ON CREMONA TRANSFORMATIONS AND QUADRATIC COMPLEXES

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1. Introduction

Quadratic complexes and Cremona transformations are classical subjects. The study of quadratic complexes goes back at least to F. Klein (see [K]). In the beginning of the 20th century, Jessop, among others, studied extensively the quadratic line complex and the associated Kummer surface (see [Je],[H2]). More recently there has been a lot of research on the subject putting it in the context of contemporary geometric invariant theory with applications to vector bundles (see [Ne],[NR] and [AL]).

Cremona transformations appear also in the 19th century. The subject was introduced by Luigi Cremona in [Cr1] and extensively developed thereafter (see for example [Ca],[No],[Cr2],[Cr3],[Jo],[Cas],[Chi]). The younger sister of the older Hudson, Hilda Hudson, wrote a comprehensive book about Cremona transformations in plane and space [H1] and, as it was the case with quadratic complexes, there has been a lot of contemporary research on the subject (for higher dimension see for example [ESB],[CK1],[CK2], [P1],[P2],[RS],[PR],[GSP2]).

But, to our knowledge, the relation between the two subjects has not been studied before. This is the aim of this paper. The connection is the following. Let Q_1 denote a smooth hyperquadric in $\mathbb{P}^5$ over the field of complex numbers, considered as the Plücker hyperquadric parameterizing lines in $\mathbb{P}^3$. A *quadratic complex* or to be more precise a *quadratic line complex* is by definition a complete intersection $X = Q_1 \cap Q_2$, with a hyperquadric $Q_2 \subset \mathbb{P}^5$ different from Q_1. We assume that X is smooth, unless stated otherwise. This means that the pencil $\lambda Q_1 + \mu Q_2$ is general, i.e., the roots of $det(\lambda Q_1 + \mu Q_2)$ are all distinct (here, by abuse of notation, Q_i represents both the quadric and its associated matrix).

Take two lines $L_1, L_2 \subset \mathbb{P}^5$, $L_1 \neq L_2$, both contained in X. Fix general 3-planes $M_i \sim \mathbb{P}^3, i = 1, 2$ in $\mathbb{P}^5$, and define projections $\pi_i : \mathbb{P}^5 \dashrightarrow M_i$, $i = 1, 2$, with centers L_1 and L_2, respectively; we assume $M_i \cap L_i = \emptyset$, $i = 1, 2$. The map $\varphi = \varphi_{L_1,L_2} := \pi_2\pi_1^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is a Cremona transformation that is, as we shall see, a so-called cubo-cubic Cremona transformation, meaning both φ and its inverse have (algebraic) degree 3. In §2 and §3 we recall quadratic complexes and cubo-cubic Cremona transformations as well as the classification of these Cremona transformations in space.

The nature of φ_{L_1,L_2} depends on the relative position of the lines L_1, L_2 we are projecting from; this map is a cubo-cubic Cremona transformation which is determinantal if L_1 and L_2 do not meet and de Jonquières otherwise. In both cases the base locus scheme contains a smooth quintic curve of genus 2 and a line. This is the first main result of the paper (see Theorem 1 in §3).

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Conversely, let C be a smooth, quintic curve of arithmetic genus 2 in $\mathbb{P}^3$. In §4, we prove that every cubo-cubic Cremona transformation φ containing C in its base locus factorizes through a quadratic complex X via two linear projections as above. Moreover the residual intersection of its base locus with C classifies such Cremona . Finally, as a consequence, we obtain a Sarkisov decomposition for these cubo-cubic Cremona transformations.

In the last section, we begin the study of some singular cases. We say that we have a *singular* quadratic complex if the pencil $\lambda Q_1 + \mu Q_2$ is not general anymore and X is singular. Starting from a singular quintic curve of arithmetic genus 2 we build examples of Cremona transformations which may be related to singular quadratic complexes or to more general three dimensional varieties. We give some relevant examples such as the three dimensional standard Cremona transformation; we give also examples showing that singular quintic curves as above may produce non cubo-cubic Cremona transformations as well.

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2. Quadratic complexes

Let Q_1 denote a smooth hyperquadric in $\mathbb{P}^5$ over the field of complex numbers, considered as the Plücker hyperquadric parameterizing lines in $\mathbb{P}^3$. A *quadratic line complex* is, as defined before, a complete intersection $X = Q_1 \cap Q_2$, with a hyperquadric $Q_2 \subset \mathbb{P}^5$ different from Q_1. An introduction to this subject may be found in [GH, chap. 6].

Let us assume that X is smooth. We know that X is a Fano rational variety whose canonical sheaf is $\omega_X = \mathcal{O}_X(-2)$.

Take two lines $L_1, L_2 \subset \mathbb{P}^5$, $L_1 \neq L_2$, both contained in X. Fix a general 3-plane $M = \mathbb{P}^3$ in $\mathbb{P}^5$, and define projections $\pi_i : \mathbb{P}^5 \dashrightarrow \mathbb{P}^3$, $i = 1, 2$, with centers L_1 and L_2, respectively; we assume $M \cap L_i = \emptyset$, $i = 1, 2$.

Lemma 1. *The restriction of π_i to $X, i = 1, 2$ induces a birational map $X \dashrightarrow \mathbb{P}^3$, which we still denote by π_i, whose inverse π_i^{-1} is given by a linear system of cubics whose base locus scheme is a smooth irreducible curve $C_i \subset \mathbb{P}^3$ of degree 5 and genus 2. In particular, $\pi_2\pi_1^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is given by a linear system of cubics vanishing on C_1.*

Proof. Fix $i \in \{1, 2\}$ and let us denote $L = L_i$ and $\pi = \pi_i$. Take a general point $y \in \mathbb{P}^3$. Consider the 2-plane $H_y := \langle L, y \rangle$ generated by L and y. We have that $H_y \cap Q_i$ is the union of L with another line L'_i. Since X is smooth, $L'_1 \cap L'_2 \not\subset L$, and for general y we have $L'_1 \neq L'_2$. Hence $L'_1 \cap L'_2$ is a point in $H_y \cap X \backslash L$, from which it follows that π is birational.

A general hyperplane section $S = \mathcal{H} \cap X$ of X is a smooth surface $S \subset \mathbb{P}^4$ which is a del Pezzo surface of degree 4, since $\omega_S = \mathcal{O}_S(-1)$. Its image via π_i is the projection of S from the point in $\mathcal{H} \cap L_i$, i.e. a (smooth) cubic surface. Thus $\pi_i^{-1} : \mathbb{P}^3 \dashrightarrow X \subset \mathbb{P}^5$, and *a posteriori* $\pi_2\pi_1^{-1}$ and $\pi_1\pi_2^{-1}$, are all given by linear systems of cubics. By blowing up X along L_i we obtain a smooth three dimensional variety X_i and a birational morphism $\sigma_i : X_i \to \mathbb{P}^3$. By construction σ_i contracts the strict transform, say R, of a line in X passing through a point of L_i; since the canonical divisor K_X of X is linearly equivalent to $-2S$, we infer $K_{X_i} \cdot R = -1$, which shows that σ_i is an extremal contraction in the sense of Mori. By the classification of extremal contractions of a smooth (projective) three dimensional variety (see [Mo, Thm. 3.3 and Cor. 3.4]) we conclude that σ_i is the blow-up of $\mathbb{P}^3$ along a smooth irreducible curve C_i; in particular all base points of $\pi_i : \mathbb{P}^3 \dashrightarrow X$ belong to C_i. Since cubic surfaces in $\mathbb{P}^3$ containing C_i correspond by π_i^{-1} to hyperplane sections of X we easily deduce that the residual intersection of two such cubic surfaces with respect to C_i is an elliptic quartic curve. By *liaison* Formulae (see [PS, Prop. 3.1]) C_i has degree 5 and genus 2. It is well known that such a curve is scheme-theoretical intersection of six cubic surfaces (see for example Proposition 3 in §4), which completes the proof. □

Remark 1. Once we know the curve C_i is smooth we do not need to use Mori's results. In fact, in this case we may deduce that $\sigma_i : X_i \to \mathbb{P}^3$ is a blow-up in a more elementary form: see for example [ESB, Prop. 1]. On the other hand, in [GH, Chap. 4, §3] the smoothness of C is deduced from general constructions on quadratic complexes.

