Foliations with radial Kupka set and pencils of Calabi-Yau hypersurfaces

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Dedicated to Professor César Camacho in occasion of his 60-th birthday

Abstract

We show that holomorphic singular codimension one foliations of the complex projective space with a Kupka singular set of radial type and verifying some global hypotheses have rational first integral. The generic elements of such pencils of hypersurfaces are Calabi-Yau.

1 Introduction

Let $\mathcal{F}$ be a codimension one holomorphic singular foliation of $\mathbf{C}P^n$. We deal with the problem of giving conditions on the singular set of $\mathcal{F}$ which imply the existence of a global rational first integral for $\mathcal{F}$, that is, that $\mathcal{F}$ is a pencil of algebraic hypersurfaces.

1.1. The foliation is defined by a class of sections $\omega \in \mathrm{H}^0(\mathbf{C}P^n, \Omega^1(d+2))$, where $d = d(\mathcal{F})$ is its degree, that is, the number of tangencies of leaves of $\mathcal{F}$ with a generic projective line. It is always possible suppose that $cod_{\mathbf{C}}(\omega)_0 \geq 2$ and we put $sing(\mathcal{F}) := \{p \in \mathbf{C}P^n | \quad \omega(p) = 0\}$. In this paper we consider dimension $n \geq 3$ and our main hypothesis is that there is a codimension two connected component of $sing(\mathcal{F})$ which is a *Kupka set* $K(\mathcal{F})$, defined by

$$K(\mathcal{F}) := \{p \in \mathbf{C}P^n | \quad \omega(p) = 0,\ d\omega(p) \neq 0\}.$$

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We consider *compact* Kupka sets of foliations of $\mathbf{C}P^n$. From [12] and references therein, it is known that there is an open covering $\{U_i\}_i$ of $K(\mathcal{F})$ and a collection of submersions $\psi_i : U_i \to \mathbf{C}^2$ with $K(\mathcal{F}) \cap U_i = \psi_i^{-1}(0,0)$, as well as a holomorhic 1-form $\eta_{pq} := px\,dy - qy\,dx$, with isolated zero at $(0,0)$, such that $\mathcal{F}_{|U_i}$ is represented by $\psi_i^*(\eta_{pq})$, where p, q are integers with $1 \leq p < q, (p,q) = 1$ or $p = q = 1$. It follows from this local product structure that $K(\mathcal{F})$ is smooth and that other components of $sing(\mathcal{F})$ do not intersect $K(\mathcal{F})$. The *transversal type* of $K(\mathcal{F})$ is given by η_{pq} and the *radial type* corresponds to $p = q = 1$; we denote in this case $K(\mathcal{F}) = R(\mathcal{F})$.

1.2. A fundamental fact in the theory of foliations of $\mathbf{C}P^n$ with Kupka set is that $\mathcal{F}$ has a global rational first integral if and only if $K(\mathcal{F})$ is a (scheme theoretically) complete intersection [8, Thm. A]. Furthermore, it is conjectured that $K(\mathcal{F})$ is always a complete intersection. There are already several partial positive answers to this conjecture. In fact, in [6, Theorem 3.5] it is proven that a codimension one foliation of $\mathbf{C}P^n$, $n \geq 3$, whose Kupka set is *not* of radial type, has a rational first integral. Also, for any transversal type, it is proven in [7] (Cor. 4.5) that the Kupka set of a foliation is a complete intersection, under the extra hypotheses $d(\mathcal{F}) \leq 2n$ *and* $n \geq 6$. At last, in [1], [2] we found the hypothesis $n \geq 6$.

In order to state our result we recall some definitions.

1.3. The *canonical bundle* of a k- dimensional algebraic variety S is defined by $K_S := \bigwedge^k TS^*$, where TS^* is the cotangent bundle.

The triviality of K_S is a main feature of *Calabi-Yau varieties*. Smooth hypersurfaces of $\mathbf{C}P^n$ with degree $n+1$ are the first examples of Calabi-Yau varieties (see [11] and [13], for instance, for the general theory of Calabi-Yau threefolds).

