CORRIGENDUM TO ON PLANE POLYNOMIAL AUTOMORPHISMS COMMUTING WITH SIMPLE DERIVATIONS

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The present note is a *corrigendum* of our paper [MePa2016]. The main results of that paper (Thms. 1 and 2) remain unchanged, but Proposition 9 and Corollary 12 therein must be replaced by Lemma 1 and Proposition 2 below. Moreover, the new proposition improves the older and readily gives Thm. 2. On the Examples 10 and 11 we may assert, in addition, that $\mathrm{Aut}(D_{-1})$ and $\mathrm{Aut}(\partial_x + (2xy + x^3)\partial_y)$ are both isomorphic to $\mathbb{K}^*$.

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Lemma 1. *Let $b(x) \in \mathbb{K}[x]$ be an arbitrary polynomial and let $a \in \mathbb{K}^*$. For any $(c, d) \in \mathbb{K} \times \mathbb{K}^*$ we consider the differential equation*

$$\mathcal{E}_{c,d}: \quad P' - aP = b(x + c) - db(x).$$

There exists a unique solution of $\mathcal{E}_{c,d}$ in $\mathbb{K}[x]$ for any pair (c, d). Moreover, if $b(x)$ has degree at least 1 and $Q \in \mathbb{K}[x]$ is a solution of both $\mathcal{E}_{c_1,d_1}$ and $\mathcal{E}_{c_2,d_2}$, then $c_1 = c_2$ and $d_1 = d_2$.

Proof. If Q_1, Q_2 are solutions of $\mathcal{E}_{c,d}$, then $Q_1 - Q_2$ is solution of $P' - aP = 0$, hence $Q_1 - Q_2 = 0$. In order to complete the proof of the first assertion we only need to show the existence of solutions stated there. Note that if $b = 0$ we already know $Q = 0$ is the only solution of $\mathcal{E}_{c,d}$ for all (c, d). Let us then write $b(x) = \sum_{i=0}^{n} b_i x^i$, with $n \geq 0$ and $b_n \neq 0$. We look for a solution $P(x) = \sum_{i=0}^{m} p_i x^i$, where $m \geq 0$ and p_m might be 0. In fact,

$$-ap_m x^m + \sum_{i=1}^{m} (ip_i - ap_{i-1})x^{i-1} = b(x + c) - db(x).$$

If $b(x + c) - db(x) = \sum_{j=0}^{r} c_j x^j$ is the null polynomial, then again $Q = 0$ is the unique solution. Otherwise that polynomial has a non-negative degree $r \leq \deg b(x)$, hence $m = r$ and we can choose $p_m = -a^{-1}c_r$. Then the equations $jp_j - ap_{j-1} = c_{j-1}$, $j = 1, \ldots, m$, can be solved by recurrence proving the required existence. To prove the last assertion we note that if Q is a solution of $\mathcal{E}_{c_1,d_1}$ and $\mathcal{E}_{c_2,d_2}$, then $b(x + c_1) - b(x + c_2) = (d_1 - d_2)b(x)$. Since the polynomial $b(x + c_1) - b(x + c_2)$ is either null or its degre is lower than the degree of $b(x)$ we deduce $d_2 = d_1$ and $b(x + c_1) = b(x + c_2)$. The last equality clearly gives $c_1 = c_2$ which completes the proof. □

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Proposition 2. *Let $D = \partial_x + (ay + b)\partial_y$ be a Shamsuddin derivation with $a \neq 0$. We have the following assertions:*

i) If $a \in \mathbb{K}^*$, then $\mathrm{Aut}(D) = \mathbb{K} \times \mathbb{K}^*$

ii) If $\deg a \geq 1$, then $\mathrm{Aut}(D)$ is isomorphic to a trivial subgroup of $\mathbb{K}^*$ which is different from $\{1\}$ if and only if either $b = 0$ or there exists a nonzero polynomial solution of the differential equation $P' - aP = (1 - d)b$.