Take two lines L_1, L_2 in the quadratic complex $X = Q_1 \cap Q_2$. Denote by $\varphi = \varphi_{L_1,L_2} := \pi_2\pi_1^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ the Cremona transformation given by Lemma 1.

In the sequel, we describe the map φ in the case where X is smooth.

We begin with some intersection theory on the resolution of the indeterminacies of φ, which will be used in the proof of Theorem 1 in next section.

Let $\sigma : W \to Z$ be the blow up of a smooth dimension 3 variety along a smooth curve C of genus g, with exceptional divisor E. The *Segre class* of C as a subscheme of Z is given by (see [F, Cor. 4.2.2])

$$\sigma_*(E - E^2 + E^3) = C - \int_C c_1(N_C Z),$$

where $N_C Z$ is the normal bundle of C in Z; here we use that the Chern class of this bundle is the inverse of the Segre class of C (smooth case). Taking into account the adjunction formula, we deduce that the self-intersections of E satisfy

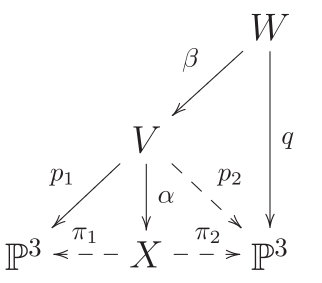
$$\sigma_*(E^2) = -C, \text{ and } \sigma_*(E^3) = K_Z \cdot C + 2 - 2g \tag{1}$$

where K_X denotes, as usual, the canonical divisor of X.

We resolve the indeterminacies of φ in two different cases:

Case (c1). $L_1 \cap L_2 = \emptyset$. Consider the blow-up $\alpha : V \to X$ of X along L_1, with exceptional divisor A; denote by $\widetilde{L_2}$ the strict transform of L_2 under α and set $p_i := \pi_i\alpha$, $(i = 1, 2)$; note that $\widetilde{L_2}$ is the base locus scheme of $p_2 = \pi_2\alpha : V \dashrightarrow \mathbb{P}^3$.

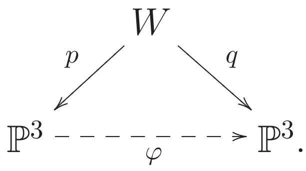
Let $\beta : W \to V$ be the blow-up of V along $\widetilde{L_2}$, with exceptional divisor B, and set $q := p_2\beta$. By construction p_1 and q are well defined and we obtain a commutative diagram:



If H_X is the restriction to X of a general hyperplane in $\mathbb{P}^5$, then p_1 and $p := p_1\beta$ are defined by the complete linear systems $|\alpha^*H_X - A|$ and $|\beta^*(\alpha^*H_X - A)|$, respectively.

Let's denote $D := \beta^*\alpha^*H_X$. By abuse of notation, we also denote by A the strict transform of A under β. In the following lemma we keep the above notations, in particular for the strict transform of exceptional divisors:

Lemma 2. *A resolution of the indeterminacies of φ is given by the following commutative diagram*



Moreover, the Picard group of W is

$$\mathrm{Pic}(W) = D\mathbb{Z} \oplus A\mathbb{Z} \oplus B\mathbb{Z},$$

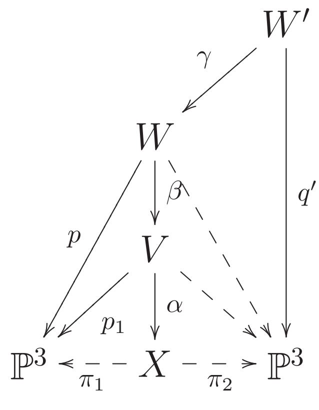
with the following intersection numbers:

$$\begin{array}{lll} A^3 = 0 & A^2 \cdot B = 0 & A^2 \cdot D = -1 \\ A \cdot B^2 = 0 & B^3 = 0 & B^2 \cdot D = -1 \\ A \cdot D^2 = 0 & B \cdot D^2 = 0 & D^3 = 4 \\ A \cdot B \cdot D = 0 & & \end{array}$$

Proof. The first assertion follows from the argument above, and the intersection numbers are obtained using formulae (1). □

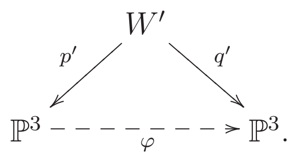
Case (c2). $L_1 \cap L_2 = \{x\}$. As before $\alpha : V \to X$ and $\beta : W \to V$ are the blow-ups of L_1 and the strict transform $\widetilde{L_2} \subset V$ of L_2 respectively. Now consider the blow-up $\gamma : W' \to W$ of W along the strict transform in W of $L_0 := \alpha^{-1}(x) \subset A$; denote by P its exceptional divisor. Note that $p_1 = \pi_1\alpha$ is a morphism but $\pi_2\alpha$ is well defined at $z \in V$ if and only if $z \notin \widetilde{L_2} \cup L_0$.

As in the former case, we obtain a commutative diagram (q' is well defined by the next Lemma)



Let $p' = p\gamma$. Keeping the above notations, we obtain the following lemma:

Lemma 3. *A resolution of the indeterminacies of φ is given by the following commutative diagram*



Moreover, the Picard group of W' is

$$\mathrm{Pic}(W') = D\mathbb{Z} \oplus P\mathbb{Z} \oplus A\mathbb{Z} \oplus B\mathbb{Z},$$

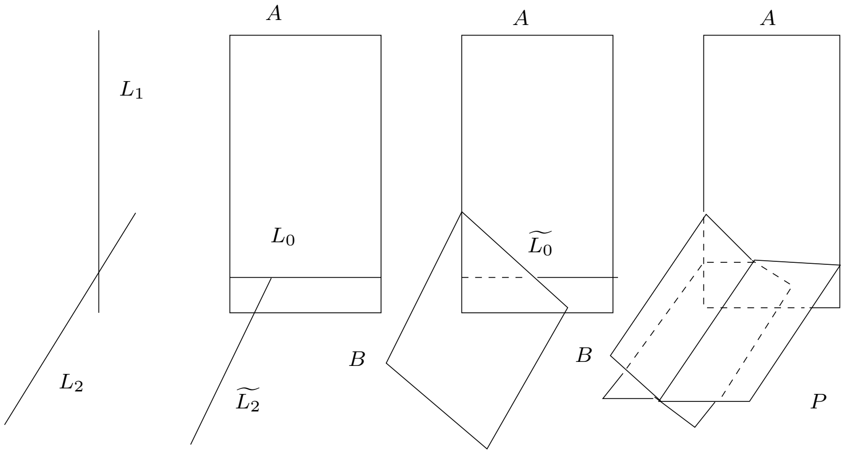


FIGURE 1. Incidences for exceptional divisors

with the following intersection numbers:

$A^3 = 1$	$A^2 \cdot B = -1$	$A^2 \cdot D = -1$	$A^2 \cdot P = 0$
$A \cdot B^2 = -1$	$B^3 = 1$	$B^2 \cdot D = -1$	$B^2 \cdot P = 0$
$A \cdot D^2 = 0$	$B \cdot D^2 = 0$	$D^3 = 4$	$D^2 \cdot P = 0$
$A \cdot P^2 = -1$	$B \cdot P^2 = -1$	$D \cdot P^2 = 0$	$P^3 = 2$
$A \cdot B \cdot D = 0$	$A \cdot B \cdot P = 1$	$A \cdot D \cdot P = 0$	$B \cdot D \cdot P = 0$

Proof. We have $p' = p_1\beta\gamma$. To prove the first assertion it suffices to show that $\pi_2\pi_1^{-1}p'$ is a morphism, that is to say, that $\pi_2\alpha\beta\gamma$ is a morphism. The linear system defining π_2 is cut out by hyperplane sections of X passing through L_2. Since the unique normal direction to L_1 at x which is a tangent direction for all such hyperplane sections is that defined by L_2, it follows that $\pi_2\alpha$ has no infinitely near base points over L_0. This proves the assertion.

For the intersection numbers we use formulae (1) relating the Segre class and adjunction formula. Note first that the last row of numbers follows from figure 1. The cubic powers may be computed taking into account the behavior of the canonical divisor in each blow-up. For the rest we use once again formulae (1); for example let us compute $A^2 \cdot P$. Recalling the notations for strict transforms of exceptional divisors we have $\gamma^*(A) = A + P$, then $A^2 \cdot P = (\gamma^*(A) - P)^2 \cdot P = -2A \cdot \gamma_*(P^2) + P^3$, and $\gamma_*(P^2) = -L_0$, $A \cdot L_0 = -1$. Thus $A^2 \cdot P = -2 + P^3 = -2 + 2 = 0$.

□

For the following section we recall the definition of special lines on X (see [GH, Chap. 6, §4]).

Definition 1. Let $L \subset X$ be a line on the quadratic complex. The line L is said to be *special* if either of the three equivalent conditions holds:

(1) $dim(\cap_{x \in L} T_x X) = 2$.
(2) the locus $T_x \cap X$ of lines in X through a generic point $x \in L$ consists of fewer than four lines.
(3) The normal bundle of $L = \mathbb{P}^1$ in X is $N_{L/X} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Remark 2. In fact, as shown in [GH, Chap. 6, §4] the normal bundle of a non special line in X is trivial, then the *speciality* of a line may be understood as the form in which it is embedded in.

3. Cremona Transformations

Let $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a rational map given by $\varphi = (f_0 : \cdots : f_3)$, where the f_i are homogeneous polynomials of the same degree d and without common factors; these polynomials define a subscheme $\mathrm{Base}(\varphi)$ of $\mathbb{P}^3$, the so-called *base locus scheme* of φ. The integer d is called the degree of φ and denoted by $deg(\varphi)$. The *Jacobian* $\mathrm{Jac}(\varphi)$ of φ is the effective divisor on $\mathbb{P}^3$ defined by the Jacobian determinant

$$\det\left(\frac{\partial f_i}{\partial x_j}\right);$$

this determinant is a homogeneous polynomial of degree $4(\deg(\varphi)-1)$. The map φ is called a *Cremona transformation* if it has a rational inverse φ^{-1}. If both the Cremona transformation and its inverse have degree 3 then it is called a *cubo-cubic* Cremona transformation.

In the following theorem, we will use, implicitly, the classification of cubo-cubic Cremona transformations of $\mathbb{P}^3$. There are essentially, three kinds of cubo-cubic Cremona transformations φ:

(1) φ is called *determinantal*, if there exists a 4×3 matrix A with linear entries such that φ is given by the four 3×3 minors of the matrix A. The inverse φ^{-1} is also determinantal.
(2) φ is *de Jonquières* if and only if the strict transform of a general line under φ^{-1} is a singular plane rational cubic curve whose singular point is fixed. For such a transformation there is always a quadric contracted onto a point, the corresponding fixed point for φ^{-1}, which is also a de Jonquières transformation.
(3) φ is *ruled* if the strict transform of a plane under φ^{-1} is a ruled cubic surface.

We recall the following results which characterize cases (1) and (3) below.

Proposition 1. *A cubo-cubic Cremona transformation is determinantal if and only if its base locus scheme is an arithmetically Cohen-Macaulay curve of degree 6 and (arithmetic) genus 3.*

Proposition 2. *A cubo-cubic Cremona transformation is ruled if and only if it is defined by a linear system, Λ say, of non normal cubic surfaces; in particular, the dimension 1 part of its base locus scheme is a union of at most 3 lines, one of which is a line of singular points for all surfaces in Λ.*

Denoting by T^D_{33}, T^J_{33} and T^R_{33} the (constructible and irreducible) sets of determinantal, de Jonquières and ruled cubo-cubic transformations respectively, these sets satisfy

$$T^D_{33}\cap T^J_{33}=\emptyset \quad , \quad T^D_{33}\cap T^R_{33}\neq\emptyset \quad , \quad T^J_{33}\cap T^R_{33}\neq\emptyset.$$

For more details on cubo-cubic transformations we refer the reader to [H1] or [P1].

Theorem 1. *Assume that the quadratic complex X is smooth and let L_1, L_2 be two (distinct) lines in X. Then the map $\varphi = \varphi_{L_1,L_2} = \pi_2\pi_1^{-1}$ is a cubo-cubic Cremona transformation such that:*

(a) The support of $\mathrm{Base}(\varphi)$ *consists of an irreducible genus 2 quintic curve C and a line L.*

(b) If $L_1\cap L_2=\emptyset$, then φ is a Cremona determinantal transformation, $\mathrm{Jac}(\varphi)$ *is the union of a quadric and a sextic irreducible surfaces,* $\mathrm{Base}(\varphi)=C\cup L$ *as schemes and L is a secant line to C which is not trisecant.*

(c) If $L_1\cap L_2\neq\emptyset$, then φ is a de Jonquières transformation, $\mathrm{Jac}(\varphi)$ *is the union of two quartic surfaces, one of these being irreducible, and the other one supported on a*

quadric. In this case the base locus scheme has an embedded point at $\pi_1(L_2) \in C \cup L$, $\mathrm{Base}(\varphi)_{red} = C \cup L$ *is a complete intersection of a cubic surface and the unique quadric surface containing* C*, and* L *is a trisecant line to* C.

Proof. Recall notations in cases (c1) and (c2) from Section 2. In Lemma 1 we proved that φ is cubo-cubic and $\mathrm{Base}(\varphi)$ contains $C \cup L$ as in our statement; this proves (a). Moreover, denoting by V_3 the 3-space generated by L_2 and a general line $\ell \subset \mathbb{P}^3$, there is a curve $\overline{C}$ of degree 3 (*a priori* not necessarily irreducible or reduced) such that

$$V_3 \cap X = V_3 \cap Q_1 \cap Q_2 = \overline{C} \cup L_2.$$

We deduce that $\pi_1(\overline{C}) = \varphi_*^{-1}(\ell)$ and that $\overline{C}$ is irreducible. It follows that $\overline{C}$ is a twisted cubic in V_3, having L_2 as a bisecant line, since the genus of $\overline{C} \cup L_2$ is 1. Therefore, the restriction of π_1 to $\overline{C}$ is injective if and only if $L_1 \cap L_2 = \emptyset$; in this case $\varphi_*^{-1}(\ell)$ is a twisted cubic in $\mathbb{P}^3$ with $\pi_1(L_2)$ as a bisecant line. Otherwise $\varphi_*^{-1}(\ell)$ is a plane singular cubic.