1.4. Let M be a n-dimensional projective smooth variety with a codimension one holomorphic singular foliation $\mathcal{F}$ with $cod_{\mathbf{C}} sing(\mathcal{F}) \geq 2$. Let $T_{\mathcal{F}}$ be the *tangent sheaf* of $\mathcal{F}$ and consider the line bundle

$$\bigwedge^{n-1} T_{\mathcal{F}}^*$$

which is the dual of the top exterior power. Remark that local sections of $\bigwedge^{n-1} T_{\mathcal{F}}^*$ correspond to top degree holomorphic forms along the leaves of $\mathcal{F}$.

1.5. We recall that a line bundle $\mathcal{L}$ of M is called *Nef* if $c_1(\mathcal{L}) \cdot C \geq 0$ for all curves C, where $c_1(\mathcal{L})$ means the first Chern class of $\mathcal{L}$. Nefness will play a fundamental role along this paper. We remark that Nefness is a weaker condition than the Ampleness condition, which has been already exploited in the field of holomorphic foliations (e.g. [12], [6]). For information on the geometrical meaning of nefness of contangent bundles of foliations we refer [16], [5], [4].

At last, we call *normal Baum-Bott indices* the usual indices ([5]) of foliations by curves induced in generic plane sections of a codimension one foliation.

Theorem :

Let $\mathcal{F}$ be a codimension one singular foliation of $\mathbf{C}P^n$, for $n \geq 3$; denote $S_{n-2}(\mathcal{F})$ its codimension two singular set.

1. *Suppose that $d(\mathcal{F}) = 2n$ and that $\mathcal{F}$ has a compact connected Kupka set of radial transversal type $R(\mathcal{F})$.*
2. *Suppose that $S_{n-2}(\mathcal{F}) \setminus R(\mathcal{F})$ has non-positive normal Baum-Bott indices.*
3. *Consider $\sigma : M \to \mathbf{C}P^n$ the blowing up along $R(\mathcal{F})$ and $\hat{\mathcal{F}}$ the transformed foliation of $\mathcal{F}$ by σ, with $codim_{\mathbf{C}}(\hat{\mathcal{F}}) \geq 2$ on M.*
4. Suppose that $\wedge^{n-1} T^*_{\hat{\mathcal{F}}}$ is Nef.

Then

1. $S_{n-2}(\mathcal{F}) = R(\mathcal{F})$ *and* $deg\, R(\mathcal{F}) = (n+1)^2$.
2. *$\mathcal{F}$ is a pencil of hypersurfaces of degree $n+1$ which are smooth along the base-locus $R(\mathcal{F})$. The generic element of the pencil is a Calabi-Yau hypersurface.*

Remark: This result was motivated by [17, Thm. 2], applied to a degree 4 foliation $\mathcal{F}$ of $\mathbf{C}P^2$. The hypotheses on $\mathcal{F}$ in that paper imply that the transformed foliation $\hat{\mathcal{F}}$ under blow up of k radial points has Morse singularities and that $T^*_{\hat{\mathcal{F}}}$ is Nef. The conclusion is that $\mathcal{F}$ is a pencil of cubics, smooth at the $k = 9$ base-points. Counterexamples, *without the Nefness condition*, are given by a one parameter family of foliations of degree 4 in the plane described in [15]. In such family, for all parameters, singularities are either radial or have local holomorphic first integral, but $k = 12$ and only a countable set of parameters corresponds to pencils of elliptic curves.

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2 Preliminaries

2.1 Adjunction formula

A codimension one foliation $\mathcal{F}$ of a n-dimensional smooth projective variety M can be represented by a non-trivial integrable section of $\Omega^1_M \otimes N_{\mathcal{F}}$ where $N_{\mathcal{F}}$ is the normal bundle of $\mathcal{F}$. For instance, in $\mathbf{C}P^n$ we have

$$N_{\mathcal{F}} = \mathcal{O}_{\mathbf{C}P^n}(d(\mathcal{F}) + 2),$$

where $d(\mathcal{F})$ is the *degree* of the foliation, that is, the degree of the tangency divisor between the leaves of $\mathcal{F}$ and a generic line.

