x_1^2 + x_2^2 + x_3^2 + x_5^2 + 2x_4x_6, \quad F = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4(x_5^2 + 2x_4x_6) + 2x_4x_5.$$

Hence one of the cones in the pencil defined by F and G is

$$K = (\lambda_1 - \lambda_4)x_1^2 + (\lambda_2 - \lambda_4)x_2^2 + (\lambda_3 - \lambda_4)x_3^2 + 2x_4x_5$$

with vertex $o = (0 : 0 : 0 : 0 : 0 : 1)$ the only point in $\mathrm{Sing}(X)$. Therefore T_oG is defined by $x_4 = 0$, from which it follows $X \cap T_oG$ has equations

$$x_1^2 + x_2^2 + x_3^3 + x_5^2 = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^3 + \lambda_4 x_5^2 = x_4 = 0.$$

Then $\sigma(\mathcal{P}) = [1111]$.

Case $(a2), \sigma(X) = [6]$: We may assume

$$G = 2x_1x_6 + 2x_2x_5 + 2x_3x_4, F = \lambda_1(2x_1x_6 + 2x_2x_5 + 2x_3x_4) + 2x_1x_5 + 2x_2x_4 + x_3^2$$

and $K = 2x_1x_5 + 2x_2x_4 + x_3^2$; then $o = (0 : 0 : 0 : 0 : 0 : 1)$ and T_oG is defined by $x_1 = 0$. Hence $X \cap T_oG$ has equations

$$x_2x_5 + x_3x_4 = \lambda_1(2x_2x_5 + 2x_3x_4) + 2x_2x_4 + x_3^2 = x_1 = 0$$

from which the assertion follows.

$\sigma(X) = [51]$: Here

$$G = x_1^2 + 2x_2x_6 + 2x_3x_5 + x_4^2, F = \lambda_1 x_1^2 + \lambda_2(2x_2x_6 + 2x_3x_5 + x_4^2) + 2x_2x_5 + 2x_3x_4,$$

then $o = (0 : 0 : 0 : 0 : 0 : 1)$ and T_oG is defined by $x_2 = 0$. We get $X \cap T_oG$ is defined by

$$x_1^2 + 2x_3x_5 + x_4^2 = \lambda_1 x_1^2 + \lambda_2(2x_3x_5 + x_4^2) + 2x_3x_4 = x_2 = 0,$$

proving the assertion.

The remaining Segre symbol is done similarly.

Case ($a3$) and **Case** ($a4$): Proof as in the preceding cases, we omit the details.

Case ($b1$): In this case

$$G = x_1^2 + x_2^2 + 2x_3x_4 + 2x_5x_6, F = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3(2x_3x_4 + 2x_5x_6) + x_3^2 + x_5^2$$

there is a cone in the pencil defined by $K = (\lambda_1 - \lambda_3)x_1^2 + (\lambda_2 - \lambda_3)x_2^2 + x_3^2 + x_5^2$, then $R = \mathrm{Sing}(X)$ is given by $x_1 = x_2 = x_3 = x_5 = 0$; set $o = (0 : 0 : 0 : a : 0 : b) \in L$. We have T_oG is given by $ax_3 + bx_5 = 0$. If $a \neq 0$ and $t := -b/a$ we get $X \cap T_oG$ is defined by the equations

$$x_1^2 + x_2^2 + 2(tx_4 + x_6)x_5 = \lambda_1 x_1^2 + \lambda_2 x_2^2 + 2\lambda_3(tx_4 + x_6)x_5 + (t^2 + 1)x_5^2 = 0.$$

If $t^2 = -1$, a straightforward computation shows $\sigma(\mathcal{P}) = [(11)11]$. Now, suppose $t^2 \neq -1$ and set $x_4' := \tau^{-1}(tx_4 + x_6)$, where τ is a square root of $t^2 + 1$. The system of equations above becomes

$$x_1^2 + x_2^2 + 2\tau x_4' x_5 = \lambda_1 x_1^2 + \lambda_2 x_2^2 + 2\lambda_3 \tau x_4' x_5 + (t^2 + 1)x_5^2 = 0.$$

By setting $x_5' := \tau x_5$ we obtain equations for $X \cap T_oG$ of the form

$$x_1^2 + x_2^2 + 2x_4'x_5' = \lambda_1 x_1^2 + \lambda_2 x_2^2 + 2\lambda_3 x_4'x_5' + (x_5')^2 = 0,$$

therefore $\sigma(\mathcal{P}) = [211]$ for such a t.

Finally, when $a = 0$ we also get $\sigma(\mathcal{P}) = [211]$.

Case $(b2)$: In this case

$$G = 2x_1x_2 + 2x_3x_5 + x_4^2 + x_6^2, F = \lambda_1(2x_1x_2 + 2x_3x_5 + x_4^2) + \lambda_2 x_6^2 + x_1^2 + 2x_3x_4.$$

There is a cone in the pencil defined by $K = (\lambda_2 - \lambda_1)x_6^2 + x_1^2 + 2x_3x_4$, then $R = \mathrm{Sing}(X)$ is given by $x_1 = x_3 = x_4 = x_6 = 0$; set $o = (0 : a : 0 : 0 : b : 0) \in R$. We have T_oG if given by $ax_1 + bx_3 = 0$. If $a = 0$ or $b = 0$, then T_oG is defined by $x_3 = 0$ or $x_1 = 0$. In the first case we see $\sigma(\mathcal{P}) = [31]$, in the second case $\sigma(\mathcal{P}) = [(21)1]$.

If $ab \neq 0$ and $t := -b/a$ we get $X \cap T_oG$ is defined by the equations

$$2(tx_2 + x_5)x_3 + x_4^2 + x_6^2 = 2\lambda_1(tx_2 + x_5)x_3 + \lambda_1 x_4^2 + \lambda_2 x_6^2 + x_3(x_3t^2 + 2x_4) = 0.$$

Setting $x_5' = x_5 + tx_2$ and $x_3' = x_3t^2 + 2x_4$ we get, as before, that the Segre symbol of the corresponding pencil in the variables x_3', x_4, x_5', x_6 is $[31]$.

Case $(b3)$ Now

$$G = x_1x_2 + x_3x_6 + x_4x_5,\ \ F = \lambda_1(2x_1x_2 + 2x_3x_6 + 2x_4x_5) + x_1^2 + 2x_3x_5 + x_4^2.$$

There is a unique cone in the pencil $\lambda F + \mu G$ given by $K = x_1^2 + 2x_3x_5 + x_4^2$ whose vertex is the singular line R which in this case has equations $x_1 = x_3 = x_4 = x_5 = 0$. Notice that $X = G \cap K$. The tangent space T_oG to G at a point $o = (0 : a : 0 : 0 : 0 : b) \in R$ is given by $ax_1 + bx_3 = 0$.

If $a = 0$, then T_oG is defined by $x_3 = 0$, from which we deduce $T_oG \cap X$ is defined by

$$x_1x_2 + x_4x_5 = x_1^2 + x_4^2 = 0,$$

and we deduce $\sigma(\mathcal{P}) = [(22)]$.

If $a \neq 0$, we may take $o = (0 : 1 : 0 : 0 : 0 : b)$, then T_oG is given by $x_1 = -bx_3$. We obtain $T_oG \cap X$ is the cone over the quartic curve

$$x_2 = x_3x_6 + x_4x_5 = b^2x_3^2 + 2x_3x_5 + x_4^2 = 0.$$

It is easy to see that it corresponds to the Segre symbol $[4]$.

Case $(b4)$: Now

$$G = 2x_1x_3 + x_2^2 + 2x_4x_6 + x_5^2, F = \lambda_1(2x_1x_3 + x_2^2 + 2x_4x_6 + x_5^2) + 2x_1x_2 + 2x_4x_5$$

There is a unique cone in the pencil $\lambda F + \mu G$ given by $K = 2x_1x_2 + 2x_4x_5$ whose vertex is the singular line R, and is given by $x_1 = x_2 = x_4 = x_5 = 0$. The tangent space T_oG to G at a point $o = (0 : 0 : a : 0 : 0 : b) \in R$ is given by $ax_1 + bx_4 = 0$.

Let us compute $T_oG \cap X$. If $a = 0$ or $b = 0$, then T_oG is defined by $x_4 = 0$ or $x_1 = 0$, respectively, and we see $T_oG \cap X$ corresponds to a pencil of Segre symbol $[(31)]$. Now suppose $ab \neq 0$. Replacing $x_1 = -\frac{b}{a}x_4 = tx_4$, we get:

$$2x_4(x_6 + tx_3) + x_2^2 + x_5^2,\ 2\lambda_1 x_4(x_6 + tx_3) + \lambda_1 x_2^2 + \lambda_1 x_5^2 + 2x_4(x_5 + tx_2)$$

Replacing $x_3' = x_6 + tx_3$ and $x_5' = x_5 + tx_2$ we deduce $G \cap T_oG$ and $F \cap T_oG$ are given by

$$2x_4x_3' + x_2^2 + x_5^2, 2\lambda_1(x_4x_3' + x_2^2 + x_5^2) + 2x_4x_5'$$

and we obtain the following quadrics in x_2, x_3', x_4, x_5':

$$G_1 = 2x_4x_3' + (1+t^2)x_2^2 + x_5'^2 - 2tx_2x_5',\ F_1 = 2\lambda_1(x_4x_3' - tx_5'x_2) + 2x_4x_5' + \lambda_1(1+t^2)x_2^2 + \lambda_1(x_5')^2$$

The associated symmetric matrix of the pencil $F_1 + sG_1$ is

$$A(s) := \begin{pmatrix} \lambda_1(1+t^2) + s(1+t^2) & 0 & 0 & -\lambda_1 t - st \\ 0 & 0 & \lambda_1 + s & 0 \\ 0 & \lambda_1 + s & 0 & 1 \\ -\lambda_1 t - st & 0 & 1 & \lambda_1 + s \end{pmatrix}$$

The determinant of $A(s)$ has a fourfold root $s = -\lambda_1$. In the 3×3 minor $A(s)_{22}$, $-\lambda_1$ is a simple root. So the Segre Symbol is $[(31)]$ because we have:

$$l_0 = 4, l_1 = 1, l_2 = 0 \text{ and so } e_0 = l_0 - l_1 = 3, e_1 = l_1 - l_2 = 1$$

where l_{4-i} is the minimum multiplicity of the root in the subdeterminant of order $4 - i$.

Case $(c1)$: Here

$$F = \lambda(x_1^2 + x_2^2 + x_3^2) + \lambda_4 x_4^2 + \lambda_5 x_5^2 + \lambda_6 x_6^2,\ G = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2$$

and the smooth conic C_2 is defined by

$$x_1^2 + x_2^2 + x_3^2 = x_4 = x_5 = x_6 = 0.$$

Take $o = (a : b : c : 0 : 0 : 0) \in C_2$. By symmetry we may suppose $a \neq 0$; set $t := -b/a, u := -c/a$, and note $t^2 + u^2 = -1$.

On the other hand, T_oG is given by the $ax_1 + bx_2 + cx_3 = 0$. To obtain equations for $X \cap T_oG$ we replace x_1 by $tx_2 + ux_3$ in the F, G and get

$$F_1 = \lambda_1(2tux_2x_3 - u^2x_2^2 - t^2x_3^2) + \lambda_4x_4^2 + \lambda_5x_5^2 + \lambda_6x_6^2,\ G_1 = 2tux_2x_3 - u^2x_2^2 - t^2x_3^2 + x_4^2 + x_5^2 + x_6^2.$$

Since $2tux_2x_3 - u^2x_2^2 - t^2x_3^2$ is a square we deduce the pencil defined by F_1 and G_1 has Segre symbol $\sigma(\mathcal{P}) = [1111]$ as we wanted.

Case $(c2)$: If $\sigma(X) = [(211)11]$ we have

$$F = \lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3(x_3^2 + x_4^2 + 2x_5x_6) + x_5^2,\ G = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5x_6.$$

Then $\mathrm{Sing}(X)$ is the rank 2 conic given by $x_1 = x_2 = x_5 = 0$ and $x_3^2 + x_4^2 = 0$. The intersection of the two lines is the special point $o = (0 : 0 : 0 : 0 : 0 : 1)$. Considering $\mathrm{CL}_o(X)$ we easily find $\sigma(\mathcal{P}) = [(11)11]$ and for a general point we find $\sigma(\mathcal{P}) = [211]$. The other cases are similar.

Cases $(c3), (c4)$: The proof here is similar as $(c2)$ and will be omitted.

Case (d): The quadratic complex is defined by

$$F = 2\lambda_1(x_1x_2 + x_3x_4 + x_5x_6) + x_1^2 + x_3^2 + x_5^2, G = x_1x_2 + x_3x_4 + x_5x_6, \tag{2}$$

so that the singular plane α is given by $x_1 = x_3 = x_5 = 0$. If $o = (0 : a : 0 : b : 0 : c) \in \alpha$, then $T_oG = \{ax_1 + bx_3 + cx_5 = 0\}$, where, by symmetry, we may suppose $a \neq 0$. By making $x_1 = tx_3 + ux_5, t := -b/a, u := -c/a$, the polynomials in (2) become

$$F_1 = 2\lambda_1(x_3(tx_2+x_4)+x_5(ux_2+x_6))+(t^2+1)x_3^2+2tux_3x_5+(u^2+1)x_5^2,\ G_1 = x_3(tx_2+x_4)+x_5(ux_2+x_6).$$

We change coordinates as $x_4' := tx_2 + x_4, x_6' := ux_2 + x_6$ which allows to describe $X \cap T_oG$, in a dimension three projective space with coordinates x_3, x_4', x_5, x_6', by the equations

$$x_3x_4' + x_5x_6' = (t^2 + 1)x_3^2 + 2tux_3x_5 + (u^2 + 1)x_5^2 = 0.$$

First suppose $(t^2+1)(u^2+1)\neq 0$ and choose square roots τ and μ from each factor there. By making

$$y_3 := \tau x_3, y_4 := \mu x_4', y_5 := \mu x_5, y_6 := \tau x_6',$$

we obtain an equivalent linear system of equations

$$y_3y_4 + y_5y_6 = y_3^2 + 2\gamma y_3y_5 + y_5^2 = 0,$$

where $\gamma := tu/\tau\mu$. Notice

$$y_3^2 + 2\gamma y_3y_5 + y_5^2 = (y_3 - \delta_1 y_5)(y_3 - \delta_2 y_5)$$

where $\delta_1 = -\gamma + \sqrt{\gamma^2-1}, \delta_1 = -\gamma - \sqrt{\gamma^2-1}$. Moreover, the plane $y_3 - \delta_i y_5 = 0$ is tangent to the smooth quadric $y_3y_4 + y_5y_6$ at the point $(0:1:0:-\delta_i)$, for $i=1,2$. We deduce $\sigma(\mathcal{P}) = [(22)]$ if $\delta_1 \neq \delta_2$ (i.e. $\gamma^2 \neq 1$) and $\sigma(\mathcal{P}) = [(211)]$ otherwise.