In the rest of the proof, we will use that through a smooth genus 2 quintic curve C in $\mathbb{P}^3$ there pass smooth cubic surfaces and a (unique) quadric (see Proposition 3 in §4).

(b) Suppose $L_1 \cap L_2 = \emptyset$. Here $L = \pi_1(L_2)$. Take general cubic surfaces, say S, S', such that $S \cap S' = C \cup L \cup \varphi_*^{-1}(\ell)$. Since a twisted cubic curve is arithmetically Cohen-Macaulay of genus 0, by *liaison* $C \cup L$ is an arithmetically Cohen-Macaulay curve of degree 6 and arithmetic genus 3, whose ideal is generated by 4 independent cubic forms which are the maximal minors of a 4×3 matrix of linear forms ([PS, §3]); in particular φ is determinantal. Using adjunction on (a smooth surface) S, we have that arithmetic genus 3 for $C \cup L$ implies L is a secant line to C which is not trisecant.

To complete the proof of (b), we use Lemma 2. Let $H_1, H_2 \subset W$ be the pullbacks of a hyperplane in $\mathbb{P}^3$, by p and q, respectively. We have

$$H_1 \sim D - A \quad , \quad H_2 \sim D - B.$$

We obtain:

$$A \cdot H_1^2 = B \cdot H_2^2 = 2, \quad B \cdot H_1^2 = A \cdot H_2^2 = 0$$

from which we deduce that $p(A)$ (resp. $q(B)$) is a quadric surface contracted by φ (resp. φ^{-1}). If E is the exceptional divisor of p_1 we have

$$K_W = p^*(K_{\mathbb{P}^3}) + B + E.$$

Note that E is irreducible. We know that $\mathrm{Jac}(\varphi^{-1})$ has degree 8. Since

$$\mathrm{Jac}(\varphi^{-1}) = q_*(E + B),$$

we obtain

$$E \cdot H_2^2 = 6,$$

showing that $\mathrm{Jac}(\varphi^{-1})$ is the union of a quadric and a sextic surfaces, both irreducible. By symmetry, $\mathrm{Jac}(\varphi)$ has the same properties.

(c) Let us assume $L_1 \cap L_2 \neq \emptyset$. In this case $\pi_1(L_2)$ is a point and $\varphi_*^{-1}(\ell)$ is a plane cubic. Then φ is de Jonquières. Moreover, as above we deduce that $C \cup L$ is arithmetically Cohen Macaulay of degree 6 and genus 2; hence L is a trisecant line to C. Since C contains a quadric, necessarily it contains L. The existence of an embedded point follows from [P2].

Let $H_1, H_2 \subset W'$ be the pullbacks of a hyperplane in $\mathbb{P}^3$, by p' and q', respectively. We have

$$H_1 \sim D - A - P \quad , \quad H_2 \sim D - B - P.$$

Since

$$A \cdot H_1^2 = B \cdot H_2^2 = 2, \;\; P \cdot H_2^2 = 0, \quad A \cdot H_2^2 = B \cdot H_1^2 = 0,$$

by lemma 3, then $p'(A)$ (respectively $q'(B)$) is a quadric surface contracted by φ (respectively φ^{-1}), and P is contracted by q'.

Note that in this case the strict transform of L_2 in V is contained in $p_1^{-1}(C) = E$, hence $\beta^*(E) = E+B$. Therefore $K_{W'} = (p')^*(K_{\mathbb{P}^3})+\gamma^*(\beta^*(E)+B)+P = (p')^*(K_{\mathbb{P}^3})+E+2B+P$. We obtain the equality $\mathrm{Jac}(\varphi^{-1}) = q'_*(2B + E)$ as divisors. We complete the proof by arguing as in the former case showing that $\mathrm{Jac}(\varphi^{-1})$ is the union of a double quadric and a quartic surfaces.

□

Remarks 3. a) As we have seen in the proof of part (b) of Theorem 1, the line L is nothing but $\pi_1(L_2)$. In part (c) of that result, the line L_2 is contracted by π_1 to a point P_0 which is the embedded point of $\mathrm{Base}(\varphi)$; the line L, coinciding with $p_1(L_0)$, is then a trisecant to C containing P_0. Thus the embedded point is in $C \cap L$.

b) The lines L_1 and L_2 may be special, this is irrelevant for Theorem 1. Indeed, all arguments (including the intersection numbers) used in the proof do not depend on whether the lines are special or not.

c) On the other hand, the quadric surface $p(A)$ containing the quintic curve $C_{L_1} = C$ (see the proof of Theorem 1) is a quadratic cone if the line L_1 is special and is smooth if L_1 is not special (see Definition 1 and Remark 2). This quadric is the variety $\mathrm{Sec}_3(C_{L_1})$ of trisecant lines of C_{L_1}.

The maps $\pi_i, i = 1, 2$, above, can be used to construct Cremona transformations in various ways. In this section, we suggested two ways in which this can be done and in the next section we state a general theorem classifying such Cremona transformations. We consider in the next two examples $X = Q_1 \cap Q_2$ such that the pencil $\lambda Q_1 + \mu Q_2$ is general (see §1). Let $x_0, \ldots, x_5$ be homogeneous coordinates on $\mathbb{P}^5$.

Example 1. Case where $L_1 \cap L_2 = \emptyset$. Let $Q_1 := (x_1x_0 - x_3x_2 + x_5x_4 = 0)$ and $Q_2 := (x_0^2 - x_1^2 + 2x_2^2 - 2x_3^2 + 4x_4^2 - 4x_5^2 = 0)$. Let $M_1 \sim \mathbb{P}^3$ be given by $x_0 = x_3 = 0$ and $M_2 \sim \mathbb{P}^3$ be given by $x_2 = x_5 = 0$. Consider the tangent space T_xX to X at $x = (1 : 1 : 1 : 1 : 0 : 0)$. We take L_1 to be one of the four lines in $T_xX \cap X$, parameterized for example by:

$$(x_0 : -3x_0 + 4x_3 : -2x_0 + 3x_3 : x_3 : \sqrt{3}(x_0 - x_3) : \sqrt{3}(x_0 - x_3)), \quad (x_0 : x_3) \in \mathbb{P}^1.$$

By intersecting X with the plane $\langle y, L_1 \rangle$ through the point $y = (0 : y_1 : y_2 : 0 : y_4 : y_5) \in M_1$ and the line L_1, we get an expression for $\pi_1^{-1}(y)$, given by six cubic polynomials . To get a line L_2 disjoint from L_1, we consider the point $y = (0 : 0 : 1 : 1 : 1 : 1)$ and take one of the four lines in $T_yX \cap X$, for example:

$$(\sqrt{3}(x_2 - x_5) : -\sqrt{3}(x_2 - x_5) : x_2 : -3x_2 + 4x_5 : -2x_2 + 3x_5 : x_5), \quad (x_2 : x_5) \in \mathbb{P}^1.$$

By intersecting M_2 with the plane $\langle z, L_2 \rangle$ through the point $z \in \mathbb{P}^5$ and the line L_2 we obtain $\pi_2(z)$. We compute $\pi_2\pi_1^{-1}(y)$ by replacing $z = \pi_1^{-1}(y)$ and obtain four cubic polynomials. We check that the line $\pi_1(L_2)$ is bisecant to the quintic curve C. Then these polynomials define a cubo-cubic determinantal Cremona transformation. It may also be checked that $\mathrm{Jac}(\varphi)$ factorizes as a quadric times a sextic.