From the exact sequence of sheaves

$$0 \to T_{\mathcal{F}} \to T_M \to \mathcal{J}_{sing(\mathcal{F})} \cdot N_{\mathcal{F}} \to 0,$$

by dualising and taking the top exterior powers, we obtain the following isomorphism of line bundles of M:

$$\bigwedge^n TM^* = \bigwedge^{n-1} T^*_{\mathcal{F}} \otimes N^*_{\mathcal{F}}.$$

Denoting $K_M := \bigwedge^n TM^*$ we have an Adjunction Formula:

$$\bigwedge^{n-1} T^*_{\mathcal{F}} = K_M \otimes N_{\mathcal{F}}.$$

2.2 Blowing up of a radial Kupka set

Let $\sigma_{R(\mathcal{F})} : M \to \mathbf{C}P^n$ be the blow up along the radial Kupka set of $\mathcal{F}$ (as remarked, $R(\mathcal{F})$ is a smooth variety). Let $E = \sigma^{-1}_{R(\mathcal{F})}(R(\mathcal{F}))$ be the exceptional divisor, and denote by $\hat{\mathcal{F}}$ the strict transform foliation of $\mathcal{F}$ under $\sigma_{R(\mathcal{F})}$ (taken with singularities of codimension ≥ 2). We assert the following line bundle isomorphism:

$$N_{\hat{\mathcal{F}}} = \sigma^*_{R(\mathcal{F})}(N_{\mathcal{F}}) \otimes \mathcal{O}_M(-2E).$$

In fact, on open sets $U = (x, y, z_1, \ldots z_{n-2})$ of the adapted covering of $R(\mathcal{F})$, with local submersion $\psi : U \to \mathbf{C}^2 = (x, y)$, $\mathcal{F}_{|U}$ can be given by a local holomorphic 1-form $\psi^*(\eta)$, where $\eta = (x + h.o.t)dy - (y + h.o.t)dx$. In local coordinates we have

$$\sigma_{R(\mathcal{F})}(x, t, z_1, \ldots, z_{n-2}) = (x, xt, z_1, \ldots, z_{n-2}) = (x, y, z_1, \ldots, z_{n-2}),$$

thus $\sigma^*_{R(\mathcal{F})}(\psi^*(\eta)) = x^2 \cdot \hat{\eta}$, where $x = 0$ is a local equation of the divisor E and $\hat{\eta}$ induces $\hat{\mathcal{F}}$. Therefore:

$$\sigma^*_{R(\mathcal{F})}(N^*_{\mathcal{F}}) = N^*_{\hat{\mathcal{F}}} \otimes \mathcal{O}_M(-2E)$$

and dualising we prove the assertion.

By other side, it is well known that

$$K_M = \sigma^*_{R(\mathcal{F})}(K_{\mathbf{C}P^n}) \otimes \mathcal{O}(E),$$

which can be verified by considering locally

$$\sigma^*_{R(\mathcal{F})}(dx \wedge dy \wedge dz_1 \ldots \wedge dz_{n-2}) = x\ dx \wedge dt \wedge dz_1 \ldots \wedge dz_{n-2}.$$

Combining these line bundle isomorphisms with the Adjunction Formula (Section 2.1) we obtain:

$$\bigwedge^{n-1} T^*_{\hat{\mathcal{F}}} = \sigma^*_{R(\mathcal{F})}(\bigwedge^{n-1} T^*_{\mathcal{F}}) \otimes \mathcal{O}_M(-E).$$

3 Proof of the Theorem

Lemma 1

Let $\mathcal{F}$ be a codimension one foliation of $\mathbf{C}P^n$ satisfying the hypotheses of the Theorem. Then $\deg R(\mathcal{F}) = (n+1)^2$.