Second, if $(t^2+1)(u^2+1) = 0$, by arguing analogously we easily get the corresponding Segre symbol is $[(22)]$ again. Hence $\sigma(\mathcal{P}) = [(211)]$ if and only if $\gamma^2 = 1$, that is, $a^2+b^2+c^2 = 0$, from which we obtain the proof in this case. $\square$

Remark 4.7. In the Proposition above we did not consider the cases where $\mathrm{Sing}(X)$ is mixed, that is, when $\sigma(X)$ contains more brackets with a cone of multiplicity ≥ 2. For example, the Segre symbols $[(22)2]$, $[(111)(11)1]$, $[(111)21], [(111)(21)], [(111)3]$ $[(211)2]$ and $[(211)(11)]$ contain two such brackets.

It is important to note that for mixed cases the nature of $\sigma(\mathcal{P})$ can not be obtained from that of the "pure cases", as can be seen in the example below.

Example 4.8. If $\sigma(X) = [(22)2]$, then $\mathrm{Sing}(X) = L \cup \{p\}$, where L a line and p a point, $p \notin L$. We will prove there exist $p_1, p_2 \in L$ such that $\sigma(\mathcal{P}) = [(22)]$ if $o = p$, $\sigma(\mathcal{P}) = [(11)2]$ for $o \in \{p_1, p_2\}$ and $\sigma(\mathcal{P}) = [22]$ otherwise.

Indeed, we may write

$$G = x_1x_2 + x_3x_4 + x_5x_6,\ F = 2\lambda_1 x_1x_2 + 2\lambda_2(x_3x_4 + x_5x_6) + x_1^2 + x_3^2 + x_5^2$$

The pencil $\lambda F + \mu G$ contains two cones given by

$$K_1 = 2(\lambda_2 - \lambda_1)(x_3x_4 + x_5x_6) + x_1^2 + x_3^2 + x_5^2,\ K_2 = 2(\lambda_1 - \lambda_2)x_1x_2 + x_1^2 + x_3^2 + x_5^2$$

with vertices a point $p = (0:1:0:0:0:0)$ and a line V_2 given by $(x_1 = x_2 = x_3 = x_5 = 0)$, respectively. The tangent space T_pG is given by $x_1 = 0$, then we easily obtain $\sigma(\mathcal{P}) = [(22)]$.

For $o = (0:0:0:a:0:b) \in V_2$, the corresponding tangent space to G is given by $ax_3 + bx_4 = 0$. If $a = 0$ then T_oG is given by $x_5 = 0$ and then $T_oG \cap X$ is defined by the equations

$$x_1x_2 + x_3x_4 = 2\lambda_1 x_1x_2 + 2\lambda_2 x_3x_4 + x_1^2 + x_3^2 = x_5 = 0$$

which shows $\sigma(\mathcal{P}) = [22]$ in this case.

Finally, if $a \neq 0$ we make $x_3 = tx_5$, with $t := -b/a$, and set $x_4' = tx_4 + x_6$. We obtain $T_oG \cap X$ is defined by the equations

$$x_1x_2 + x_4'x_5 = 2\lambda_1 x_1x_2 + 2\lambda_2 x_4'x_5 + x_1^2 + (t^2+1)x_5^2 = 0 \tag{3}$$

in the dimension 3 linear space of coordinates x_1, x_2, x_4', x_5.

On one hand, it $t^2 \neq -1$ we choose a square root τ of t^2+1 and by changing coordinates as $y_1 := x_1, y_2 := x_2$, $y_4 := \tau^{-1}x_4'$ and $y_5 := \tau x_5$ we obtain an equivalent system of equations

$$y_1y_2 + y_4y_5 = 2\lambda_1 y_1y_2 + 2\lambda_2 y_4y_5 + y_1^2 + y_5^2 = 0$$

which shows $\sigma(\mathcal{P}) = [22]$.

On the other hand, if $t^2 = -1$ the system of equations in (3) becomes

$$x_1x_2 + x_4'x_5 = 2\lambda_1x_1x_2 + 2\lambda_2x_4x_5 + x_1^2 = 0;$$

by making a change of coordinates keeping x_1 and x_2 unchanged and in such a way that $x_4'x_5$ is a sum of squares of the other two coordinates we easily deduce $\sigma(\mathcal{P}) = [(11)2]$.

Now we consider lines in X passing through a smooth point.

Proposition 4.9. *Let X be a quadratic complex. For a point $x \in X$ we denote by $\ell(x)$ the number of lines in X passing through x. We have:*

(a) for $o \in \mathrm{Sing}(X)$, any line in X intersects a line of the ruling of $\mathrm{CL}_o(X)$.

(b) $x \notin \mathrm{Sing}(X)$ implies $\ell(x) \geq 1$ and either $\ell(x) \leq 4$ or $\ell(x) = \infty$; it follows that X is covered by lines. Moreover, in the last case we have exactly one of the following situations:

($b1$) $T_xX \cap X$ is either a plane or the union of a plane with a line passing through x.

($b2$) $T_xX \cap X$ is the union of two planes intersecting along a line passing through x.

($b3$) $T_xX \cap X$ is an irreducible quadratic cone with vertex at x.

Proof. Statement (a) is promptly obtained: for a line $L \subset X \setminus \{o\}$ there exists $p \neq o$ in $L \cap T_oG$; the line $\overline{op}$ is contained in $X \cap T_oG = \mathrm{CL}_o(X)$.

To prove (b) note that $X \cap T_xX$ is the intersection of two quadric cones in $T_xX = \mathbb{P}^3$, with vertex x. It is a cone whose directrix is the intersection of two coplanar conics. This intersection may be a set of at most four points, a line, a line and a point, two lines or a smooth conic from which we have the assertions. □

Remark 4.10. From Propositions 4.4 and 4.6 we obtain quadratic complexes with a unique singular point but which contain either a plane or two planes intersecting along a line. Indeed, it suffices to consider complexes with a singular point o such that a directrix of $\mathrm{CL}_o(X)$ contains either a line or two lines intersecting at a point.

The remark together with the following examples show all possibilities in Proposition 4.9 really occur.

Example 4.11. Let $X = F \cap G$ be given by the (smooth) quadrics

$$F := x_1^2 + x_2^2 + x_3^2 + x_6x_4 + x_5^2, \quad G := x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_4x_5 + x_5x_6.$$

Take the smooth point $x = (0:0:0:0:0:1)$ in X. Then T_xF and T_xG are given by $x_4 = 0$ and $x_5 = 0$. It follows that $T_xX \cap X = T_xF \cap F \cap T_xG \cap G$ is the cone with vertex x over the smooth conic in X with equation $x_1^2 + x_2^2 + x_3^2 = 0$ contained in the plane V given by $x_4 = x_5 = x_6 = 0$; notice that V does not pass through x and it is contained in T_xX. This conic is the singular locus of X.

Furthermore, note that $F - G$ is a cone in the pencil, with vertex V, $V \cap X$ is the conic above, and the discriminant of the pencil has -1 as triple root and three other simple roots. Then $\sigma(X) = [(111)111]$.

Example 4.12. Consider the quadratic complex defined by

$$K = x_1x_2 + x_4x_5, G = x_1x_3 + x_2x_4 + x_5x_6.$$

In this case $\mathrm{Sing}(X)$ is the line $x_1 = x_2 = x_4 = x_5 = 0$.

Let $x = (1:0:0:0:0:0)$. Hence x is a smooth point with T_xX given by $x_2 = x_3 = 0$. Therefore $X \cap T_xX$ is the union of the plane $x_2 = x_3 = x_5 = 0$ and the line $x_2 = x_3 = x_4 = x_6 = 0$.

On the other hand, if $x = (0:1:0:0:0:0)$, then T_xX is given by $x_1 = x_4 = 0$ and $X \cap T_xX$ is the union of the two planes $x_1 = x_4 = x_5x_6 = 0$. The Segre symbol of this quadratic complex X is $[(321)]$.

Example 4.13. Consider the quadratic X such that $\mathrm{Sing}(X)$ is a plane α. According to the table in §1 we know $\sigma(X) = [(222)]$ and we can assume $X = F \cap G$ where

$$F = 2a(x_1x_2 + x_3x_4 + x_5x_6) + x_1^2 + x_3^2 + x_5^2 \text{ and } G = 2x_1x_2 + 2x_3x_4 + 2x_5x_6,$$

so that the singular plane α is given by $x_1 = x_3 = x_5 = 0$.

On the other hand, the pencil $\lambda F + \mu G$ contains the cone $x_1^2 + x_3^2 + x_5^2$, which is the join of α and the conic $\overline{C}$ given by $x_2 = x_4 = x_6 = x_1^2 + x_3^2 + x_5^2 = 0$.

Set $o = (0:u:0:v:0:w) \in \alpha$. The tangent space T_oG given by $ux_1 + vx_3 + wx_5 = 0$, intersects $\overline{C}$ in two occasionally coinciding points, p, q say; these are coincident when $(u:0:v:0:w) \in \overline{C}$. By construction the lines $\overline{op}$ and $\overline{oq}$ are contained in X. It follows that X is the union of these pairs of lines as o varies in α.

We conclude that for every point in α there passes two lines (counted with multiplicity) meeting $\overline{C}$ and contained in X.

4.2. **Planes in the quadratic complex.** Let X be a singular quadratic complex. If $\alpha \subset X$ is a plane in X we know that there exists $o \in \alpha \cap \mathrm{Sing}(X)$ (Corollary 4.3), we deduce $\alpha \subset \mathrm{CL}_o(X)$.

Conversely, if $o \in \mathrm{Sing}(X)$ and $R \subset \mathrm{CL}_o(X)$ is a line with $o \notin R$, then we get $\overline{op} \subset X$ for all $p \in R$. Thus, the plane $\alpha := \langle o, R\rangle$ is contained in X.

The following result follows readily from the argument above and Propositions 4.4 and 4.6.

Proposition 4.14. *X contains a plane if and only if there exists $o \in \mathrm{Sing}(X)$ such that a directrix D of $\mathrm{CL}_o(X)$ contains a line. Moreover, if $\sigma(X)$ contains only one bracket of multiplicity ≥ 2, this occurs when $\sigma(X)$ is one of the following:*

$$[6], [(51)], [(21)111], [(11)1111], [(33)], [(42)], [(321)], [(411)], [(222)].$$

Example 4.15. Consider the pencil generated by

$$K_1 = x_1x_3 - x_2x_4 \text{ and } G = x_1x_3 + x_2x_4 - 2x_5x_6.$$

Its Segre symbol is $[(11)(11)(11)]$. The cones in the pencil are K_1 and

$$K_2 = x_1x_3 - x_5x_6 \text{ and } K_3 = x_2x_4 - x_5x_6.$$

The corresponding vertices of these 1-cones are the lines V_1, V_2, V_3 given, respectively, by

$$x_1 = x_2 = x_3 = x_4 = 0,\ x_1 = x_3 = x_5 = x_6 = 0,\ x_2 = x_4 = x_5 = x_6 = 0.$$

Note that X has 6 singular points, the intersection that we get intersecting G with $V_i, i = 1, 2, 3$:

$$p_0 = (1:0:0:0:0:0)\ \ p_1 = (0:0:1:0:0:0)\ \ p_2 = (0:0:0:1:0:0)$$

$$p_3 = (0:1:0:0:0:0)\ \ p_4 = (0:0:0:0:0:1)\ \ p_5 = (0:0:0:0:1:0)$$

On the other hand, we may see that X is the image of the rational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ defined by (see [AGP, Example 4])

$$\phi = (x^2y : xy^2 : yzw : xzw : xyw : xyz).$$

A straightforward computation shows ϕ^{-1} is the projection of X from the line $L = \overline{p_1p_2}$ that goes through the singular points p_1 and p_2.

Note that the quintic curve C in this case is reducible, a union of 5 distinct lines, and that the quadratic complex X has six ordinary double points.

Let us explain how we get the five lines of the quintic C. A plane generated by 3 singular points, one in each vertex, is contained in X. Hence, the quadratic complex contains the following 8 planes:

$$024, 025, 034, 035, 124, 125, 134, 135$$

where 024 means $p_0p_2p_4$ and similarly with the other planes. The $\mathbb{P}^3$ generated by the line L and the planes $024, 025, 134, 135$ project into 4 lines in $\mathbb{P}^3$ from L. The remaining planes are contracted since they contain the line L. The remaining component of C is the projection of $\langle L, V_1\rangle$.

5. Classification of the associated quintics

Let $X = F \cap G \subset \mathbb{P}^5$ be a singular quadratic line complex as in § 2.2. Let $L \subset X$ be a line, $L \not\subset \mathrm{Sing}(X)$, to which we associate a projection $\pi : X \dashrightarrow M = \mathbb{P}^3$ as in Proposition 3.3, $C = \mathrm{Base}(\pi^{-1})$.

Proposition 5.1. *Let $p \in L$ be a point such that $T_pF \cap T_pG \cap X$ has dimension 2. Then, we have at least one of the following:*
(*a*) *C is reducible;*
(*b*) *C is supported on a line;*
(*c*) *Every dimension 2 component of $T_pF \cap T_pG \cap X$ is a plane containing L.*
It follows that if $L \cap \mathrm{Sing}(X) \neq \emptyset$, then C is either reducible or non reduced.

Proof. Suppose (c) does not occur. Then the image of the cone $T_pF \cap T_pG \cap X$ by π contains an irreducible component of C, say C_0, with $\deg C_0 \leq 4$. If C is irreducible we may write the 1-cycle associated to C as $C = mC_0$ for a positive integer a. Then $5 = \deg C = m \deg C_0$. We deduce $m = 5$ and C_0 is a line, which proves the statement.

For the last assertion suppose $p \in \mathrm{Sing}(X)$. If C satisfies (a) or (b) there is nothing to prove. Otherwise C satisfies (c); in this case the directrix of the cone $\mathrm{CL}_p(X)$ contains only lines, at least two (see Propositions 4.4 and 4.6). It follows that C contains at least a line as a component. ☐

In the following (§ 5.1, § 5.2 and § 5.3) we describe the components of the ACM quintic curve C of arithmetic genus 2, base locus of the linear system of cubics defining π^{-1}, giving a classification in terms of the Segre symbols $\sigma(X)$, first when $\mathrm{Sing}(X)$ is a discrete set, then when it has dimension 1 and finally when it is a plane.