Example 2. Case where $L_1 \cap L_2 \neq \emptyset$. Keep Q_1, Q_2 and L_1 as before. Let now L_2 be the line in X containing the point x of L_1, with parametrization as L_1 but changing the sign in the last two coordinates. Consider the new projection π_2 on M_1 with center L_2. The composition with π_1^{-1} is a cubo-cubic Cremona transformation φ, and the intersection of $L = T_xX \cap M$ with the quintic curve C has three points, so L is trisecant to C. It follows that φ is a de Jonquières transformation. It may also be checked that $\mathrm{Jac}(\varphi)$ factorizes as the square of a quadric times a quartic.

Example 3. Case where L_1 is a special line. Keep the quadratic complex X defined by Q_1 and Q_2 as above, and let $z = (-2 : 2 : 2\sqrt{5} : -2\sqrt{5} : -4 : 4) \in X$. Then the tangent space T_zX intersects X in two lines, L_1 and L_2, such that one of these lines, say L_1, contains the point $w = (2 : 2 : \sqrt{5} : \sqrt{5} : 1 : 1)$. The tangent spaces T_zQ_1 and T_wQ_2 coincide, so the line L_1 is special, as it can be verified since in this case one has $\dim(\bigcap_{p\in L_1} T_pX) = 2$. On the other hand $\dim(\bigcap_{p\in L_2} T_pX) = 1$, so L_2 is not a special line. The Cremona transformation built from the projections from L_1 and L_2 to M defined by $x_0 = x_1 = 0$ is, as expected, a de Jonquières cubo-cubic since L_1 and L_2 meet. This may be verified by the factorization of the Jacobian, as before. In this example the quadric surface of trisecants to the quintic curve C_{L_1} given by the first projection is a quadratic cone.

4. ACM quintic curves

In this section, the data we begin with is a smooth genus 2 quintic curve in $\mathbb{P}^3$ and we consider Cremona transformations containing this curve in the base locus, to obtain its relation with a quadratic complex.

In the sequel we write ACM for arithmetically Cohen-Macaulay.

Proposition 3. *Let $C \subset \mathbb{P}^3$ be a smooth genus 2 quintic curve, and $\mathcal{J}_C$ the ideal sheaf associated to C. There exist irreducible homogeneous polynomials g, f_1, f_2 of degrees $2, 3, 3$, respectively, such that:*

a) $\mathcal{J}_C$ is generated by g, f_1, f_2.

b) The surfaces $F_i := V(f_i), i = 1, 2$ are smooth.

c) The pencil generated by f_1, f_2 cuts out over $Q := V(g)$ the family of trisecant lines of C. The induced rational fibration $Q \dashrightarrow \mathbb{P}^1$ cuts out either a g_3^1 or a g_2^1 on C depending on whether Q is smooth or not.

Moreover, we have

$$h^0(\mathcal{J}_C(2)) = 1,\ h^0(\mathcal{J}_C(3)) = 6. \tag{2}$$

Proof. For $n \geq 2$ consider the exact sequence

$$0 \longrightarrow \mathcal{J}_C(n) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(n) \longrightarrow \mathcal{O}_C(n) \longrightarrow 0$$

Then we have $h^0(\mathcal{O}_C(n)) \geq \binom{n+3}{3} - h^0(\mathcal{J}_C(n))$, the Riemann-Roch Theorem implies that $h^0(\mathcal{O}_C(n)) = 5n + 1 - 2$, and therefore

$$h^0(\mathcal{J}_C(2)) \geq 1,\ h^0(\mathcal{J}_C(3)) \geq 6$$

hence C is properly contained in a complete intersection of type $(2, 3)$. On the other hand, by *liaison* theory ([PS, §3]) we know that C is an ACM curve which is the theoretical scheme defined by the intersection of a unique quadric Q and two cubic surfaces; in particular the equalities in (2) hold. The smoothness of the F_i's may be assumed by [PS, Prop. 4.1]. By comparing the genus formula for a complete intersection with that of an effective divisor on a smooth surface, we conclude that this line is a trisecant line of C. Now, when the cubic surfaces describe the linear system containing C, the residual intersections with Q are the trisecant lines to C, and it follows that Q is the union of these lines. Finally, consider the two trisecant lines S_1, S_2 such that

$$V(f_i) \cap Q = C \cup S_i, i = 1, 2.$$

Since the rational map $f_1/f_2 : Q \dashrightarrow \mathbb{P}^1$ extends to $Q-(S_1 \cup S_2)$ (there are no embedded points in a complete intersection), then S_1 and S_2 are linearly equivalent, as divisors on Q. This proves the proposition.

□

By Proposition 3, the homogeneous polynomials

$$X_0g, X_1g, X_2g, X_3g, f_1, f_2 \in \mathbb{C}[X_0, \dots, X_3], \tag{3}$$

generate the ideal sheaf $\mathcal{J}_C$ and the linear space $\mathrm{H}^0(\mathcal{J}_C(3))$.

Proposition 4. *Let $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ be the rational map associated to the linear system $\mathbb{P}\mathrm{H}^0(\mathcal{J}_C(3))$, and $\sigma : Z = \mathbb{B}l_C(\mathbb{P}^3) \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along C with E its exceptional divisor.*

There is a commutative diagram

$$\begin{array}{ccc} Z & & \\ {\scriptstyle\sigma}\downarrow & \searrow^{\psi} & \\ \mathbb{P}^3 & \overset{\phi}{\dashrightarrow} & \mathbb{P}^5. \end{array}$$

Moreover, the morphism ψ is birational onto its image $X := \psi(Z)$, which is a smooth codimension 2 subvariety of $\mathbb{P}^5$ of degree 4.

Proof. The first assertion is clear since σ resolves the indeterminacies of ϕ.

Let H be the divisor in Z defined as the pullback of a general hyperplane in $\mathbb{P}^3$.

The morphism ψ is defined by the complete linear system $|3H - E|$. By formulae (1),

$$(3H - E)^3 = 4, \tag{4}$$

then the degree of X divides 4. Since X is not contained in a hyperplane, $\deg X = 4$. It follows that ψ is birational onto X.

To prove that X is smooth, take a sufficiently general plane $\Pi \subset \mathbb{P}^3$, i.e. transversal to the quintic C and not containing any trisecant to C. The restriction of ϕ to Π, induces a rational map $\Pi \dashrightarrow \mathbb{P}^5$, defined by plane cubics curves through five points in general position. Hence $\psi_*\sigma_*^{-1}(\Pi)$ is a (smooth) del Pezzo surface of degree 4, which is a hyperplane section of X containing the line into which Q is contracted. For each point of X there is such a smooth hyperplane section. This completes the proof.

□

Remarks 4. (a) The rational map ψ is birational and contracts the strict transforms by σ of trisecants to C onto the points of a line, say $L_1 \in \mathbb{P}^5$. In the case where $\mathrm{Sec}_3(C)$ is a smooth quadric, one ruling of this quadric is given by the trisecant lines to C and the lines in the second ruling are bisecant to C. The strict transform by σ of each line of this second ruling is sent by ψ, isomorphically, onto L_1.

(b) The subvariety X of $\mathbb{P}^5$ is in fact a complete intersection of two hyperquadrics, as we will see in a more general setup with C not necessarily smooth (see Proposition 5).