Proof:

If $n > 3$ any extra component of $Sing(\mathcal{F})_{n-2}$ would intersect $R(\mathcal{F})$, contradicting the local product structure of $\mathcal{F}$ along $R(\mathcal{F})$. Then it suffices to consider two cases:

(1) $Sing_{n-2}(\mathcal{F}) = R(\mathcal{F})$ and

(2) $n = 3$ and $Sing_1(\mathcal{F}) \neq R(\mathcal{F})$.

Case (1). Let H_2 be a generic 2-plane in $\mathbf{C}P^n$, so that H_2 cuts transversally all irreducible components of $Sing_{n-2}(\mathcal{F})$. The tangencies of H_2 with $\mathcal{F}$, at non singular points, give rise to Morse type singularities of $\mathcal{G} = i^*(\mathcal{F})$, where $i : H_2 \to \mathbf{C}P^n$ is the inclusion. Then the singularities of $\mathcal{G}$ are the tangencies of $\mathcal{F}$ with H_2, denoted $\{q_1, \ldots, q_r\}$ and the points of $R(\mathcal{F}) \cap H_2 = \{p_1, \ldots, p_m\}$, where $m = \deg\, R(\mathcal{F})$, the p_j's beeing radial singularities of $\mathcal{G}$. By the Baum-Bott formula we get:

$$\sum_i BB(\mathcal{G}, q_i) + \sum_j BB(\mathcal{G}, p_j) = (d(\mathcal{G}) + 2)^2 = (d(\mathcal{F}) + 2)^2.$$

On the other hand, the Morse singularities have zero Baum-Bott indices and for a radial singularity the index is 4. Therefore we obtain

$$4 \deg\, R(\mathcal{F}) = (d(\mathcal{F}) + 2)^2 = 4(n+1)^2.$$

Case (2). Let

$$Sing_1(\mathcal{F}) = R(\mathcal{F}) \cup X,$$

where $X \neq \emptyset$ has non-positive normal Baum-Bott indices. Keeping notations of case 1, let

$$R(\mathcal{F}) \cap H_2 = \{p_1, \ldots, p_m\}, \quad X \cap H_2 = \{r_1, \ldots, r_s\}$$

and $q_1, \ldots, q_r$ Morse points of $\mathcal{G} = i^*(\mathcal{F})$. Again Baum-Bott formula gives:

$$\begin{aligned} (d(\mathcal{F}) + 2)^2 &= \sum_i BB(\mathcal{G}, q_i) + \sum_j BB(\mathcal{G}, p_j) + \sum_k BB(\mathcal{G}, r_k) \\ &\leq 4 \deg\, R(\mathcal{F}). \end{aligned}$$

That is

$$\deg\, R(\mathcal{F}) \geq 16.$$

Let $\sigma_{R(\mathcal{F})} : M \to \mathbf{C}P^3$ be the blowing up of the Kupka set and $\mathcal{H} = \sigma^*_{R(\mathcal{F})}(H_2)$ the total transform of H_2. Since $\bigwedge^2 T^*_{\hat{\mathcal{F}}}$ is Nef, by a theorem of Kleiman [14]

$$0 \leq c_1(\bigwedge^2 T^*_{\hat{\mathcal{F}}})^2 \cdot \mathcal{H}.$$

If $E = \sigma^{-1}_{R(\mathcal{F})}(R(\mathcal{F}))$, we have (Section 2.2):

$$\bigwedge^2 T^*_{\hat{\mathcal{F}}} = \sigma^*_{R(\mathcal{F})}(\bigwedge^2 T^*_{\mathcal{F}}) \otimes \mathcal{O}_M(-E).$$