This classification is obtained in the following way. First the center of projection, the line L, is chosen through a singular point $o \in \mathrm{Sing}(X)$ and a point p in a directrix D of the cone $\mathrm{CL}_o(X)$. Propositions 4.6 and 4.4 give the general and special positions for p to consider. Next, since lines in X intersecting L correspond to points in the quintic C these propositions give the part of C coming from lines going through o. This part of C, image by π of $\mathrm{CL}_o(X)$, is contained in a plane since $L \subset \mathrm{CL}_o(X) \subset T_oG \simeq \mathbb{P}^4$. The complete results are obtained by computing the cubic polynomials defining π^{-1} and the primary decomposition of the ideal generated by them.

5.1. **Quadratic complexes with isolated singularities.**

Recall that the brackets for which $\mathrm{Sing}(X)$ has isolated singularities, and to which the following result refers, are classified in the list given in Table 1.

Theorem 5.2. *Let C be the base locus of the linear system of cubics defining π^{-1} as above. Assume $\mathrm{Sing}(X)$ is discrete and let $o \in \mathrm{Sing}(X) \cap L$. We also denote by D a directrix of the*

cone $\mathrm{CL}_o(X)$. *Then* C *contains a plane curve and we get the following assertions about the genus two quintic curve* C, *depending on the choice of* o *and* $\{p\} = L \cap D$:
(*a*) *If* $\sigma(X)$ *is* [21111], [3111], [(11)1111] *or* [(21)111], *then* C *consists of an elliptic cubic curve lying in a plane, and a conic (smooth in the two first cases and of rank 2 in the other cases) in a different plane; these curves intersect in two points in the first and third cases and in a double point in the other cases.*
(*b*) *If* $\sigma(X)$ *is* [411] *(resp.* [51]*), then* D *is a quartic with a singular point* s *and we have two cases:*

(*b*1) *If* L *goes through* s *then the quintic* C *consists of a smooth conic with a double structure union a secant (resp. tangent) line.*

(*b*2) *If* L *goes through a general point of* D *then* C *consists of a nodal (resp. cuspidal) cubic curve union a smooth conic through the singular point of the cubic.*
(*c*) *If* $\sigma(X)$ *is* [6], *then* D *is a twisted cubic union a tangent line* $C_3 \cup T$. *If* L *goes through* $C_3 \cap T$ *(resp.* $C_3 \backslash T$*; resp.* $T \backslash C_3$*), then* C *consists of a smooth conic with a double structure union a tangent line (resp. two not coplanar smooth conics meeting in a point union a tangent line to one of them at this point; resp. a cuspidal cubic union a smooth conic through the cusp).*
(*d*) *If* $\sigma(X)$ *is* [(31)11] *(resp.* [(41)1]*) then* D *is the union of two not coplanar smooth conics meeting in two points (resp. a double point) and we have two cases:*

(*d*1) *If* L *goes through a general point of* D *then* C *consists of a smooth conic union two lines one of them with a double structure which is secant (resp. tangent) to the conic, the other line meeting only the double line.*

(*d*2) *If* L *goes through a singular point of* D, *then* C *consists of a rank 2 conic with a double structure and a general secant line (resp. a coplanar line through the singular point of the conic).*
(*e*) *If* $\sigma(X)$ *is* [(51)], *then the directrix* D *consists of a smooth conic* C_2 *union a rank 2 conic* $\ell_1 \cup \ell_2$ *intersecting in a point* $s \in C_2$. *If* L *goes through a general point of* C_2 *then* C *consists of three coplanar and concurrent lines, one of them with a double structure, union a fourth line meeting the double line. If* L *goes through a general point on the lines* $\ell_i, i = 1, 2$, *then* C *consists of a smooth conic union a tangent line with a double structure union one more line meeting the double line. If* L *goes through* s, *then* C *consists of two meeting lines, one with a double structure and the other with a triple structure.*

Proof. First of all recall that the projection of $\mathrm{CL}_o(X)$ is contained in a plane so C contains a plane curve.

The different cases to consider for the line $L = \overline{op}$ follow from Propositions 4.6, part (a), and 4.4.

We prove (a) and (b), the proof of the other cases are similar.
(a) According to Proposition 4.6 (a), in all these cases D corresponds to the Segre symbol [1111]. It follows it is a nonsingular elliptic quartic curve by Proposition 4.4(a). Since we project with center a line L meeting D, then the image of D is an elliptic plane cubic C_3. We deduce $C = C_3 \cup C_2$ where C_2 is a degree 2 curve.

On the other hand, recall that C is an ACM quintic curve of arithmetic genus 2 (Proposition 3.3), it follows there exists a unique quadric Q containing C. Then $Q = H_1 \cup H_2$ and C_3 is contained in one of these planes, say H_1. Since C is contained in irreducible cubic surfaces (in fact smooth), then C_2 is contained in H_2 and is a conic, maybe singular.

Since the arithmetic genus of $C_3 \cup C_2$ is 2, we deduce that these curves intersect in (counting multiplicities) two points.

The different cases in the statement occur as we may verify by a direct computation using a computer algebra system.

(b) If $\sigma(X)$ is [411] (resp. [51]), Proposition 4.6 implies D corresponds to the Segre symbol $\sigma(\mathcal{P}) = [211]$ (resp. [31]). According to Proposition 4.4 the curve D is a rational quartic C_4 with an ordinary double point (resp. a cusp). We then have two cases:

If L intersects D at the singular point, then it projects to a smooth conic with a double structure, and the quintic C contains also a secant line (resp. a tangent line) to this conic.

If L intersects D in a smooth point, then it projects to a plane cubic curve with a node (resp. cusp). As we saw in the proof of case (a) we have C is the union of this curve and a conic C_2. In this case C_2 is smooth and goes through the singular point of the cubic curve.

The two cases can be computed using a computer algebra system to find first the six cubic polynomials given by π^{-1} and defining C, then the primary decomposition of the ideal generated by them to obtain the components, multiple structure and their mutual intersections.

For example, for $\sigma(X) = [51]$, the singular point of X is $o = (0:0:0:0:0:1)$. By using equations for $\mathrm{CL}_o(X)$ as in Proposition 4.6, case $(a2)$, we may choose D as $\mathrm{CL}_o(X) \cap \{x_6 = 0\}$. The singular point of D is $p = s = (0:0:0:0:1:0)$. If we project from $L = \overline{op}$ to $M \simeq \mathbb{P}^3$ defined by $(x_5 = x_6 = 0)$, disjoint from L, then we compute π^{-1} by substituting the equations of the plane $\langle L, m\rangle$, $m \in M$, into the equations of X and eliminating x_5 and x_6. We obtain

$$\pi^{-1} = (2x_1x_2^2 : 2x_2^3 : 2x_2^2x_3 : 2x_2^2x_4 : -x_2(\lambda x_1^2 + 2x_3x_4) : -x_1^2x_2 + \lambda x_1^2x_3 + 2x_3^2x_4 - x_2x_4^2),$$

where $\lambda = \lambda_1 - \lambda_2$ is a nonzero parameter.

The primary decomposition gives the components of C, a reduced line defined by (x_2, x_3) and a smooth conic defined by $(x_2, \lambda x_1^2 + 2x_3x_4)$ with a double structure whose ideal is generated by x_2^2, $\lambda^2x_1^2x_3 + 2x_2x_3x_4 + 2\lambda x_3^2x_4 - \lambda x_2x_4^2$, $\lambda^2x_1^2x_2 + 2x_2^2x_4 + 2\lambda x_2x_3x_4$, $\lambda^2x_1^4 + 2x_1^2x_2x_4 + 2\lambda x_1^2x_3x_4 + 2x_2x_4^3$. Notice that the line is tangent to the conic. This proves $(b1)$ for $\sigma(X) = [51]$.

A general point of D may be parametrized by

$$p(t) := (t : 0 : 1 : -\frac{\lambda t^2}{2} : -\frac{\lambda t^4}{8} - \frac{t^2}{2} : 0).$$

since D is defined by $(\lambda x_1^2 + 2x_3x_4 = x_1^2 + 2x_3x_5 + x_4^2 = x_2 = x_6 = 0)$. Now we take the line $L = L_t = \overline{op(t)}$ and project to $M = (x_3 = x_6 = 0)$. We may suppose without loss of generality that $\lambda = 1$. By a computation analogous to the preceding one we obtain the components for $C = C_t$: if $t \neq 0$, a cuspidal cubic defined by

$$x_2 = (t^2+2)x_1^2x_4 + 2tx_1x_4^2 + 2x_4^3 - 2x_1^2x_5 = 0$$

where the cusp is $(x_1 : x_2 : x_4 : x_5) = (0:0:0:1)$, and a smooth conic defined by

$$8tx_1 - (t^4+4t^2)x_2 + 8x_4 = -(t^8 + 8t^6 + 16t^4)x_2^2 + (16t^4 + 64t^2)x_2x_4 - 64x_4^2 - 128t^2x_2x_5 = 0;$$

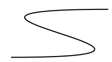
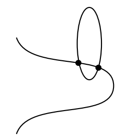
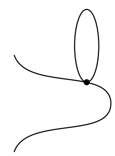
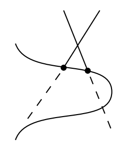
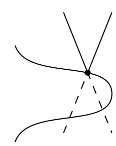
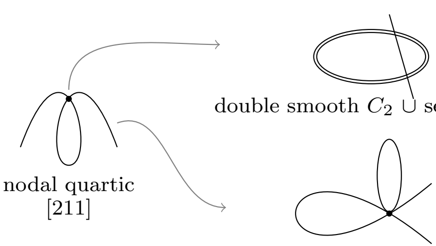
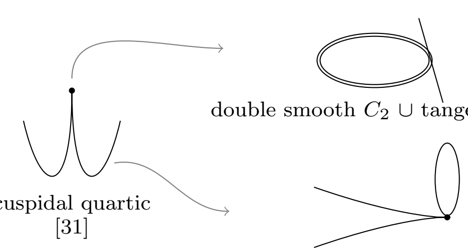
if $t = 0$, $x_2 = x_1^2x_4 + x_4^3 - x_1^2x_5 = 0$ and $x_4 = x_1^2 + 2x_2x_5 = 0$, respectively. The conic meets the cubic at its cusp, and this proves $(b2)$ for $\sigma(X) = [51]$.

□

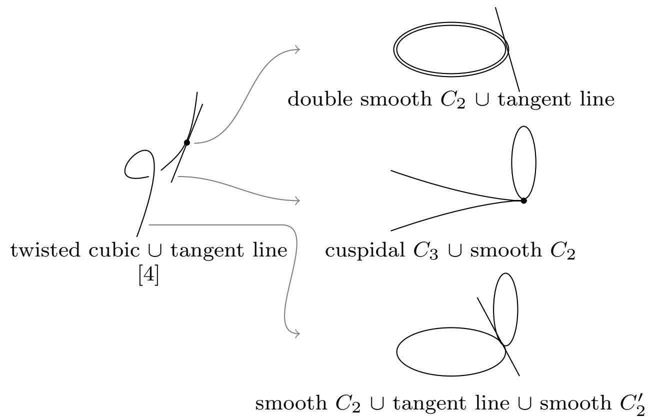
Remark 5.3. A special case where C contains a line as one of its components is when X contains a plane α with $o \in \alpha$ and $L \not\subset \alpha$.

Graphically, the conclusions of Theorem 5.2 may be summarized in the following table of pictures.

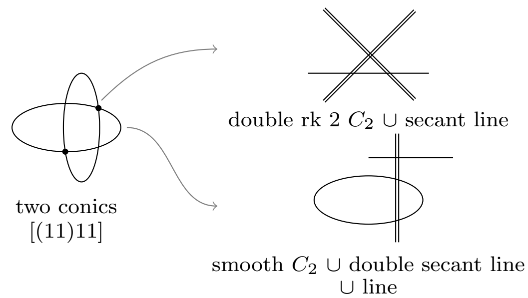
Table of results for $\dim(\mathrm{Sing}(X))$=0 :

$\sigma(X)$	Vert(X)	directrix D	quintic curve C
[21111]	1 point	 elliptic quartic [1111]	 elliptic $C_3 \cup$ smooth C_2
[3111]	1 point	 elliptic quartic [1111]	 elliptic $C_3 \cup$ smooth C_2
[(11)1111]	2 points	 elliptic quartic [1111]	 elliptic $C_3 \cup$ rk 2 C_2
[(21)111]	1 double point	 elliptic quartic [1111]	 elliptic $C_3 \cup$ rk 2 C_2
[411]	1 point	 nodal quartic [211]	double smooth $C_2 \cup$ secant line; nodal $C_3 \cup$ smooth C_2
[51]	1 point	 cuspidal quartic [31]	double smooth $C_2 \cup$ tangent line; cuspidal $C_3 \cup$ smooth C_2

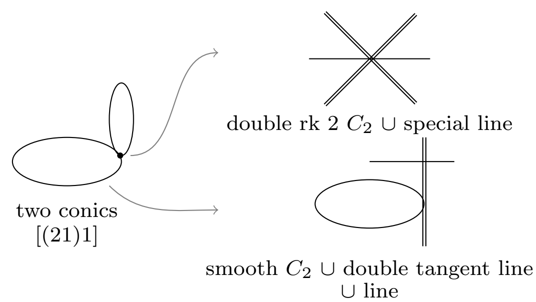
[6] 1 point



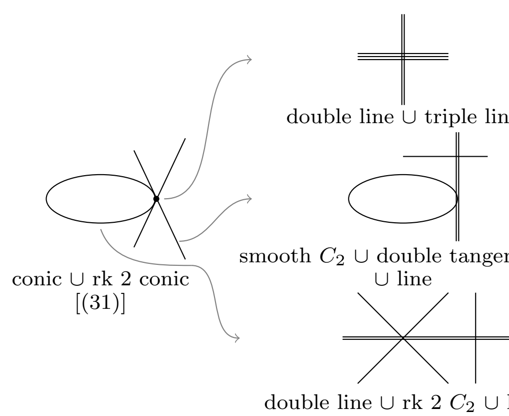
[(31)11] 1 double point



[(41)1] 1 double point



[(51)] 1 double point



5.2. **Quadratic complexes singular along a dimension 1 subvariety.** Now we suppose $\dim \mathrm{Sing}(X) = 1$. Since $\mathrm{Vert}(X)$ is supported on $\mathrm{Sing}(X)$, Table 1 gives the list of cases we consider in this paragraph. We have $\mathrm{Vert}(X)$ is either a line, a smooth conic, a rank 2 conic or a rank 1 conic.

First suppose $\mathrm{Vert}(X) = R$ is a line. Let $L \subset X$ be a line with $L \not\subset \mathrm{Sing}(X)$, $L \cap \mathrm{Sing}(X) \neq \emptyset$ and $\pi : X \dashrightarrow \mathbb{P}^3 = M$ the projection with center L and M general.