The following result proves that from the construction in Theorem 1 we obtain every Cremona transformation whose base locus scheme contains a smooth genus 2 quintic curve.

Theorem 2. *Let $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a cubo-cubic Cremona transformation whose base locus scheme* $\mathrm{Base}(\varphi)$ *contains a smooth genus 2 quintic curve C.*

The map φ factorizes through a quadratic complex X, via two linear projections $\pi_{L_i} : \mathbb{P}^5 \dashrightarrow \mathbb{P}^3$ with centers two lines L_1, L_2 contained in X.

Proof. We keep the notations from Proposition 4. From this result and Remarks 4(b), we know that $X = \psi(Z)$ is a quadratic complex and the image by ψ of the strict transform of the quadric Q is a line L_1. This line is the center of a linear projection which is the inverse map of $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$

There exist homogeneous polynomials $f_0, f_1, f_2, f_3 \in \mathbb{C}[x, y, z, w]$ of degree 3, without common factors, such that

$$\varphi = (f_0 : f_1 : f_2 : f_3).$$

Since $\mathrm{Base}(\varphi)$ contains C the linear space, over $\mathbb{C}$, generated by these polynomials is a four dimensional subspace of $\mathrm{H}^0(\mathcal{J}_C(3))$. Then we may factorize φ through ϕ by way of a linear projection π_{L_2} whose center $L_2 \subset \mathbb{P}^5$ is a line. By Lemma 1 we only need to prove that $L_2 \subset X$.

Suppose $L_2 \not\subset X$. Then L_2 intersects X in, say, $k \geq 0$ points. We know that $k \leq 2$ since X is a complete intersection of two hyperquadrics. Then a 2-plane, general among those containing L_2, intersects $X \backslash L_2$ in $4 - k$ points, contradicting the birationality of φ. □

Theorem 2 shows that any cubo-cubic Cremona transformation, whose base locus contains a smooth genus 2 quintic curve C, factorizes through $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ via a projection of $\mathbb{P}^5$ with center a line in X. Since the projection of a quadratic complex X from a line therein always leads to a birational map $X \dashrightarrow \mathbb{P}^3$, we may describe the nature of the Cremona transformations as in the Theorem above, in terms of what information from C that line carries. We will do this in Theorem 3 below. To begin with, we show that given a line $L_1 \subset X$, there are two families of lines in X intersecting it:

i) those coming from the normal directions at points of C which yield de Jonquières transformations, a dimension 1 family.

ii) those coming from bisecants L to C which yield determinantal transformations when $L \not\subset Q$, and a linear automorphism otherwise, a dimension 2 family.

To fix some notation, let $\sigma : Z \to \mathbb{P}^3$ be the blow-up of a smooth quintic genus 2 curve C, as before, and $\psi = \phi\sigma$. Take two points $x, y \in C$ not contained in the same trisecant to C. Denote by E_x the rational curve $\sigma^{-1}(x)$ and by $S_{x,y}$ the strict transform by σ of the line which is bisecant to C at the points x, y.

Lemma 4. *Let $\overline{L}$ be a line in $\mathbb{P}^5$. Then $\overline{L} \subset X$ if and only if either $\overline{L} = \psi(E_x)$ or $\overline{L} = \psi(S_{x,y})$ for $x, y \in C$.*

Proof. Let $\mathcal{H}$ be a general hyperplane in $\mathbb{P}^5$. Since $\psi^*(\mathcal{H}) = 3H - E$ and $(3H - E) \cdot E_x = (3H - E) \cdot S_{x,y} = 1$, then we see that $\psi(E_x)$ and $\psi(S_{x,y})$ are lines in X.

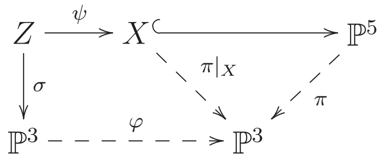
For the converse assertion denote by $L_1 \subset X$ the line onto which the strict transform of Q is contracted under ψ.

First suppose that $\overline{L} \subset X$ is a line which intersects L_1. If $\overline{L} = L_1$ then it is $\psi(S_{x,y})$ for a ruling in Q when this quadric is smooth (remark 4(a)), and is $\psi(E_x)$ where x is the vertex when the quadric is singular. If $\overline{L} \neq L_1$ we already know that $\overline{L}$ is contracted under π_{L_1} onto a point of C and then it is $\psi(E_x)$ for a point $x \in C$.

It remains to deal with the case where $\overline{L} \cap L_1 = \emptyset$. The 3-space generated by these lines intersects X along four lines, the two additional lines meet $\overline{L}$ and L_1. It follows that the projection of $\overline{L}$ is a bisecant to C and this completes the proof.

□

Theorem 3. *Let $\overline{L} \subset \mathbb{P}^5$ be a line. Consider a rational map $\varphi = \varphi_{\overline{L}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by the commutative diagram:*



where $\pi = \pi_{\overline{L}} : \mathbb{P}^5 \dashrightarrow \mathbb{P}^3$ is a projection with center $\overline{L}$. Then φ is a Cremona transformation if and only if $\overline{L} \subset X$. Moreover

a) If $\overline{L} = \psi(S_{x,y})$, then φ is determinantal of bidegree (3,3) when $\sigma(S_{x,y}) \not\subset \mathrm{Sec}_3(C)$, and it is a linear automorphism otherwise.

b) If $\overline{L} = \psi(E_x)$, then φ is a de Jonquières transformation.

Proof. The first assertion is contained in the last part of the proof of Theorem 2. We know also that φ is either an automorphism or a cubo-cubic transformation.

Theorem 1 together with Lemmas 1 and 4 give part (a) of the Theorem.

Let $\overline{L} = \psi(E_x)$, where $x \in C$. The pullback of a general hyperplane containing $\overline{L}$, by ψ, is a smooth surface, S say, containing E_x. Then $F := \sigma(S)$ has a singular point at x. This point is necessarily a double point and the trisecant line in $F \cap Q$ is forced to go through that point. Assertion (b) follows from part (c) of Theorem 1 (see also part (a) of Remarks 3).

□

Remark 5. We have seen that the strict transform on Z of the quadric Q containing C is contracted by ψ onto a line (the equation of Q is given by $g = 0$ in (3)). This line is the line $L_1 \subset X$ appearing in Theorem 2. On the other hand the line $L_2 \subset X$ in that theorem is $\overline{L}$ in both cases (a) and (b).

Sarkisov decomposition. We keep the notations from the last sections. Theorems 1 and 2 imply a very simple geometric description of the cubo-cubic transformations arising from our construction. Indeed, we conclude that such a Cremona transformation may be obtained from $\mathbb{P}^3$ as a product of two elementary transformations or links: first one blows up a smooth quintic curve C of genus 2 and contracts the strict transform of the quadric containing C onto the line L_1 in the Fano variety X; second we blow up X along L_2 and contract the union of lines on X touching L_2, which is a surface, onto the smooth quintic curve contained in $\mathrm{Base}(\varphi^{-1})$, in $\mathbb{P}^3$. These elementary transformations are special cases of the so-called *Sarkisov links*: they are links of type II. This description does not depend on whether the lines L_1 and L_2 intersect or not. However, according to a theorem of Corti ([Co] or [Ma]) on the algorithm to reach the end of the Sarkisov program, we have also other possibilities to obtain it. We may summarize this result (with the notations of Theorem 2):

Corollary 1. *Let $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ be a Cremona transformation arising from a quadratic complex. Then φ admits a decomposition as a product of two Sarkisov links of type II:*

$$\begin{array}{ccccccc} & & Z_1 & & Z_2 & & \\ & \swarrow^{\sigma_1} & & \searrow^{\psi_1} \quad \swarrow^{\psi_2} & & \searrow^{\sigma_2} & \\ \mathbb{P}^3 & & & X & & & \mathbb{P}^3 \\ & & & \underset{\varphi}{\dashrightarrow} & & & \end{array} .$$

5. On the singular case

Now we want to generalize the above construction. Given a singular ACM quintic curve of genus 2, our goal is to obtain Cremona transformations. In most cases we obtain codimension 2 quartics in $\mathbb{P}^5$, some of them being singular quadratic complexes and some which are not necessarily quadratic complexes.