Hence

$$0 \leq [c_1(\sigma^*_{R(\mathcal{F})}(\bigwedge^2 T^*_{\mathcal{F}}))^2 - 2 \cdot c_1(\sigma^*_{R(\mathcal{F})}(\bigwedge^{n-1} T^*_{\mathcal{F}})) \cdot E + E^2] \cdot \mathcal{H} =$$

$$= c_1(\bigwedge^2 T^*_{\mathcal{F}})^2 \cdot H_2 + E^2 \cdot \mathcal{H} =$$

$$= c_1(\bigwedge^2 T^*_{\mathcal{F}})^2 \cdot H_2 - \deg\ R(\mathcal{F})$$

where we have used the Projection Formula [9, Prop. 2.5]. By section 2.1 we get:

$$\bigwedge^2 T^*_{\mathcal{F}} = \mathcal{O}_{\mathbf{C}P^3}(d(\mathcal{F}) - 2)) = \mathcal{O}_{\mathbf{C}P^3}(4)$$

and therefore we obtain:

$$0 \leq 16 - \deg\ R(\mathcal{F}).$$

$\square$

Lemma 2

Under the hypotheses of the Theorem, if S is a hypersurface containing $R(\mathcal{F})$ then $\deg\ S \geq n+1$*; moreover, if* $\deg\ S = n+1$ *then S is smooth along $R(\mathcal{F})$.*

Proof:

Suppose $R(\mathcal{F}) \subset S$ and consider a generic plane section $C_S := S \cap \Pi$. Lemma 1 gives

$$\deg\ R(\mathcal{F}) = (n+1)^2$$

and C_S contains the points $p_1 \ldots, p_{(n+1)^2}$ of $R(\mathcal{F}) \cap \Pi$. Denote $\nu(C_S, p_i) \geq 1$ the algebraic multiplicity of C_S at points of $R(\mathcal{F}) \cap \Pi$.

Keeping the notation of Lemma 1, let $\hat{\mathcal{F}}$ be the transformed foliation by $\sigma_{R(\mathcal{F})}$. Denoting $\hat{C}_S$ the strict transform of C_s by $\sigma_{R(\mathcal{F})}|_{\hat{\Pi}} : \hat{\Pi} \to \Pi$, where $\hat{\Pi}$ is the strict transform of Π and denoting E_i's the exceptional curves of $E \cap \hat{\Pi}$, we have

$$\hat{C}_S = \left(\sigma_{R(\mathcal{F})}|_{\hat{\Pi}}\right)^* (C_S) - \sum_{i=1}^{(n+1)^2} \nu(C_S, p_i) \cdot E_i$$

and hence

$$c_1 \left(\bigwedge^{n-1} T^*_{\hat{\mathcal{F}}}\right) \cdot \hat{C}_S =$$

$$= c_1 \left(\sigma^*_{R(\mathcal{F})}(\bigwedge^{n-1} T^*_{\mathcal{F}}) \otimes \mathcal{O}_M(-E)\right) \cdot \hat{C}_S =$$

$$= (n+1)\cdot \deg\, C_S - \sum_{i=1}^{(n+1)^2} \nu(C_S, p_i).$$

The Nefness of $\bigwedge^{n-1} T^*_{\hat{\mathcal{F}}}$ implies deg $C_S \geq n+1$ with equality only if $\nu(C_S, p_i) = 1$ for all i.

$\square$

Remark that by the same reasoning we prove that there is no projective line $l \subset \Pi$ passing by more than $(n+1)$ points of $R(\mathcal{F}) \cap \Pi$, there is no conic of Π passing by more than $2\cdot(n+1)$ points of $R(\mathcal{F})\cap\Pi$, etc.

We recall some known definitions that shall be used in the proof of the next proposition (for more details see [18, Chap. II]).

Let V be a 2-bundle on $\mathbf{C}P^n$ with *even* first Chern class $c_1(V)$. Put

$$V_{norm} := V\left(-\frac{c_1(V)}{2}\right);$$

in this case $c_1(V_{norm}) = 0$.

By definition V is *stable* if $H^0(\mathbf{C}P^n, V_{norm}) = 0$. And V is *semistable* if

$$H^0(\mathbf{C}P^n, V_{norm}) \neq 0 \quad \textit{and} \quad H^0(\mathbf{C}P^n, V_{norm}(-1)) = 0.$$

On the other hand, the *discriminant* of V is the integer number

$$\Delta(V) = c_1(V)^2 - 4\cdot c_2(V).$$

It is invariant with respect to tensoring with $\mathcal{O}(k)$; in particular $\Delta(V) = \Delta(V_{norm})$

Proposition 1: *Under the hypotheses of the Theorem, $R(\mathcal{F})$ is a complete intersection of hypersurfaces of degree $n+1$.*

Proof:

Firstly we show that $R(\mathcal{F})$ is contained in some hypersurface S with deg $S = n+1$.