Under these conditions, we have:

Theorem 5.4. *Let C be the base locus of the linear system of cubics defining π^{-1}. We assume $L \cap R = \{o\}$. We also denote by D the directrix of the cone $\mathrm{CL}_o(X)$. We get the following assertions about the genus two quintic curve C, depending on the choice of o and $\{p\} = L \cap D$.*
(a) *Suppose $\sigma(X) = [(22)11]$. Then o has two special positions r_1, r_2, and we have:*

$(a1)$ *If $o \in \{r_1, r_2\}$, then according to the choice of $p \in D$ we have that C consists of a rank 2 conic with a double structure, and a general chord; or a smooth conic union one of its chords with a double structure and a line.*

$(a2)$ *If o is general, then C consists of a plane nodal cubic curve union two lines through the node.*
(b) *Suppose $\sigma(X) = [(32)1]$. Then o has one special position and there are two possibilities:*

$(b1)$ *If $o = r$ is special, then C consists of a smooth conic union one of its tangents with a double structure union a skew line through the point of tangency.*

$(b2)$ *If o is general, then C consists of a plane cuspidal cubic curve union two skew lines going through the cusp.*
(c) *Suppose $\sigma(X) = [(42)]$. Then o has one special position r and we have:*

$(c1)$ *If $o = r$, then according to the choice of $p \in D$ we have that C consists of a line with a triple structure union a line with a double structure; or a rank 2 conic, each of its components with a double structure union a special chord; or a line with a triple structure union two simple lines.*

$(c2)$ *If o is general, then C consists of a smooth conic union a tangent line union two other lines through the point of tangency; or a plane cuspidal cubic curve union two skew lines through the cusp.*
(d) *Suppose $\sigma(X) = [(33)]$. Then o has no special positions and there are two possibilities:*

$(d1)$ *C consists of a conic union one of its tangent lines with a double structure union a line going through the point of tangency.*

$(d2)$ *C consists of four concurrent lines, one of them with a double structure.*

Proof. As we said before, we keep notations as in Proposition 4.6 and its proof and each time use this result together with Proposition 4.4.
(a) By Propositions 4.6 and 4.4) there exist $r_1, r_2 \in R$ such that D is a rational quartic curve with a node when $o \notin \{r_1, r_2\}$ and two skew conics intersecting at two points when $o \in \{r_1, r_2\}$. Note that R and the center L of projection belong to the ruling of $\mathrm{CL}_o(X)$.

$(a1)$ Suppose D consists of two skew smooth conics intersecting at p_1, p_2. It is easy to see that R goes through one of them, p_1 say. It follows that L can go through p_2 or can intersect one of the conics outside $\{p_1, p_2\}$. In the first case each of these conics projects to a line and in the second case one of these conics projects to a (smooth) conic and the other one to a chord. More precisely:

In the first case, and according to the proof of case $(b1)$ of Proposition 4.6, we have $o = (0 : 0 : 0 : 1 : 0 : -i)$, $p_1 = (0 : 0 : 0 : 0 : 0 : 1) \in R$, $p_2 = (0 : 0 : i : 1 : 1 : -i) \in D$. Using some computer algebra system we may calculate π^{-1} to obtain the degree 3 homogeneous polynomials defining C. Recalling that the saturated ideal of an ACM quintic curve is always generated by a quadric and two cubic polynomials, we can compute the ideal of C to be

$$\mathcal{I}(C) = (x_3^2, x_3(x_1^2 - 2x_2^2), ix_1^2x_3 + ix_2^2x_3 + x_6x_1^2 - 2x_6x_2^2),$$

in terms of the coordinates of the 3-space defined by $(x_4 = x_5 = 0)$.

We get double structures supported on the lines $x_3 = x_1 \pm \sqrt{2}x_2 = 0$, and a simple line $x_3 = x_6 = 0$. When L is generated by o and a general point $p \in D$, D the union of two smooth conics the computations are similar and we omit the details.

$(a2)$ Suppose D is a rational quartic, then R contains its singular point, therefore L intersects D at a smooth point. Then π projects $\mathrm{CL}_o(X)$ onto a nodal plane cubic curve union of two lines through the node. The details, which we omit, are similar to some we give below, for example $(b1)$ and $(b2)$.

(b) In this case, for the special point $r \in R$, the curve D is the union of two skew irreducible conics each in a tangent plane to the other, intersecting in a point of R. For a general point of R we have a rational quartic curve with a cusp belonging to R.

$(b1)$ Let $L = \overline{op}$ with $o = r$ and p a general point of D. Then the part of C coming from $\mathrm{CL}_o(X)$ consists of a smooth conic union one of its tangent lines with a double structure; since C is a quintic curve it must contain another simple line.

More precisely, with the notations of Proposition 4.6$(b2)$, let $o = r = (0 : 0 : 0 : 0 : 1 : 0)$ be the special point. It follows that $T_oG = (x_3 = 0)$ and D is the union of two conics in the planes given by $(x_1 + x_6 = 0)$ and $(x_1 - x_6 = 0)$, where without loss of generality we are assuming that $\lambda_2 = -1$ and $\lambda_1 = 0$. One of the conics can be parametrized by $p(t) = (1 : -\frac{1}{2}(t^2+1) : 0 : t : 0 : 1)$.

Let $L_t = \overline{op(t)}$ and $M = \mathbb{P}^3$ given by $(x_1 = x_5 = 0)$. As before we may compute π^{-1} and deduce the primary decomposition associated to C. We get the conic with ideal $(x_3, -x_4^2 + x_2x_6 + tx_4x_6)$, union the tangent line (x_3, x_6) with a double structure, union the simple line $(2x_4 - tx_6, t^2x_3 - 2x_4)$ if $t \neq 0$ or (x_4, x_6) if $t = 0$, going through the point of tangency $(x_2 : x_3 : x_4 : x_6) = (1 : 0 : 0 : 0)$.

$(b2)$ Let $o = (0 : 1 : 0 : 0 : b : 0)$ be a general point of $R = \mathrm{Sing}(X)$. It follows that T_oG is given by $x_1 = -bx_3$. Intersecting the generators of X with T_oG and $x_2 = 0$ to avoid the point o, we get the rational singular quartic curve D in $\mathbb{P}^3$ with coordinates x_3, x_4, x_5, x_6, given by the two equations:

$$b^2x_3^2 + 2x_3x_4 - x_6^2 = 2x_3x_5 + x_4^2 + x_6^2 = 0$$

Outside its singular point $(0 : 0 : 1 : 0)$, the rational quartic curve D can be parametrized by:

$$x_3 = 1, \ \ x_4 = \frac{1}{2}(t^2 - b^2), \ \ x_5 = -\frac{1}{8}t^4 + \frac{1}{4}b^2t^2 - \frac{1}{8}b^4 - \frac{1}{2}t^2, \ \ x_6 = t$$

We then consider a line L joining a general point $o \in R$ with $p_b(t) \in D$ as above and the $\mathbb{P}^3 = M$ given by $x_2 = x_3 = 0$. Projecting X with center L onto M we get the genus 2 quintic curve $C_{t,b}$. It decomposes as a plane cuspidal cubic curve $\mathcal{C}_{t,b}$ union two skew lines $\ell_{t,b}, \ell'_{t,b}$ going through the cusp as follows:

$$\mathcal{C}_{t,b} = (x_1, 2x_4^3 - 2tx_4^2x_6 + (-b^2 + t^2 + 2)x_4x_6^2 + 2x_5x_6^2)$$

$$\ell_{t,b} = (x_4 + (b - t)x_6, x_1 + x_6) \text{ and } \ell'_{t,b} = (x_4 - (b + t)x_6, x_1 - x_6)$$

(c) In this case (see case (b3) of Proposition 4.4), there exists a special position $r \in R$ such that D is an open polygonal of three lines, the middle one with a double structure. The singular line R intersects D in a point r_0 belonging only to the middle line. For general $o \in R$, the curve D is a twisted cubic union a tangent line and accordingly we have:

$(c1)$ If we take the center of projection L as a line through the special point $o = r$ and a point $p \in D \setminus \{r_0\}$, then according to the choice of $p \in D$, we have the various possibilities for the quintic C, namely: if p is the intersection of the middle component with one of the other two, then C is the union of two lines, one of them supporting a triple structure and the other a double structure; if p is a general point of the middle component of $D \setminus \{r_0\}$, then C is the open polygonal of three lines, the middle one with a triple structure; if p is a general point of the extreme lines of D then C is the union of a rank 2 conic supporting a double structure union a simple line through the singular point. The details are similar to the previous cases we have already worked out and are omitted.

($c2$) If the point $o = (0 : 1 : 0 : 0 : 0 : b)$ is general in R, then the directrix D is a twisted cubic C_3 union a tangent line T. On obtains $C_3 \cap T = \{r\}$ (the special point in R) and we may parametrized $C_3 \setminus \{r\}$ by

$$p_b(t) = (0 : 0 : 1 : t : -\frac{t^2 + b^2}{2} : \frac{t^3 + b^2 t}{2}).$$

We then consider the line L joining o to $p_b(t)$ and $M = \mathbb{P}^3$ given by $x_2 = x_3 = 0$ and obtain, as before, the quintic curve $C_{t,b}$. The ideal $\mathcal{I}(C_{t,b})$ is generated by $bx_1^2 + tx_1x_4 + x_1x_5$, $x_1(x_1^2 + x_4^2)$ and $-2x_1^2x_6 - 2x_4^2x_6 + t^2x_1^2x_4 + b^2x_1^2x_4 + (b^2 + t^2)x_4^3 - 2tx_1^2x_5 + 2tx_4^2x_5 + 4bx_1x_4x_5 + 4x_4x_5^2$.

The primary decomposition of the ideal gives that, in this case, the quintic is a non singular conic in the plane $x_1 = 0$ union a tangent line to the conic in that plane union two other skew lines going through the point of tangency.

If the center of projection L contains a general point of the tangent T then the quintic C is the union of a cuspidal cubic and two lines through the cusp. We omit the details.

(d) In this case D is the union of a conic and two lines, intersecting it at a point $r_0 \in R$. We have two possibilities: either L contains a point of the conic or of one of the two lines, different from r_0. In the former case C consists of four concurrent lines, one having a double structure, and in the latter case it consists of a conic, one of its tangent lines with a double structure and a skew line through tangency point. We omit the details of the computations. □

Now suppose $\mathrm{Vert}(X)$ is a smooth (resp. rank 2; resp. rank 1) conic. Recall that $\mathrm{Vert}(X)$ is supported on $\mathrm{Sing}(X)$.

Theorem 5.5. *Let C be the base locus of the linear system of cubics defining π^{-1}. We assume $L \cap \mathrm{Sing}(X) = \{o\}$. We also denote by D the directrix of the cone $\mathrm{CL}_o(X)$. We get the following assertions about the genus two quintic curve C, depending on the choice of o and $\{p\} = L \cap D$:*
(a) *Suppose $\sigma(X) = [(111)111]$. The quintic C is an elliptic plane cubic union a skew line with a double structure, through a point of the cubic.*
(b) *Suppose $\sigma(X) = [(211)11]$. Then o has one special position r and we have:*

($b1$) *If $o = r$, then C consists of a smooth conic union a secant line supporting a triple structure.*

($b2$) *If o is general, then C consists of a plane nodal cubic curve union a skew line through the node, the line supporting a double structure.*

(c) *Suppose $\sigma(X) = [(311)1]$. Then o has one special position r and we have:*

($c1$) *If $o = r$, then C consists of the union of two lines one of them with a fourfold structure.*

($c2$) *If o is general, then C consists of a plane cuspidal cubic curve union a skew line through the cusp, the line supporting a double structure.*

(d) *Suppose $\sigma(X) = [(411)]$. Then o has one special position r and we have:*

($d1$) *If $o = r$, then according to the choice of $p \in D$ the curve C consists of a unique line supporting a fivefold structure, or the union two lines one of them supporting a fourfold structure.*

($d2$) If o is general then according to the choice of $p \in D$, the curve C consists of a plane cuspidal cubic curve union a skew line through the cusp supporting a double structure, or a smooth conic union a tangent line union a skew line, through the point of tangency, supporting a double structure.

(e) Suppose $\sigma(X) = [(221)1]$. Then, o has two special positions r_1, r_2, and we have:

($e1$) If $o \in \{r_1, r_2\}$, then C consists of two lines one of them with a fourfold structure.

($e2$) If o is general, then C consists of a conic union a secant with a triple structure.

(f) Suppose $\sigma(X) = [(321)]$. Then o has one special position r and we have:

($f1$) If $o = r$ then C consists of the union two lines one of them supporting a fourfold structure.

($f2$) If o is general, then according to the choice of $p \in D$, the curve C consists of a smooth conic union a tangent line supporting a triple structure, or the union of three concurrent lines one of these with a triple structure.

Proof. Again we keep notations as in Proposition 4.6 and its proof and each time use this result together with Proposition 4.4.

(a) In this case $\mathrm{Sing}(X)$ is a smooth conic. As follows from Proposition 4.6($c1$) the directrix D is a smooth elliptic quartic curve for all $o \in \mathrm{Sing}(X)$. We give the proof for general o and p.

Without loss of generality and to simplify computations, we may choose the λ's and the variables in such a way that $X = K \cap G$, where

$$K = x_4^2 - x_5^2 + 4x_6^2,\ G = x_1^2 - x_2^2 - x_3^2 + x_4^2 + x_5^2 - 2x_6^2.$$

Hence $\mathrm{Sing}(X)$ is defined by $(x_1^2 - x_2^2 - x_3^2 = x_4 = x_5 = x_6 = 0)$ and may be parametrized by $o = (b^2+1 : b^2-1 : 2b : 0 : 0 : 0)$. Therefore T_oG is defined by $(b^2+1)x_1 - (b^2-1)x_2 - 2bx_3 = 0$.

On the other hand, we may take $D = T_oG \cap X \cap H$, where H is the hyperplane not containing o, defined by $(x_1 = x_2)$; then D may be formally parametrized by

$$p_b(t) = (b\sqrt{2(t^4 - t^2 + 1)} : b\sqrt{2(t^4 - t^2 + 1)} : \sqrt{2(t^4 - t^2 + 1)} : t^2 - 1 : t^2 + 1 : t).$$

If $b \neq 0$ and $t \neq 0$, we may choose $M = \mathbb{P}^3$ defined by $(x_3 = x_6 = 0)$. As in the proofs of preceding results we compute π^{-1} and deduce the following generators for $\mathcal{I}(C_{t,b})$:

$$((t^2-1)x_4 - (t^2+1)x_5)((1-b^2)x_2 + (b^2+1)x_1)$$

$$t(x_4^2 - x_5^2)((1-b^2)x_2 + (b^2+1)x_1)$$

$$(x_4 + x_5)(2x_4x_5 + x_1^2 - x_2^2) + t^2(x_4 - x_5)(2x_4x_5 - x_1^2 + x_2^2) + b\sqrt{2(t^4 - t^2 + 1)}(x_1 - x_2)(x_4^2 - x_5^2).$$

For $t^4 - t^2 + 1 \neq 0$, by putting $b = b'/\sqrt{2(t^4 - t^2 + 1)}$ and multiplying the two first generators above by $t^4 - t^2 + 1$ we obtain generators for $\mathcal{I}(C_{t,b'})$ whose coefficients are polynomials in t and b'. From this we obtain the primary decomposition of $\mathcal{I}(C_{t,b'})$ verifying statement (a).