According to the Peskine-Szpiro Deformation Theorem [PS, Thm. 6.2], the family of ACM quintic curves of genus 2 is parameterized by a scheme $\mathcal{S}$ over $\mathbb{C}$, which is a dense open set of a projective space, and there exists an $\mathcal{S}$-scheme $\mathcal{C}$ of codimension 2 in ${\mathbb{P}^3}_{\mathcal{S}} = \mathbb{P}^3 \times \mathcal{S}$, flat over $\mathcal{S}$, such that

a) the ideal sheaf $\mathcal{J}_{\mathcal{C}}$ admits a minimal resolution

$$0 \longrightarrow \mathcal{O}^2_{\mathbb{P}^3_{\mathcal{S}}}(-4) \longrightarrow \mathcal{O}_{{\mathbb{P}^3}_{\mathcal{S}}}(-2) \oplus \mathcal{O}^2_{\mathbb{P}^3_{\mathcal{S}}}(-3) \longrightarrow \mathcal{J}_{\mathcal{C}} \longrightarrow 0 . \tag{5}$$

b) If $s \in \mathcal{S}$, the fiber $\mathcal{C}(s)$ is an ACM subscheme of codimension 2 of $\mathbb{P}^3 = \mathbb{P}^3 \times \{s\}$, in such a way that its minimal resolution is obtained from (5) by tensorizing with $\mathbb{C}(s)$ over $\mathcal{O}_{\mathcal{S}}$.

c) If C is an ACM codimension 2 subscheme of $\mathbb{P}^3$, the ideal sheaf $\mathcal{J}_C$ admitting a minimal resolution of the form

$$0 \longrightarrow \mathcal{O}^2_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}^2_{\mathbb{P}^3}(-3) \longrightarrow \mathcal{J}_C \longrightarrow 0 , \tag{6}$$

then there exists a point $s \in \mathcal{S}$, such that $C = \mathcal{C}(s)$ and the resolution (6) is obtained from (5) by tensorizing with $\mathbb{C}(s)$ over $\mathcal{O}_{\mathcal{S}}$. Moreover, the set

$$\mathcal{S}_{sm} := \{s \in \mathcal{S} : \mathcal{C}(s) \text{ is smooth}\}$$

is a dense open subset of $\mathcal{S}$.

We consider the blow-up $\Sigma : \mathcal{Z} = \mathbb{B}l_{\mathcal{C}}(\mathbb{P}^3_{\mathcal{S}}) \to \mathbb{P}^3_{\mathcal{S}}$ of $\mathbb{P}^3_{\mathcal{S}}$ along $\mathcal{C}$. We may define a rational map $\Phi : \mathbb{P}^3_{\mathcal{S}} \dashrightarrow \mathbb{P}^5_{\mathcal{S}}$ and complete the following commutative diagram with a morphism Ψ into $\mathbb{P}^5_{\mathcal{S}}$.

$$\begin{array}{ccc} \mathcal{Z} & & \\ {\scriptstyle\Sigma}\downarrow & \searrow^{\Psi} & \\ \mathbb{P}^3_{\mathcal{S}} & \overset{\Phi}{\dashrightarrow} & \mathbb{P}^5_{\mathcal{S}} \end{array} \tag{7}$$

Denote $\mathcal{X} := \Psi(\mathcal{Z})$. Now consider the set

$$\mathcal{S}_0 := \{s \in \mathcal{S} : \Psi_s \text{ is generically finite}\} \supset \mathcal{S}_{sm}.$$

If $s_0 \in \mathcal{S}_0$, then $\Psi_{s_0} : \mathcal{Z}(s_0) \to \mathcal{X}(s_0)$ is a dominant morphism onto a dimension 3 variety.

Proposition 5. *Let s_0 be a point in $\mathcal{S}_0$. Then Ψ_{s_0} is a birational morphism and $\mathcal{X}(s_0)$ is a 3-dimensional variety of degree 4. Moreover, if $\mathcal{X}(s_0)$ is normal, it is a complete intersection of two hyperquadrics in $\mathbb{P}^5 \times \{s_0\}$.*

Proof. Fix a point $s \in \mathcal{S}_{sm}$. We take the line T joining s_0 to s and define $T_0 := T \cap \mathcal{S}_0$. This is a regular (integral) scheme whose generic point lies in $\mathcal{S}_{sm}$. Moreover, if $\mathcal{X}(s_0)$ is normal, then $\mathcal{X}$ is a flat family over an open set of T_0 containing s, since an algebraic family, without multiple fibers, of normal varieties is flat.

On the other hand, we know that there exists at least an element $s \in \mathcal{S}_{sm}$, such that $\mathcal{X}(s)$ is a complete intersection, and for which the morphism $\Phi_s : \mathcal{Z}(s) \to \mathcal{X}(s)$ is birational.

By the conservation of numbers for flat deformations (see [F, Cor. 10.2.1]), Proposition 4 implies that Ψ_{s_0} is birational and $\mathcal{X}(s_0)$ has degree 4. A flat deformation of the intersection of 2 quadrics is a complete intersection as long as the limit remains normal (see [AV]). It follows that $\mathcal{X}(s)$ is an intersection of two hyperquadrics when $\mathcal{S}(s_0)$ is a normal variety. □

In particular, we have the following corollary:

Corollary 2. *If $s \in \mathcal{S}_{sm}$ then $\mathcal{X}(s)$ is a smooth complete intersection of two quadric hypersurfaces in $\mathbb{P}^5 \times \{s\}$.*

Specializing quadratic complexes. The deformation method, after Theorems 1 and 3, gives us a way to specialize quadratic complexes, thought as a complete intersection of two smooth hyperquadrics in $\mathbb{P}^5$. In fact, every (smooth) quadratic complex may be associated to an ACM smooth quintic in $\mathbb{P}^3$ by fixing a diagram as in Theorem 2. On the other hand, as we saw, an ACM deformation of such a quintic leads to a complete intersection of two hyperquadrics in $\mathbb{P}^5$, provided that this variety is normal. Therefore, the universal diagram (7) may be used to obtain a deformation of a quadratic complex to a singular normal one.

To construct degenerated ACM quintics we may use *liaison* theory. We know that such a quintic C is linked to a line under a complete intersection of a quadric and a cubic surface. The saturated ideal defining C is minimally generated by a quadratic polynomial q and two cubic polynomials f_1, f_2. Thus, to obtain a (not necessary smooth) quadratic complex we may do the following. Fix a line $L \subset \mathbb{P}^3$ and take polynomials q and f of degree 2 and 3 respectively, without common factors, vanishing along L. For example, if L has equations $x = y = 0$, then $q = a_0x + b_0y$, $f = f_1 = a_1x + b_1y$, where the a_0, a_1 and b_0, b_1 are homogeneous polynomials. The generators q, f_1 and f_2 are the maximal minors of the matrix (see [PS, §3]):

$$N := \begin{pmatrix} y & -x \\ a_0 & b_0 \\ a_1 & b_1 \end{pmatrix}$$

We now give several relevant examples dealing with Cremona transformations having a singular quintic curve in the base locus. Some of them admit a description in terms of a quadratic complex, as the so-called standard Cremona transformation, a determinantal not ruled transformation which is related to a classical quadratic complex with six singular points. We also give examples which can be associated to a 3-dimensional variety not contained in any hyperquadric.