In [7] it is proven that Kupka sets are subcanonically embedded. From Serre's construction the normal bundle of $R(\mathcal{F})$ extends as a rank two vector bundle V of $\mathbf{C}P^n$, having a holomorphic section s with an exact sequence:

$$0 \to \mathcal{O}_{\mathbf{C}P^n} \xrightarrow{\cdot s} V \to \mathcal{J}_{R(\mathcal{F})}(d(\mathcal{F})+2) \to 0 \tag{1}$$

where $\mathcal{J}_{R(\mathcal{F})}$ is the ideal sheaf associated to $R(\mathcal{F})$ and

$$c(V) = 1 + (d(\mathcal{F}) + 2) \cdot \mathbf{h} + \deg\, R(\mathcal{F}) \cdot \mathbf{h}^2 \in \mathbf{Z}[\mathbf{h}]/\mathbf{h^{n+1}} \simeq \mathbf{H}^*(\mathbf{C}P^n, \mathbf{Z});$$

here $c(V)$ denote the *total Chern class* of V.

Since $d(\mathcal{F}) = 2n$ and $\deg\, R(\mathcal{F}) = (n+1)^2$, we get:

$$\Delta(V) = c_1(V)^2 - 4 \cdot c_2(V) = 0$$

According to [3] it follows that V is non stable, that is $H^0(V_{norm}) \neq 0$.

Tensorising (1) by $\mathcal{O}_{\mathbf{C}P^n}(-n-1)$ and taking the long exact sequence of cohomology we get:

$$H^0(\mathcal{O}(-n-1)) = 0 \to H^0(V_{norm}) \simeq H^0(\mathcal{J}_{R(\mathcal{F})}(n+1)) \to 0 = H^1(\mathcal{O}(-n-1)),$$

from which it follows the existence of S.

Secondly we prove that $R(\mathcal{F})$ is a complete intersection.

Together with Lemma 2 we conclude that $n+1$ is the *minimum* degree of hypersurfaces containing $R(\mathcal{F})$. By [18, Lemma 1.3.4] this is equivalent to the semistability of V.

Let τ be a non-trivial holomorphic section of V_{norm} and $(\tau)_0$ its scheme of zeroes. We assert that $(\tau)_0$ has *not* codimension one. In fact, if $(\tau)_0$ is a hypersurface of degree k, then from τ we obtain a holomorphic section of $V_{norm}(-k)$, contradicting the semistability of V.

We consider the exact sequence

$$0 \to \mathcal{O} \xrightarrow{\cdot\tau} V_{norm} \to \mathcal{J}_Z \to 0$$

where $Z = (\tau)_0$ (either Z is empty or has codimension two). If we suppose that Z is not empty, then $\deg Z = c_2(V_{norm})$. But $c_1(V_{norm}) = 0$ and $\Delta(V_{norm}) = \Delta(V) = 0$ gives:

$$c_2(V_{norm}) = 0.$$

Then we conclude that $Z = \emptyset$. Hence V_{norm} is defined by an extension of line bundles on $\mathbf{C}P^n$. Then it splits as a sum of line bundles ([18, Chap. I, §2]); the same is true for V. As known, V splits if and only if $R(\mathcal{F})$ is a complete intersection.

$\square$

After Proposition 1, we conclude that $\mathcal{F}$ coincide with a pencil of degree $n+1$ hypersurfaces using [8].

For $n = 3$, applying Bertini Theorem we conclude that the singularities of $\mathcal{F}$ of dimension 1 intersect the base locus $R(\mathcal{F})$, violating the local product structure of $R(\mathcal{F})$.

Remark: At this point we see that Case (2) in the proof of Lemma 1 in fact does not exist.

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