(b) In this case $\mathrm{Sing}(X)$ is a rank 2 conic, the union of two lines, whose intersection r is a special point.

($b1$) If $o = r$ the directrix D consists of two smooth conics meeting in two points lying in $\mathrm{Sing}(X)$. Then $L = \overline{rp}$, for p a smooth point in D. It follows C consists of a conic, projection of one of the conics, union a secant line, projection of the other conic. This line supports a triple structure. More precisely:

We can take $X = K \cap G$ with

$$K = x_1^2 - x_2^2 + x_5^2 \quad G = x_1^2 + x_2^2 + x_3^2 - x_4^2 + 2x_6x_5$$

Then $\mathrm{Sing}(X)$ is given by the two lines: $x_1 = x_2 = x_5 = x_3^2 - x_4^2 = 0$. The special singular point is the intersection of these two lines $o = r = (0:0:0:0:0:1)$ and a directrix D of $\mathrm{CL}_o(X)$ may be given by $(x_5 = x_6 = x_1^2 - x_2^2 = x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0)$. Then we may parametrize one of the conics, avoiding one of the singular points of D, by $p(t) := (t\sqrt{2} : t\sqrt{2} : t^2 - 1 : t^2 + 1 : 0 : 0)$, with $t \neq 0$ to avoid the other singular points of D. By taking $M = \mathbb{P}^3$ defined by $x_1 = x_6 = 0$ we compute, as before, the quintic C_t. We obtain the conic given by

$$(x_5, (t^2 - 1)x_2x_3 - tx_3^2 - (t^2 + 1)x_2x_4 + tx_4^2)$$

union the secant line (x_5, x_2) with the triple structure given by

$$(x_5^3, x_2x_5, 2tx_2x_3^2 - 2tx_2x_4^2 + (t^2 - 1)x_3x_5^2 - (t^2 + 1)x_4x_5^2, x_2^2).$$

This proves $(b1)$.

$(b2)$ For o general the quintic consists of a nodal cubic curve union a skew line through the node supporting a double structure.

If the point $o = (0:0:1:\pm 1:0:b)$ is general we find D is a rational quartic curve, with an ordinary singular point at $(0:0:0:0:0:1)$, given by $x_1^2 - x_2^2 + x_5^2 = 0$ and $x_1^2 + x_2^2 - b^2x_5^2 + 2x_6x_5 = 0$. Parametrizing this curve and considering the center of projection $L = \overline{op}$,, $p = p(t) \in D$ and $M = (x_3 = x_4 = 0)$, we can compute the quintic C_t and find it does not depend on the parameter b and is the plane nodal cubic curve

$$(x_5, -(t^2 + 1)x_1^2x_2 + (t^2 - 1)x_1x_2^2 - tx_1^2x_6 + tx_2^2x_6)$$

union the skew line going through the node :

$$(2tx_2 - (t^2 + 1)x_5, (-t^2 + 1)x_1 + (t^2 + 1)x_2 - 2tx_5)$$

with the double structure given by:

$$((-4t^2)x_2^2 + (4t^3 + 4t)x_2x_5 - (t^4 + 2t^2 + 1)x_5^2, (-t^2 + 1)x_1 + (t^2 + 1)x_2 - 2tx_5).$$

(c) The proof here is similar to (b).

(d) In this case $\mathrm{Sing}(X)$ is a rank 2 conic, the union of two lines, whose intersection r is a special point.

$(d1)$ If $o = r$, a directrix D consists of a rank 2 conic with a double structure. If the line $L = \overline{rp}$, with p the singular point of D, let's show that the quintic C consists of a line with a fivefold structure. If $p \in D$ general, then C consists of two meeting lines, one with a fourfold structure.

The equations of X can be taken to be $K = 2x_1x_3 + x_2^2$ and $G = 2x_1x_4 + 2x_2x_3 + x_5^2 - x_6^2$. $\mathrm{Sing}(X)$ is the rank 2 conic given by $x_1 = x_2 = x_3 = x_5^2 - x_6^2 = 0$ and the special point is the intersection of the two lines $o = r = (0:0:0:1:0:0)$. The intersection $\mathrm{CL}_o(X)$ with the hyperplane $(x_4 = 0)$ gives a directrix D, which is the rank 2 conic with double structure $x_1 = x_4 = x_2^2 = 2x_2x_3 + x_5^2 - x_6^2 = 0$; notice that $\mathrm{Sing}(X) \cap D$ is supported on $\{(0:0:0:0:1:\pm 1)\}$. Projecting from the line $L = \overline{op}$, with $p = (0:0:1:0:0:0)$ being the intersection of the two lines of D, to $M = \mathbb{P}^3 = (x_3 = x_4 = 0)$, we find the quintic C is the line $(x_1 = x_2 = 0)$ with the fivefold structure given by the ideal

$$(x_1^2, x_2^3 - x_1x_5^2 + x_1x_6^2, x_1x_2^2).$$

If, however, we take $o = r$ to be special, $p(t)$ to be general in one of the lines of the support of D, and M defined by $(x_4 = x_5 = 0)$, we find that the quintic C_t consists, for each $t \neq 0$, of the union the two lines defined by the ideals $(tx_2 - x_6, x_1)$ and (x_2, x_1), the second with the fourfold structure given by

$$(x_2^4, -tx_2^3 + x_2^2x_6 + 2x_1x_3x_6 - tx_1x_6^2, -2tx_1x_2x_3 + t^2x_1x_2x_6 - x_2^2x_6 - 2x_1x_3x_6 + tx_1x_6^2, x_1x_2^2, x_1^2).$$

We note that if $t = 0$ the line $\overline{op(0)}$ is contained in $\mathrm{Sing}(X)$ and then π^{-1} is not birational in this case.

$(d2)$ If o is general in $\mathrm{Sing}(X)$, then D consists of a twisted cubic union a tangent line. For $L = \overline{op}$, then according to the choice of $p \in D$ we have the two possibilities for the quintic as in the statement.

The proof is similar to other ones that have appeared before and we omit the details.

(e) This is also similar to the proofs that have been done elsewhere.

(f) Here $\mathrm{Vert}(X)$ is a rank one conic. There is a special position $o = r \in \mathrm{Sing}(X)$. In this case a directrix D of $\mathrm{CL}_o(X)$ consists of a double rank 2 conic, union of two lines meeting at a point $p_0 \in \mathrm{Sing}(X)$. For every line $L = \overline{rp}$, $p \in D \setminus \{p_0\}$, the quintic C consists of two lines, one with a fourfold structure.

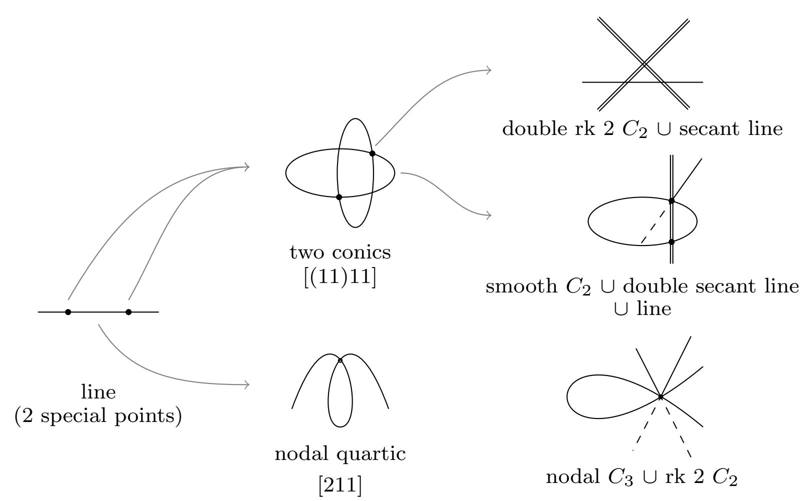
If $o \in \mathrm{Sing}(X)$ is general, then D consists of a smooth conic C_2 union two lines meeting C_2 in a point $p_1 \in \mathrm{Sing}(X)$. When we project from $L = \overline{op}$, $p \in D \setminus \{p_1\}$, the quintic C is either a union of three lines meeting in a point, one of them supporting a triple structure, if $p \in C_2$; or is a smooth conic union a tangent line supporting a triple structure if p lies in either one of the two lines in D. We omit the details similar to what we prove in the other items. $\square$

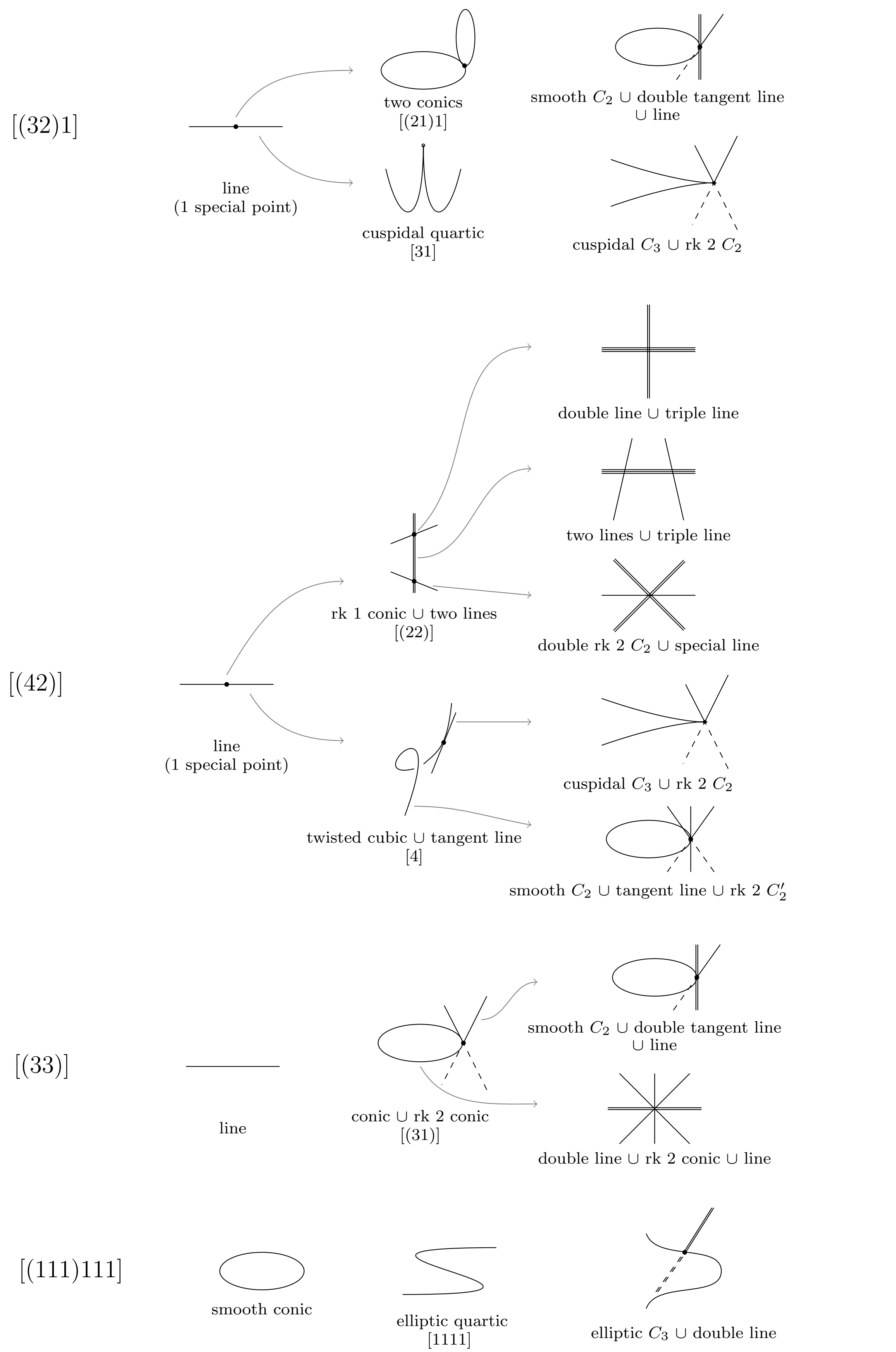
Graphically, the conclusions of Theorems 5.4 and 5.5 may be summarized in the following table of pictures.