Proof. We keep all notations from [MePa2016]. Let $\rho = (f, g) \in \mathrm{Aut}(D)$. First we assert that $f = \sum_{i=0}^{n} f_i y^i = f_0 = x + c$ and $g = \sum_{j=0}^{m} g_j y^j = g_0 + dy$ for suitable $c \in \mathbb{K}, d \in \mathbb{K}^*$ and $g_0 \in \mathbb{K}[x]$. Indeed, $n > 0$ contradicts $f_n' + naf_n = 0$ in the top equality in [MePa2016, (2)]; hence $f = x + c$ for some $c \in \mathbb{K}$, and then $m \geq 1$ because ρ is an automorphism. Since the Jacobian determinant of ρ belongs to $\mathbb{K}^*$ the assertion follows. Furthermore, from the bottom equality in [MePa2016, (2)] we obtain

$$g_0' + bd = a(x + c)g_0 + b(x + c) \quad \text{and} \quad a(x) = a(x + c); \tag{1}$$

here $a(x + c)$ (analogously $b(x + c)$) denotes the polynomial a, thought of as a polynomial function, composed with $f = x + c$. Since $\rho^\ell \in \mathrm{Aut}(D)$ for all $\ell \geq 1$ we deduce that if $\deg a \geq 1$, then $c = 0$, and if $a \in \mathbb{K}^*$, then (1) is equivalent to the unique equality

$$g_0' - ag_0 = b(x + c) - db(x); \tag{2}$$

Now, assume $a \in \mathbb{K}^*$ and define a map $\eta : \mathbb{K} \times \mathbb{K}^* \to \mathrm{Aut}_{\mathbb{K}}(\mathbb{K}[x, y])$ by doing $(c, d) \mapsto (x+c, dy+P_{c,d}(x))$, where $P_{c,d}(x)$ is the unique solution given by Lemma 1. By construction, the image of η is precisely $\mathrm{Aut}(D)$. Since that map is clearly injective, to prove assertion (i) it suffices to show η is a homomorphism. In fact, by composing $\eta(c_1, d_1)$ with $\eta(c_2, d_2)$ we get the element

$$(x + c_1 + c_2, d_1 d_2 y + d_1 P_{c_2,d_2}(x) + P_{c_1,d_1}(x + c_2)) \in \mathrm{Aut}(D).$$

Thus the polynomial $g_0 = d_1 P_{c_2,d_2}(x) + P_{c_1,d_1}(x + c_2)$ satisfies the differential equation $\mathcal{E}_{c_1+c_2,d_1d_2}$ from which we deduce $g_0 = P_{c_1+c_2,d_1d_2}$, as required. Next, assume $\deg a \geq 1$ (hence $c = 0$). From $g_0' + bd = ag_0 + b$ it follows g_0 is a solution of the differential equation $P' - aP = (1 - d)b$. Note that if such a solution exists, then it is unique; let us denote P_d such a solution. Then we may define an injective homomorphism $\mathrm{Aut}(D) \to \mathbb{K}^*$, by mapping $(x, dy + P_d) \in \mathrm{Aut}(D)$ to d. Finally, assume $\mathrm{Aut}(D) \neq \{1\}$. Then there exists $d \neq 1$ such that $(x, dy + P_d) \in \mathrm{Aut}(D)$. Hence $P_d = 0$ if and only if $b = 0$. If $P_d \neq 0$ and $\alpha \in \mathbb{K}^*$, then αP_d is a solution of the differential equation $P' - aP = (1 - e)b$, with $e := 1 - \alpha(1 - d)$, so $(x, (1 - \alpha(1 - d))y + P_{1-\alpha(1-d)}) \in \mathrm{Aut}(D)$ if $\alpha(1 - d) \neq 1$, from which the proof follows. □

References

[MePa2016] L. G. Mendes and I. Pan, *On plane polynomial automorphisms commuting with simple derivations*, J. of Pure and Appl. Algebra, Vol. 221, N. 4 (2016), pp. 875-882.