Example 4. We choose $L = (x = y = 0) \subset \mathbb{P}^3$, $q = xy$ and $f_1 = (x+y)zw$. Then

$$N = \begin{pmatrix} y & -x \\ y & 0 \\ zw & zw \end{pmatrix}$$

Hence $f_2 = yzw$ from which we obtain the 6 degree 3 generators for $\mathcal{J}_C$

$$qx, qy, qz, qw, f_1, f_2.$$

A set of degree 3 generators of the ideal $\mathcal{J}_C$ is then given by $x^2y, xy^2, xyz, xyw, yzw - xzw, yzw$, or also $x^2y, xy^2, xyz, xyw, xzw, yzw$. Therefore we may consider the rational map

$$\phi = (x^2y : xy^2 : yzw : xzw : xyw : xyz),$$

whose image is a complete intersection $X \subset \mathbb{P}^5$ of the two hyperquadrics of equations

$$x_0x_2 - x_1x_3 = x_0x_2 - x_4x_5 = 0.$$

Consider the line $L_2 := (x_2 = x_3 = x_4 = x_5 = 0) \subset X$. By projecting from X with center L_2 we obtain a birational map $\pi_{L_2} : X \dashrightarrow \mathbb{P}^3$ such that $\pi_{L_2}\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ is the *Standard cubo-cubic transformation*

$$\varphi = (yzw : xzw : xyw : xyz).$$

Note that the quintic curve C in this case is reducible, a union of 5 distinct lines, and that the quadratic complex X has six ordinary double points. It is the well known *Tetrahedral Quadratic Complex.*

In the two following examples, we show that the method above may fail to give a singular quadratic complex when the ACM quintic curve is very singular. If the variety $X = \psi(Z) \subset \mathbb{P}^5$ obtained from such a curve is still irreducible and of codimension 2, it has degree 4 and it may be used to produce cubic transformations. As we will also see, we may lose symmetry when we leave the context of (singular) quadratic complex.

Example 5. We choose $L = (x = y = 0) \subset \mathbb{P}^3$, $q = xz + yw$ and $f_1 = qx + y^3$. Then

$$N = \begin{pmatrix} y & -x \\ z & w \\ q & y^2 \end{pmatrix}.$$

Hence $f_2 = -qw + y^2z$, from which we obtain the 6 degree 3 generators for $\mathcal{J}_C$

$$qx, qy, qz, qw, f_1, f_2.$$

Another set of generators is then

$$qx, qy, qz, qw, y^3, y^2z.$$

Therefore we may consider the rational map

$$\phi = (qx : qy : qz : qw : y^3 : y^2z);$$

denote by X its image. A straightforward computation shows that there are no quadratic forms vanishing along X.

On the other hand, putting $x = y$ and letting x tend to 0 we observe that X contains the line

$$L_2 := (x_0 = x_1 = x_4 = x_5 = 0).$$

Projecting from X with this line as center, on the 3-space $(x_2 = x_3 = 0)$ we deduce that the rational map $\varphi := \pi_{L_2}\phi$ is defined by

$$\varphi = (qx : qy : y^3 : y^2 z).$$

Finally, we note that this map is birational whose inverse is

$$\varphi^{-1} = (xz : yz : yw : -xw + y^2).$$

This is a ruled cubo-quadric transformation. Moreover, φ is determinantal with associated matrix as follows:

$$\begin{pmatrix} y & 0 & 0 \\ -x & 0 & y \\ 0 & z & -w \\ 0 & -y & -x \end{pmatrix}$$

Example 6. Now consider $q = xz + yw$ and $f_1 = x^3 + y^2 z$. As in the former example we obtain $f_2 = x^2 w - y^2 z$. We obtain a codimension 2 subvariety of $\mathbb{P}^5$ which is not contained in any hyperquadric. By projecting with center the line

$$L_2 := (x_0 = x_1 = x_4 = x_5 = 0) \subset X,$$

we obtain the rational map $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by

$$\varphi = ((xz + yw)x : (xz + yw)y : x^3 + y^2 z : x^2 w - y^2 z).$$

It is a cubo-quintic Cremona transformation whose inverse map is

$$\varphi^{-1} = ((zy^3 + zx^3 - y^2x^2 + wy^3)x : (zy^3 + zx^3 - y^2x^2 + wy^3)y : (-yw + x^2)x^3 : (y^2 + wx)x^3).$$

Last, we give an example of a cubo-cubic Cremona transformation which arises from a singular quadratic complex whose base locus scheme contains a not generically reduced ACM quintic curve.

Example 7. Consider the singular quadratic complex X defined by the intersection of the two hyperquadrics

$$Q_1 = (x_0 x_1 - x_2 x_3 + x_4 x_5 = 0), \quad Q_2 = (x_0^2 - x_1^2 + x_2^2 = 0).$$

Choose the line $L_1 = (x_0 = x_1 - x_2 = x_3 = x_4 = 0)$ in X, parameterized by $(0 : x_1 : x_1 : 0 : 0 : x_5)$ and the 3-space $M \sim \mathbb{P}^3 = (x_2 = x_4 - x_5 = 0)$ parameterized by $(y_0 : y_1 : 0 : y_3 : y_4 : y_4)$. Then, by intersecting with X the 2-plane generated by L_1 and a point of M, and eliminating the parameters of L_1 we obtain π_1^{-1}, the rational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ whose components are generators for the quintic curve C:

$$\phi = (2y_0y_1y_4 : (y_0 + y_1)^2 y_4 : (y_0^2 - y_1^2)y_4 : 2y_1y_3y_4 : 2y_1y_4^2 : -y_0(y_0^2 + y_1^2) + (y_0^2 - y_1^2)y_3).$$

Projecting from X with center the line $L_2 = (x_2 - \imath\, x_0 = x_1 = x_3 = x_5 = 0)$, and composing with ϕ, we obtain the cubo-cubic Cremona transformation

$$\varphi = (\imath\, y_4(-\imath\, y_1 + y_0)^2 : (y_0^2 + y_1^2)y_4 : 2y_1y_3y_4 : -y_0^3 - y_0y_1^2 + y_0^2y_3 - y_1^2y_3)$$

with inverse obtained in the same way

$$\varphi^{-1} = (y_4(y_0^2 + y_1^2) : \imath\, y_4(-\imath\, y_1 + y_0)^2 : 2y_0y_3y_4 : -y_0y_1^2 - y_1^3 - \imath\, y_0^2y_3 + \imath\, y_1^2y_3).$$

The Jacobian of φ is $-12(y_0^3 + y_0y_1^2 - y_0^2y_3 + y_1^2y_3)y_1(-\imath\, y_1 + y_0)^2 y_4^2$.

A last computation shows that the linear space generated by the entries of φ is generated by the maximal minors of the following matrix

$$\begin{pmatrix} y_0 - \imath y_1 & 0 & 2y_1 \\ -y_2 & y_0 + \imath y_1 & 0 \\ 0 & -y_1 & y_0 - y_2 \\ 0 & 0 & -y_3 \end{pmatrix}$$

Thus φ is a determinantal cubo-cubic transformation.

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