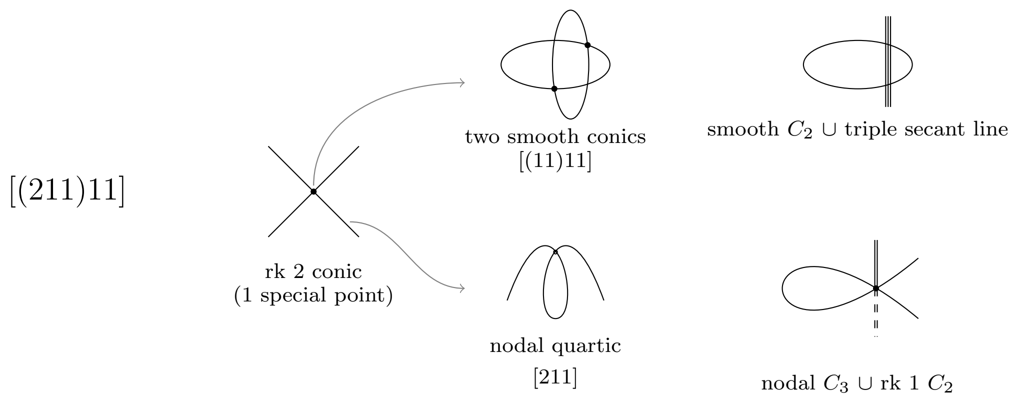
Table of results for $\dim(\mathrm{Sing}(X)) = 1$

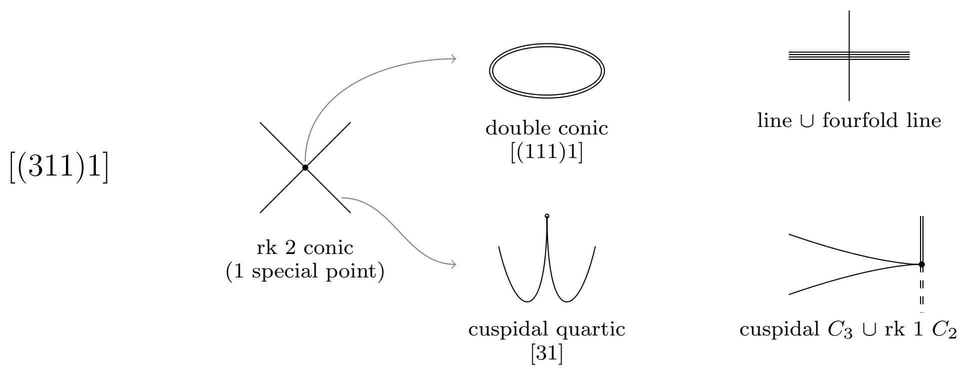
$\underline{\sigma(X)}$ | $\underline{\mathrm{Vert}(X)}$ | $\underline{\text{directrix } D}$ | $\underline{\text{quintic curve } C}$

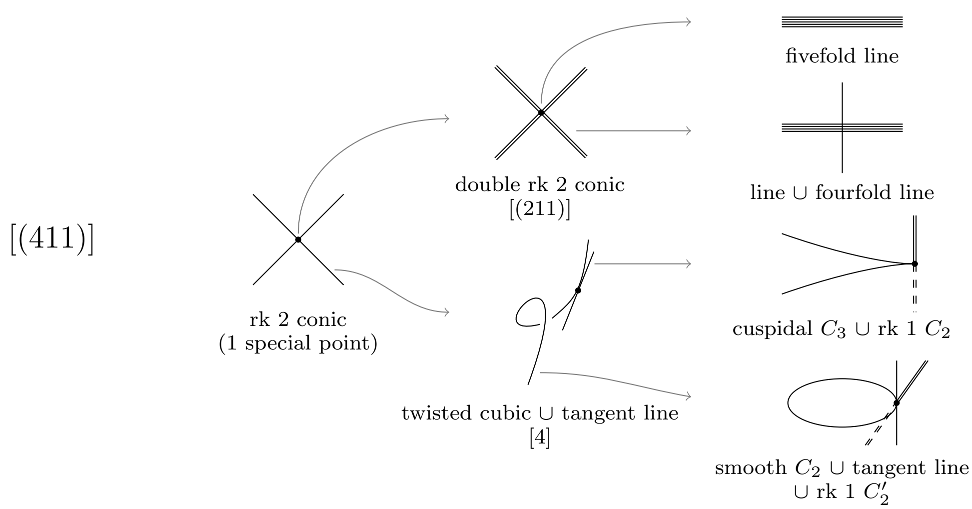
[(22)11]

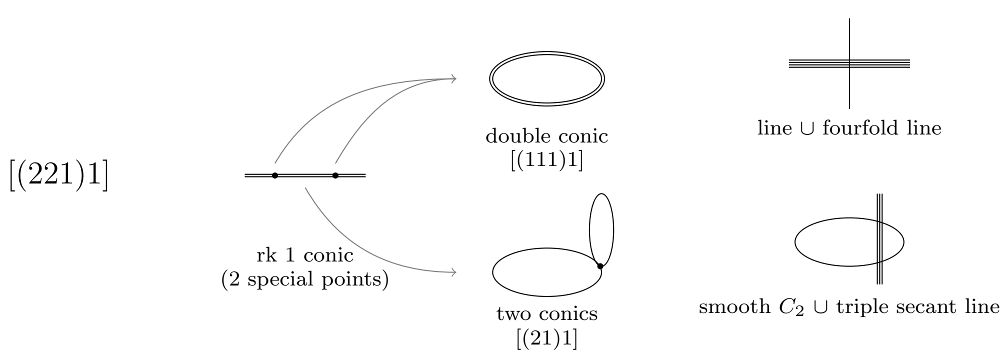



[(32)1]
line
(1 special point)
two conics
[(21)1]
cuspidal quartic
[31]
smooth C_2 ∪ double tangent line
∪ line
cuspidal C_3 ∪ rk 2 C_2
[(42)]
line
(1 special point)
rk 1 conic ∪ two lines
[(22)]
twisted cubic ∪ tangent line
[4]
double line ∪ triple line
two lines ∪ triple line
double rk 2 C_2 ∪ special line
cuspidal C_3 ∪ rk 2 C_2
smooth C_2 ∪ tangent line ∪ rk 2 C_2'
[(33)]
line
conic ∪ rk 2 conic
[(31)]
smooth C_2 ∪ double tangent line
∪ line
double line ∪ rk 2 conic ∪ line
[(111)111]
smooth conic
elliptic quartic
[1111]
elliptic C_3 ∪ double line

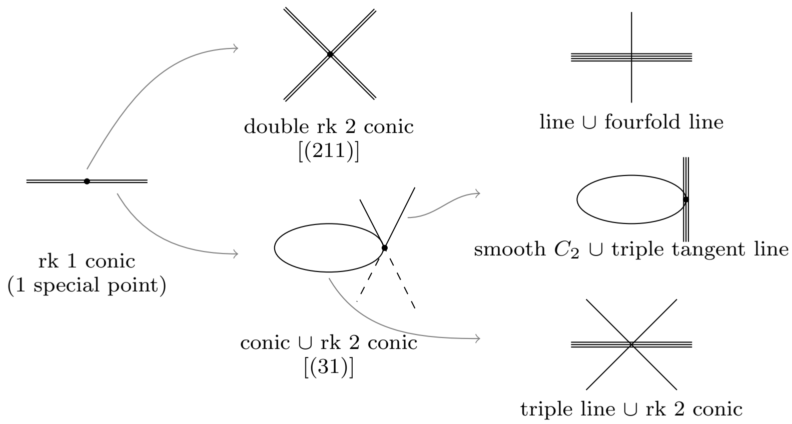

[(211)11]
rk 2 conic
(1 special point)
two smooth conics
[(11)11]
smooth C_2 ∪ triple secant line
nodal quartic
[211]
nodal C_3 ∪ rk 1 C_2


[(311)1]
rk 2 conic
(1 special point)
double conic
[(111)1]
line ∪ fourfold line
cuspidal quartic
[31]
cuspidal C_3 ∪ rk 1 C_2


[(411)]
rk 2 conic
(1 special point)
double rk 2 conic
[(211)]
fivefold line
line ∪ fourfold line
twisted cubic ∪ tangent line
[4]
cuspidal C_3 ∪ rk 1 C_2
smooth C_2 ∪ tangent line
∪ rk 1 C_2'


[(221)1]
rk 1 conic
(2 special points)
double conic
[(111)1]
line ∪ fourfold line
two conics
[(21)1]
smooth C_2 ∪ triple secant line

[(321)]



5.3. **Quadratic complexes singular along a plane.** Finally, we consider the case where $\mathrm{Sing}(X)$ is a plane. Then we have $\sigma(X) = [(222)]$, and we may take $X = K \cap G$ with $K = x_1^2 + x_3^2 + x_5^2$ and $G = x_1x_2 + x_3x_4 + x_5x_6$.

The complex X is singular along the plane α given by $x_1 = x_3 = x_5 = 0$, with a conic of special singular points $C_\alpha \subset \alpha$ given by $x_1 = x_3 = x_5 = x_2^2 + x_4^2 + x_6^2 = 0$. Let $\bar{C}$ be the conic directrix of the cone K given by $x_2 = x_4 = x_6 = x_1^2 + x_3^2 + x_5^2 = 0$. We then have:

Lemma 5.6. (*a*) *Consider the map* $\varphi : \bar{C} \longrightarrow C_\alpha$ *given by* $(a : 0 : b : 0 : c : 0) \mapsto (0 : a : 0 : b : 0 : c)$. *Then we have*

$$X = \bigcup_{\bar{o} \in \bar{C}} < \bar{o}, t_{\varphi(\bar{o})} >$$

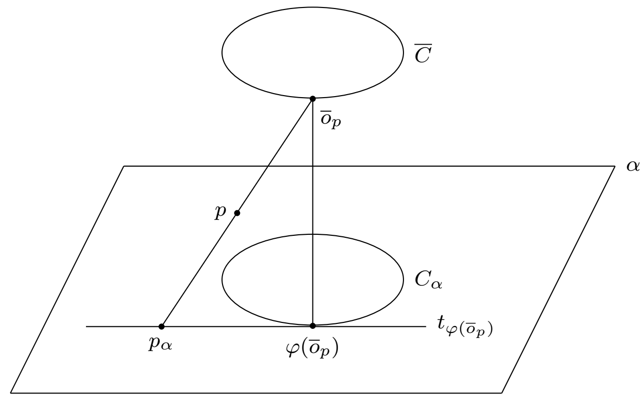
where $t_{\varphi(\bar{o})}$ *is the tangent line to* C_α *at* $\varphi(\bar{o})$ *and* $< \bar{o}, t_{\varphi(\bar{o})} >$ *is the plane spanned by* $\bar{o}$ *and* $t_{\varphi(\bar{o})}$.

The set of these planes and α *is the set of all the planes contained in* X. *In particular every line in* X *meets* α.

(*b*) *The complex* X *parametrizes the set of lines in* $\mathbb{P}^3$ *intersecting a smooth conic. If* $\alpha^* \subset \mathbb{P}^3$ *denotes the plane containing this smooth conic* C_{α^*}, *then the singular plane* α *parametrizes the set of lines in* α^*, *and* C_α *parametrizes the set of tangent lines to* C_{α^*}.

(*c*) *The group of automorphisms of* X, $Aut(X)$, *acts transitively on* C_α, *on* $\alpha \setminus C_\alpha$, *and on* $X \setminus \alpha$.

Proof. (*a*) Let $\bar{o} = (a : 0 : b : 0 : c : 0)$ be a point in $\bar{C}$, then $a^2 + b^2 + c^2 = 0$, and $\varphi(\bar{o}) = (0 : a : 0 : b : 0 : c) \in \bar{C}_\alpha$. Since $T_{\bar{o}}X$ is given by $ax_1 + bx_3 + cx_5 = ax_2 + bx_4 + cx_6 = 0$, then $T_{\bar{o}}X \cap X$ contains the plane $< \bar{o}, t_{\varphi(\bar{o})} >$. If $p \in X \setminus (\alpha \cup \bar{C})$ there are points $\bar{o}_p \in \bar{C}$ and $p_\alpha \in \alpha$ such that p is in the line $\overline{\bar{o}_p p_\alpha} \subset X$. Then $p \in < \bar{o}_p, t_{\varphi(\bar{o}_p)} >$, and this is the only plane in X through p. We deduce (*a*).



(b) Now we want to view G as the Plücker quadric parametrizing the lines in $\mathbb{P}^3$. If we let the Plücker coordinates be $(p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}) = (y_1 : y_2 : y_3 : y_4 : y_5 : y_6)$, then we compose with the automorphism A of $\mathbb{P}^5$ given by

$$(y_1 : y_2 : y_3 : y_4 : y_5 : y_6) \mapsto (y_3 : y_4 : y_5 : -y_2 : y_6 : y_1) = (x_1 : x_2 : x_3 : x_4 : x_5 : x_6).$$

This takes the usual Plücker quadric given by $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = y_1y_6 - y_2y_5 + y_3y_4 = 0$ to G given by $x_1x_2 + x_3x_4 + x_5x_6 = 0$. Let $(x : y : z : w)$ be coordinates for $\mathbb{P}^3$ and consider the plane $\alpha^* \subset \mathbb{P}^3$ given by $w = 0$ and the conic $C_{\alpha^*} \subset \alpha^*$ given by $x^2 + y^2 + z^2 = w = 0$. Let $(a : b : c : 0) \in C_{\alpha^*}$ be an arbitrary point and take the line joining it to $(0 : 0 : 0 : 1)$, and let us compute its image under the Plücker embedding Φ followed by A. We find:

$$\Phi \begin{pmatrix} a & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0 : 0 : a : 0 : b : c) \overset{A}{\mapsto} (a : 0 : b : 0 : c : 0)$$

Keeping in mind that $a^2+b^2+c^2 = 0$ we find that the cone of lines in $\mathbb{P}^3$ with vertex $(0 : 0 : 0 : 1)$ and directrix the conic C_{α^*} maps to the conic $\bar{C} \subset \mathbb{P}^5$ in the plane $x_2 = x_4 = x_6 = 0$.

In a similar way, we prove that the set of tangent lines to C_{α^*} (the dual curve) maps to the curve $C_\alpha \subset \mathbb{P}^5$, in the plane $x_1 = x_3 = x_5 = 0$:

$$\Phi \begin{pmatrix} a & b & 1 & 0 \\ b & -a & 0 & 0 \end{pmatrix} = (1 : -b : 0 : a : 0 : 0) \overset{A}{\mapsto} (0 : a : 0 : b : 0 : 1),$$

which is in C_α if $a^2 + b^2 + 1 = 0$. Similarly for the two missing points of C_{α^*} when $c = 0$.

$$\Phi \begin{pmatrix} 1 & \pm i & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (0 : 1 : 0 : \pm i : 0 : 0) = (0 : \mp i : 0 : 1 : 0 : 0) \overset{A}{\mapsto} (0 : 1 : 0 : \pm i : 0 : 0).$$

Therefore, if $t_{(a:b:c:0)}$ is the tangent line to C_{α^*} at $(a : b : c : 0) \in C_{\alpha^*}$, we have $\Phi(t_{(a:b:c:0)}) = (c : -b : 0 : a : 0 : 0) \overset{A}{\mapsto} (0 : a : 0 : b : 0 : c) = \varphi((a : 0 : b : 0 : c : 0))$, where $\varphi : \bar{C} \longrightarrow C_\alpha$ is defined in (a).

In the same way we see that the set of lines in α^* maps to α, and for each point of C_{α^*} the set of lines in $\mathbb{P}^3$ meeting this point maps to the plane in X different from α through a point of C_α. Then (b) follows from (a).

(c) Any automorphism of $\mathbb{P}^3$ stabilizing the conic C_{α^*} stabilizes the set of lines meeting this conic, so it induces an automorphism of X, by the description of X given in (b). The subgroup of such automorphisms of $\mathbb{P}^3$ stabilizes also the plane α^*, and acts transitively on $\mathbb{P}^3 \setminus \alpha^*$. It follows that it acts transitively on the set of tangent lines to C_{α^*} as well as on the set of lines in α^* not tangent to C_{α^*}, and on the set of lines not contained in α^* meeting C_{α^*}. The result then follows from (b). □

Theorem 5.7. *Let X be a quadratic complex with $\sigma(X) = [(222)]$. For any singular point $o \in \alpha$, special or general, and for any line L in X through o not contained in α, the base locus C of the linear system of cubics defining π_L^{-1} consists of two meeting lines, one of them supporting a fourfold structure.*

Proof. First note that by the description and the properties of X given in the previous lemma, for each point $o \in \alpha$, the subgroup of $Aut(X)$ fixing o acts transitively on the set of lines in X through o not contained in α. In fact such a line in X parametrizes a pencil of lines in $\mathbb{P}^3$ with focus a point of C_{α^*}, and the pencil is in a plane different from α^* and containing the line ℓ_o in α^* corresponding to o. The line ℓ_o is a tangent line or a secant line to C_{α^*} if o is a special or a general singular point respectively, and the focus of each pencil of lines we consider is a point in $\ell_o \cap C_{\alpha^*}$. Since the subgroup of automorphisms of $\mathbb{P}^3$ considered in the proof of (c) acts transitively on $\mathbb{P}^3 \setminus \alpha^*$, then the subgroup stabilizing also ℓ_o acts transitively on the set of pencils of lines we consider.

Since $Aut(X)$ also acts transitively on C_α and on $\alpha \setminus C_\alpha$, then it is sufficient to consider only one special singular point o_1 and one general singular point o_2 , and for each one a line L in X not contained in α trough that point.

Let $o_1 = (0 : 1 : 0 : i : 0 : 0) \in C_\alpha$, and $L = \overline{o_1 p}$, with $p = (1 : 0 : i : 0 : 0 : 0) \in \mathrm{CL}_{o_1}(X)$. Then $L \not\subset \alpha$ and we find that the ideal of the base locus C of π_L^{-1} is

$$\mathcal{I}(C) = (x_3^2, x_3(x_3^2 + x_5^2), x_4x_3^2 - x_4x_5^2 + 2x_6x_5x_3)$$

in terms of the coordinates of the 3-space given by $x_1 = x_2 = 0$. It follows that C consists of two lines, with ideals (x_3, x_4) and (x_3, x_5), the second one with the fourfold structure:

$$(x_3^2, x_5^3, x_3x_5^2, -x_4x_5^2 + 2x_3x_5x_6).$$

Let $o_2 = (0 : 0 : 0 : 0 : 0 : 1) \in \alpha \setminus C_\alpha$, and $L = \overline{o_2 p}$, with $p = (1 : 0 : i : 0 : 0 : 0) \in \mathrm{CL}_{o_2}(X)$. Then $L \not\subset \alpha$ and we find that the ideal of the base locus C of π_L^{-1} is

$$\mathcal{I}(C) = (x_3x_5, x_5(x_3^2 + x_5^2), ix_2(x_3^2 + x_5^2) + x_4(x_3^2 - x_5^2))$$

in terms of the coordinates of the 3-space given by $x_1 = x_6 = 0$. It follows that C consists of two lines, with ideals $(x_5, x_2 - ix_4)$ and (x_3, x_5), the second one with the fourfold structure:

$$(x_3x_5, x_3^3, x_5^3, x_2x_3^2 - ix_3^2x_4 + x_2x_5^2 + ix_4x_5^2).$$

This completes the proof. □

6. Cremona transformations and quadratic complexes

6.1. **Generalities.** A *Cremona transformation* of $\mathbb{P}^3$ is a birational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. Such a ϕ is defined by four pairwise relatively prime homogeneous polynomials of the same degree; this common degree is the *degree* of ϕ which we denote by $\deg(\phi)$, and the linear system associated to ϕ is, by definition, the linear system whose general member is the zero scheme of a linear combination of these four polynomials. The pair $(\deg(\phi), \deg(\phi^{-1}))$ is, by definition, the *bidegree* of ϕ.

If $Y \subset \mathbb{P}^3$ is an irreducible subvariety along which ϕ restricts birationally, the strict transform $\phi_*(Y)$ of Y, with respect to ϕ, is the closure of $\phi(Y - \mathrm{Base}(\phi))$.

One may prove $\deg(\phi)$ (respectively, $\deg(\phi^{-1})$) is the degree of the strict transform of a general plane (respectively, general line).

As always in this work let X be a singular quadratic complex. We take two lines $L_1, L_2 \subset X$ in such a way that $L_i \not\subset \mathrm{Sing}(X)$, $i = 1, 2$. Denote by $\pi_i : X \dashrightarrow M_i \simeq \mathbb{P}^3$ the projection from X with center L_i, $i = 1, 2$. We identify M_1 and M_2 with a fixed 3-space $\mathbb{P}^3$. Since π_1 and π_2 are birational, we know $\varphi_{12} := \pi_2 \circ \pi_1^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is a Cremona transformation.

The Cremona transformations of bidegree $(2, 2)$ and $(3, 3)$ were extensively studied by classic geometers (see for example [Cr1], [Cr2], [Co], [HU], or [P], [PRV] for more recent references). For the convenience of the reader we briefly describe the different types of such transformations.

Transformations of bidegree $(2, 2)$. The base locus scheme of such a transformation is supported on a conic and a point which does not belong to its plane. This conic may be smooth or have rank 1 or 2, and the point may be either a "proper" point or an infinitely near point lying over the conic. In fact, up to change of coordinates at the domain and the target, there are seven types of transformations of bidegree $(2, 2)$ (see [PRV, Tableau 1]). Furthermore, the inverse of such a map belongs to the same class. The set of all Cremona transformations of bidegree $(2, 2)$ is an irreducible variety of dimension 26 ([PRV, Pro. 2.4.1]).

Transformations of bidegree $(3, 3)$. The set of Cremona transformations of bidegree $(3, 3)$ is the union of three irreducible varieties of dimensions 39, 38 and 31, respectively. The largest variety consists of transformations whose base locus scheme is an ACM sextic curve of arithmetic genus 3; such a transformation is called *determinantal*, since such a scheme is defined by the maximal minors of a 4×3 matrix of linear forms. The dimension 38 variety consists of transformations ϕ whose base locus scheme has an embedded point (then it is not ACM) and may be characterized by saying that $\phi_*(R)$ is a singular cubic curve for a general line R; we call ϕ a *de Jonquières* transformation. Finally, the smallest variety consists of transformations whose associated linear system consists of cubic surfaces with a line of double points. By Bertini Theorem this line is common to all members of that linear system; we call such a transformation a *ruled* transformation (of bidegree $(3, 3)$) since a cubic surface of $\mathbb{P}^3$ with a line of double points is necessarily ruled. While there is no Cremona transformation which is both determinantal and de Jonquières, there exist ruled ones belonging to these two classes (see [P]).

Definition 6.1. *Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation. We say ϕ comes from quadratic complexes if there exist a quadratic complex X and lines $L_1, L_2 \subset X$ such that $\phi = \pi_2 \circ \pi_1^{-1}$. In this case, we also say ϕ comes from X, from (X, L_1) or even from (X, L_1, L_2).*

If $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ comes from a quadratic complex, as in Definition 6.1, then it is associated to a quadratic complex X, the two lines $L_1, L_2 \subset X$ and the two ACM quintic curves $C_1, C_2 \subset \mathbb{P}^3$, base loci of the projections π_1 and π_2, respectively; notice that in this case $L_i \not\subset \mathrm{Sing}(X)$ for $i = 1, 2$.

Proposition 6.2. *Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation coming from a quadratic complex X, with associated lines L_1, L_2 and ACM quintic curves C_1, C_2. We have:*

(a) If L_1 and L_2 intersect at a singular point of X, then ϕ has bidegree $(2, 2)$.

(b) If $L_1 \cap L_2 \cap \mathrm{Sing}(X) = \emptyset$, then ϕ has bidegree $(3, 3)$.

(c) If $\deg(\phi) = 3$, then $C_1 \subset \mathrm{Base}(\phi)$.

(d) If $\deg(\phi) = 2$, then there exists a plane $H \subset \mathbb{P}^3$ such that $C_1 \subset H \cup \mathrm{Base}(\phi)$ and H contains (at least) a component of C_1.

Proof. Let $H \subset \mathbb{P}^3$ be a general plane. The strict transform $(\pi_2^{-1})_*(H)$ is a quartic surface $S := \mathcal{H} \cap X$ where $\mathcal{H}$ is a hyperplane in $\mathbb{P}^5$ general among those containing L_2. Notice that L_1 intersects S at a unique point. Then $(\pi_1)_*(S) \subset \mathbb{P}^3$ is a surface of degree $4 - \mu$ where μ is the multiplicity of S at that point, $1 \leq \mu \leq 2$. If L_1 intersects L_2 at a singular point of X, then $\mu = 2$ and $\deg(\phi) = 2$. Otherwise $\deg(\phi) = 3$. By symmetry, we obtain the proof of (a) and (b) may be proved similarly.

We know $\phi = (f_1 : f_2 : f_3 : f_4)$ for homogeneous polynomials f_i's of degree $\deg(\phi)$. Since $\phi = \pi_2 \circ \pi_1^{-1}$, we deduce there are $f'_1, f'_2, f'_3, f'_4 \in \mathrm{H}^0(\mathcal{I}_{C_1}(3))$, where $\mathcal{I}_{C_1}$ is the ideal sheaf associated to C_1 (see Proposition 3.3), such that ϕ coincides with the rational map $\phi' : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined as $\phi' = (f'_1 : f'_2 : f'_3 : f'_4)$.

If $\deg(\phi) = 3$, then $f_1, f_2, f_3, f_4 \in \mathrm{H}^0(\mathcal{I}_{C_1}(3))$ and (c) follows.

If $\deg(\phi) = 2$, then there exists a linear form h such that hf_i is proportional to f'_i for all i, from which it follows $C_1 \subset H \cup \mathrm{Base}(\phi)$, where $H = (h = 0)$. On the other hand, the dimension 1 part of $\mathrm{Base}(\phi)$ defines a 1-cycle of degree 2: indeed, for two general quadrics Q_1, Q_2 in the linear system associated to ϕ, we have $Q_1 \cap Q_2 = \mathrm{Base}(\phi) \cup D$, where D is a smooth conic. Since $\deg(C_1) = 5$, we conclude H contains some component of C_1 which proves (d) and completes the proof of the proposition.

□

Remark 6.3. (a) There are Cremona transformations of bidegree $(2, 3)$ and $(2, 4)$. The proposition above implies they do not come from quadratic complexes.

(b) If ϕ comes from quadratic complexes, then ϕ^{-1} comes too. More precisely, if ϕ comes from (X, L_1, L_2), then ϕ^{-1} comes form (X, L_2, L_1).

6.2. **Cremona transformations coming from quadratic complexes.**

6.2.1. *Bidegree* $(3, 3)$ *case.* If $Y \subset \mathbb{P}^3$ is a pure dimension 1 subscheme, we denote by $\mathrm{Cyc}(Y)$ the 1-cycle defined by it.

Theorem 6.4. *Let X be a quadratic complex and let $L_1, L_2 \subset X$ be lines with $L_i \not\subset \mathrm{Sing}(X)$, $i = 1, 2$. Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation of degree 3 which comes from (X, L_1, L_2). Then:*

(a) If $L_1 \cap L_2 = \emptyset$, then ϕ is determinantal. In this case $L := \pi_1(L_2)$ is a line and $\mathrm{Base}(\phi) = C_1 \cup L$.

(b) If $L_1 \cap L_2 \neq \emptyset$, then ϕ is a de Jonquières transformation. In this case there exists a line $L \subset \mathbb{P}^3$ such that $\mathrm{Cyc}(\mathrm{Base}(\phi)) = C_1 + L$ *is the complete intersection of a cubic surface and a quadric, both containing C_1, and $\pi_1(L_2)$ is an embedded point of* $\mathrm{Base}(\phi)$.

Furthermore, ϕ is ruled if and only if $\sigma(X) = [(222)]$.

Proof. Parts (a) and (b) may be proven as in [AGP, Thm. 1].

Suppose ϕ is ruled and $\sigma(X) \neq [(222)]$. The first assumption implies there is a line $L_0 \subset \mathbb{P}^3$ such that all $S \in \Lambda_\phi$ is singular along L_0.

On the other hand, the second assumption implies $\dim \mathrm{Sing}(X) \leq 1$. Therefore, if $\mathcal{H} \subset \mathbb{P}^5$ is a general hyperplane containing L_2, the hyperplane section $\mathcal{H} \cap X$ of X is a normal surface: indeed, otherwise Bertini Theorem applied to π_2 implies $\mathcal{H} \cap X$ is singular along L_2, and then X should be singular there, which is not possible. We deduce $(\pi_1)_*(\mathcal{H} \cap X)$ is also normal (Lemma 3.2). By construction $(\pi_1)_*(\mathcal{H} \cap X) \in \Lambda_\phi$ which gives a contradiction.

Conversely, suppose $\sigma(X) = [(222)]$. Then $\dim \mathrm{Sing}(X) = 2$. Since Λ_ϕ is contained in the linear system associated to π_1, the assertion follows from Proposition 3.3. □

Remark 6.5. In the case where $\mathrm{Base}(\phi)$ is contained in a smooth surface of $\mathbb{P}^3$, the genus formula implies that the line whose existence is assured by Theorem 6.4 intersects C_1 at a point scheme of either length 2 or length 3, depending on whether ϕ is determinantal or de Jonquières.

Example 6.6. Keep notations as in the proof of Theorem 5.5, part (d1), and take L_1 to be the line $\overline{rp}$ therein. If π_1 is the projection with center L_1 to the 3−space given by $x_3 = x_4 = 0$, with coordinates x_1, x_2, x_5, x_6, we find

$$\pi_1^{-1} = (2x_1^3 : 2x_1^2x_2 : -x_1x_2^2 : x_2^3 - x_5^2x_1 + x_6^2x_1 : 2x_5x_1^2 : 2x_6x_1^2)$$

Now we consider a second projection $\pi_2 : X \longrightarrow M'$ centered at a line $L_2 \subset X$ to see what kind of bidegree $(3,3)$ Cremona transformation we get. There are two possibilities:

(1) $L_1 \cap L_2 = \emptyset$. We can take for instance $L_2 = (x_1 : 0 : 0 : 0 : x_5 : x_5) \subset X$ and M' the 3-subspace given by $x_1 = x_5 = 0$. By identifying $M = M'$ we get:

$$\pi_2\pi_1^{-1} = (2x_1^2x_2 : -x_1x_2^2 : x_2^3 - x_5^2x_1 + x_6^2x_1 : 2x_5x_1^2 - 2x_6x_1^2)$$

We find the base locus of this map is a sextic that decomposes as a single line union a fivefold line.

(2) $L_1 \cap L_2 \neq \emptyset$. More interestingly we can take $L_2 = (0 : 0 : x_3 : 0 : x_5 : x_5)$ intersecting L_1 at $(0 : 0 : 1 : 0 : 0 : 0)$ and project to $M = (x_3 = x_5 = 0)$. We get:

$$\pi_2\pi_1^{-1} = (2x_1^3 : 2x_1^2x_2 : x_2^3 - x_5^2x_1 + x_6^2x_1 : 2x_5x_1^2 - 2x_6x_1^2)$$

We find the base locus of this map is a line with a sixfold structure with an embedded point. Here is the ideal defining the sixfold structure supported at (x_1, x_2):

$$(x_2^6, -x_2^3 + x_1x_5^2 - x_1x_6^2, x_1x_2^3, x_1^2).$$

And here is the primary ideal with associated prime $(x_1, x_2, x_5 - x_6)$:

$$(x_5 - x_6, x_2^3, x_1^2x_2, x_1^3).$$

In the first case we get a determinantal transformation and in the second a de Jonquières.

Example 6.7. Consider X with $\sigma(X) = [(222)]$ and defined by way of

$$F = x_1^2 + x_3^2 + x_5^2 \text{ and } G = x_1x_2 + x_3x_4 + x_5x_6.$$

If L_1 and L_2 are given by $x_2 = x_3 = x_6 = 0, x_5 = ix_1$ and $x_1 = x_4 = x_6 = 0, x_5 = ix_3$, respectively, and $M = M'$ is given by $x_1 = x_4 = 0$, then

$$\pi_2\pi_1^{-1} = (ix_3(x_3^2 + x_5^2), -i(x_2x_3^2 + x_2x_5^2 + ix_6x_3^2 - ix_6x_5^2), -2ix_3^2x_5 - x_3(x_3^2 - x_5^2), 2x_3x_5x_6)$$

is a ruled determinantal Cremona transformation. A ruled de Jonquières transformation may be obtained analogously.

6.2.2. *Bidegree* $(2,2)$ *case.* Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation of bidegree $(2,2)$ which comes from (X, L_1, L_2) where $\sigma(X)$ is one of the given in Table 1. Then $L_1 \cap L_2 = \{o\} \subset \mathrm{Sing}(X)$. Denote by $\pi_o : X \dashrightarrow \mathcal{Q} \subset \mathbb{P}^4$ the map induced by the projection of $\mathbb{P}^5$ from o to a general 4-space $\mathbb{P}^4$ containing M, its image $\mathcal{Q}$ is then a hyperquadric in $\mathbb{P}^4$. If p_i denotes the image of L_i under π_o, $i = 1, 2$, we denote by $\pi_{p_i} : \mathcal{Q} \dashrightarrow \mathbb{P}^3$ the map induced by the projection of $\mathbb{P}^4$ from p_i, $i = 1, 2$.

Since $\phi = \pi_2 \circ \pi_1^{-1}$ we deduce there is a commutative diagram of the form

$$\begin{tikzcd}
& X \arrow[d, dashed, "\pi_o"] \arrow[ddl, dashed, "\pi_1"'] \arrow[ddr, dashed, "\pi_2"] & \\
& \mathcal{Q} \arrow[dl, dashed, "\pi_{p_1}"] \arrow[dr, dashed, "\pi_{p_2}"'] & \\
\mathbb{P}^3 \arrow[rr, dashed, "\phi"] & & \mathbb{P}^3
\end{tikzcd} \tag{4}$$

Note that since ϕ is birational p_1 and p_2 must be smooth points in $\mathcal{Q}$. From this description it is clear that ϕ is not defined only at $\pi_{p_1}(p_2)$ and at points lying in the image under π_{p_1} of lines in $\mathcal{Q}$ going through p_1. We deduce that $\mathrm{Base}(\phi)$ is defined by the ruling structure of the cone of lines in $\mathcal{Q}$ going through p_1 together with an additional structure supported on a special point; this special point is infinitely near to a point in the support of $\mathrm{Base}(\phi)$ if and only if the line $\overline{p_1p_2}$ is contained in $\mathcal{Q}$.

Recall that the singular set of a reduced and irreducible hyperquadric in $\mathbb{P}^4$ may be empty or consists of a point or a line; in particular such a hyperquadric is normal. In the first case the cone of lines going through a smooth point, p say, is a quadratic cone in its tangent space at p; in the second case it is the union of two planes intersecting along the line joining p with the singular point; in the last case it is the "double" plane spanned by p and the singular line.

Let $\eta : \mathbb{B}l_o(\mathbb{P}^5) \to \mathbb{P}^5$ be the blowing up of $\mathbb{P}^5$ at o; then $\pi_o\eta$ is a morphism which identifies the exceptional divisor E of η with $\mathbb{P}^4$. Denote by $\widetilde{X} \subset \mathbb{B}l_o(\mathbb{P}^5)$ the strict transform of X by η. We set $\mathrm{Sing}^{(1)}(X) := \mathrm{Sing}(\widetilde{X}) \cap E$ and say a point in $\mathrm{Sing}^{(1)}(X)$ is an *infinitely near* singularity of X.

Lemma 6.8. *Let X, $\mathcal{Q}$, p_1, p_2 be as in diagram (4). Then every singular point in $\mathcal{Q}$ is the projection under π_o of a line going through o which either is contained in $\mathrm{CL}_o(X)$ or meets another singularity (possibly infinitely near). Moreover, if K is the (unique) cone of multiplicity 2 in the pencil defining X, then $\pi_o(K \setminus \{o\}) = \mathcal{Q}$.*

Proof. Let $T \subset \mathbb{P}^5$ be a line going through o and not contained in $X = F \cap G$. Then T meets $X \setminus \{o\}$ in k points with $k = 0$ or $k = 1$. Assume $\pi_o\eta$ maps the strict transform $\widetilde{T}$ of T in a point of $\mathcal{Q}$.

Let us prove the first statement.

First suppose $k = 1$ and denote by p the point in $T \cap (X \setminus \{o\})$. Since $\mathcal{Q}$ is normal, Zariski Main Theorem implies the restriction of π_o to X defines a local isomorphism at p. We deduce $\pi_o(T)$ is singular in $\mathcal{Q}$ if and only if $p \in \mathrm{Sing}(X)$.

Now suppose $k = 0$. By construction we have $\mathcal{Q} = \pi_o(X \setminus \{o\}) \cup \pi_o\eta(\widetilde{X} \cap E)$. Since $\pi_o\eta$ contracts $\widetilde{T}$ to a point in $\mathcal{Q}$ we deduce $\widetilde{T} \cap \widetilde{X}$ consists of a point, p' say. Therefore the restriction of $\pi_o\eta$ to $\widetilde{X}$ induces a local isomorphism at p'. We conclude that $\pi_o\eta(p')$ is singular in $\mathcal{Q}$ if and only if $p' \in \mathrm{Sing}^{(1)}(X)$.

For the last assertion we notice that $\pi_o(K \setminus \{o\})$ is a dimension 3 irreducible variety containing $\pi_o(X \setminus \{o\}) = \mathcal{Q}$. $\square$

It follows (recall Table 1):

Corollary 6.9. *If* $\mathrm{Vert}(X)$ *is a point (resp. two points, a double point or a line; resp. a conic or a plane), then* $\mathrm{Sing}(\mathcal{Q})$ *is empty (resp. consists of a point; resp. is a line).*

By using Lemma 6.8 together with the geometric description of $\mathrm{CL}_o(X)$ for each possible Segre symbol we may obtain the classification of bidegree $(2,2)$ Cremona transformation coming from the corresponding quadratic complex. To simplify we only give the following:

Theorem 6.10. *Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation of bidegree $(2,2)$. There exists $\tau_1, \tau_2 \in \mathrm{Aut}(\mathbb{P}^3)$ such that $\tau_1\phi\tau_2$ comes from a singular quadratic complex.*

Proof. It suffices to prove that if $\phi = \pi_{p_2}\pi_{p_1}^{-1}$ comes from the diagram (4), then we obtain seven non projectively equivalent types for $\mathrm{Base}(\phi)$. We keep notations from diagram (4) and employ Lemma 6.8 and its corollary without explicit reference to them.

First suppose $\mathrm{Vert}(X)$ is a point, $o \in \mathrm{Sing}(X)$. Then $\mathcal{Q}$ is smooth. In this case the directrix of $\mathrm{CL}_o(X)$ may contain a line or not (Propositions 4.4 and 4.6). Then we may choose lines in X going through o in such a way that their projection under π_o define points p_1, p_2 which may either belong to a same line in $\mathcal{Q}$ or not. Since the cone of lines in $\mathcal{Q}$ going through a point is a cone over a smooth conic we deduce $\mathrm{Base}(\phi)$ consists of a conic together with either an infinitely near point not belonging to the conic plane, when $\overline{p_1p_2} \subset \mathcal{Q}$, or a point outside the conic plane. Hence we have two non projectively equivalent types of base locus schemes for ϕ.

Now suppose $\mathrm{Vert}(X)$ consists of two points (counting multiplicity) or it is a line, $o \in \mathrm{Sing}(X)$. Then $\mathcal{Q}$ has a unique singular point and the directrix of $\mathrm{CL}_o(X)$ may contain a line or not. Moreover, a linear component in this directrix may either intersects $\mathrm{Sing}(X)$ or not. We deduce that $\mathcal{Q}$ has a unique singularity and p_1, p_2 may be chosen either to be in the same line which may go through the singularity or not, or do not lie in a same line of $\mathcal{Q}$. Since the cone of lines in $\mathcal{Q}$ going through a smooth point is a cone over a rank 2 conic we deduce $\mathrm{Base}(\phi)$ consists of a rank 2 conic together with either an infinitely near point not belonging to the conic plane lying over a smooth (resp. the singular) point of the conic, or a point outside the conic plane. The two first cases occur when the line $\overline{p_1p_2} \subset \mathcal{Q}$ does not contain (resp. contains) the singular point of $\mathcal{Q}$. Hence we have three new non projectively equivalent types of base locus schemes for ϕ.

By considering the case where $\mathrm{Vert}(X)$ is a conic or a plane we likewise obtain two new non projectively equivalent types of base locus schemes for ϕ which completes the proof. ☐

Example 6.11. If $\sigma(X) = [(222)]$ we may suppose $K = (x_1^2 + x_3^2 + x_5^2 = 0)$. If $o = (0:1:0:0:0:0)$, by projecting onto $\mathbb{P}^4$ with coordinates x_1, x_3, x_4, x_5, x_6 we obtain $\mathcal{Q} = (x_1^2 + x_3^2 + x_5^2 = 0) \subset \mathbb{P}^4$, which is singular along the line $R = (x_1 = x_3 = x_5 = 0)$ in $\mathbb{P}^4$; note that a line in $\mathcal{Q}$ necessarily meets R: indeed, a hyperplane section of $\mathcal{Q}$ which is general among those containing a line $L \neq R$ is a quadratic cone containing L in its ruling. If $L_1, L_2 \subset \mathbb{P}^5$ are lines going through o, then their projection under π_o gives $p_1, p_2 \in \mathcal{Q} - R$. Consider $\varphi = \pi_{p_2}\pi_{p_1}^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

The strict transform of a general hyperplane in $\mathbb{P}^4$ under $\pi_{p_1}^{-1}$ is a general hyperplane section of $\mathcal{Q}$ going through p_1. Hence it is a quadratic cone in a 3-dimensional linear space of $\mathbb{P}^4$ which does not contain p_2. Therefore the strict transform of a general plane under φ is a quadratic cone whose vertex varies along the line $R_o := \pi_{p_2}(R) \subset \mathbb{P}^3$. Moreover, since every line joining p_2 to a point in R is contained in $\mathcal{Q}$ we deduce that R_o is the dimension 1 part of $\mathrm{Base}(\varphi)$. From [PRV, Thm. 3.1.1], and with notations as *loc. cit*, it follows φ is a bidegree $(2,2)$ Cremona transformation of type either $\tan^{[2]}(//)$, when $\overline{p_1p_2} \subset \mathcal{Q}$, or $\mathrm{gen}^{[2]}(//)$ otherwise.

Finally, with notations as in § 5.3, if $o \in C_\alpha$, then $\langle L_1, L_2 \rangle \subset X$ from which we deduce $\overline{p_1p_2} \subset \mathcal{Q}$, that is φ is of type $\tan^{[2]}(//)$. If $o \in \alpha \backslash C_\alpha$, then we may choose L_1, L_2 with either $\langle L_1, L_2 \rangle \subset X$ or $\langle L_1, L_2 \rangle \cap X = L_1 \cup L_2$; hence $\overline{p_1p_2} \subset \mathcal{Q}$ or $\overline{p_1p_2} \not\subset \mathcal{Q}$, in other words, φ is of type $\tan^{[2]}(//)$ or $\mathrm{gen}^{[2]}(//)$, respectively.

References

[AL] D. Avritzer and H. Lange, *Moduli spaces of quadratic complexes and their singular surfaces*, Geom. Dedicata 127 (2007), 177-179.

[AGP] D. Avritzer, G. Gonzalez-Sprinberg, I. Pan, *On Cremona Transformations and Quadratic Complexes*, Rendiconti del Circolo Matematico di Palermo,Vol. **57** (2008), 353-375 .

[Co] F. Conforto, *Le superficie razionali*, Zanichelli, Bologna, 1939.

[Cr1] L. Cremona, *Sulle trasformazione geometriche delle figure piane*, Bologna Mem. (2) 2 (1863), 621-630, and (2) 5 (1865), 3-35.

[Cr2] L. Cremona, *Über die Abbildung algebraischer Fläschen* , Math. Ann. 4 (1871), 213-230.

[DGPS] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 3-1-3 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2011).

[GH] Ph. Griffiths, J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons, 1978.

[H] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, Sixth edition 1993.

[HP] W.V.D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, Vol. II, Cambridge Univ. Press, 1952.

[HU] H. Hudson, *Cremona Transformations in Plane and Space*, Cambridge Univ. Press, 1927.

[J] C.M. Jessop, *A Treatise on the Line Complex*. Cambridge Univ. Press 1903.

[P] I. Pan, *Sur les transformations de Cremona de bidegré (3,3)*, l'Enseignement Math., t. 43 (1997), 285-297.

[PRV] I. Pan, F. Ronga, T. Vust, *Transformations birationnelles quadratiques de l'espace projectif complexe à trois dimensions*, Ann. Inst. Fourier **51**, 5 (2001), 1153-1187.

[PS] C. Peskine, L. Szpiro, *Liaison des variétés algébriques. I*, Inventiones Math. 26 (1974), 271-302.

Dan Avritzer Departamento de Matemática-UFMG, Belo Horizonte, MG, Brasil
dan@mat.ufmg.br

Gerard Gonzalez-Sprinberg Institut Fourier-UJF, BP 74, 38402 St-Martin-d'Hères, France
gonsprin@ujf-grenoble.fr

Ivan Pan Centro de Matemática-UDELAR, Montevideo, Uruguay
ivan@cmat.edu.uy