BIRATIONAL GEOMETRY OF FOLIATIONS ASSOCIATED TO SIMPLE DERIVATIONS

GAËL COUSIN, LUÍS GUSTAVO MENDES, AND IVÁN PAN

ABSTRACT. We propose a study of the foliations of the projective plane induced by simple derivations of the polynomial ring in two indeterminates over the complex field. These correspond to foliations which have no invariant algebraic curve nor singularities in the complement of a line. We establish the position of these foliations in the birational classification of foliations and prove the finiteness of their birational symmetries. Most of the results apply to wider classes of foliations.

1. INTRODUCTION AND RESULTS

> "(...) mais nous ne serons satisfaits que quand on aura trouvé un certain groupe de transformations (par exemple de transformations de Cremona) qui jouera, par rapport aux équations différentielles, le même rôle que le groupe des transformations birationnelles pour les courbes algébriques. Nous pourrons alors ranger dans une même classe toutes les transformées d'une même equation."
>
> — Henri Poincaré, L'Avenir des Mathématiques

At least since Poincaré [Poi91], the study of the algebraic subsets that are left invariant by a given plane polynomial vector field is known to be a difficult matter. For instance, the example $rx\partial_x + y\partial_y, r \in \mathbb{Q}$ shows that it is in general impossible to bound the degree of the invariant algebraic curves in terms of the degrees of the vector field's coefficients. In commutative algebra, the vector fields that preserve no nontrivial algebraic subset of the affine plane correspond to the so-called simple derivations and there is an extensive literature dedicated to the production of such examples, see [Sha77, Jor81, Now94, MMON01, Cou03, BLL03, Now08, GL12, Kou12], among others.

This is also an active field of study in foliation theory, where one considers the extension of the foliation to the projective plane. A key result in this context is the work of Jouanolou [Jou79] that exhibited a family of examples of foliations without any invariant algebraic curve in the projective plane and deduced the (Baire) genericity of such examples. Compare [LR03] for a generalization. In the opposite direction, the study of foliations in the neighborhood of invariant

---

divisors provides important information on the foliation [CLPT] and the study of such divisors remains crucial for the study of algebraic or Liouvillian integrability of foliations [CLNP16].

At the turn of the century, the birational geometry of foliations has been developed [Bru00, McQ01, Men00] and one has a birational classification *à la* Enriques-Kodaira for foliations of projective surfaces. The goal of this article is to explain how the recent tools in foliation theory allow to classify geometrically the simple derivations of $\mathbb{C}[x, y]$ and to study their symmetries. We will also present a set of examples found throughout the commutative algebra literature and study their relationships.

In algebra, a derivation of the ring $\mathbb{C}[x, y]$ is said to be *simple* if it does not fix globally any nontrivial proper ideal. It corresponds to a polynomial vector field of $\mathbb{C}^2$ without zeroes and without algebraic trajectories.

For any derivation ∂, the isotropy group $\mathrm{Aut}(\partial)$ is composed by the $\mathbb{C}$-automorphisms $\rho : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ which verify

$$\rho\partial = \partial\rho.$$

Although there exist derivations with infinite isotropy group, the main result of [MP17] is that $\mathrm{Aut}(\partial)$ *is trivial for any simple derivation.*

Take $\rho \in \mathrm{Aut}(\mathbb{C}[x, y])$, $R : \mathbb{C}^2 \to \mathbb{C}^2$ the polynomial automorphism associated to ρ and let ω_∂ be the dual 1-form to the vector field $\partial = f\,\partial_x + g\,\partial_y$ (i.e. $\omega_\partial = g\,dx - f\,dy$). Then $\rho\partial = \partial\rho$ is equivalent to

$$R^*(\omega_\partial) = Jac(R) \cdot \omega_\partial$$

where $Jac(R) \in \mathbb{C}^*$ is the Jacobian determinant of R. A less restrictive condition is that

$$R^*(\omega_\partial) = c \cdot \omega_\partial,$$

for some $c \in \mathbb{C}^*$ (depending on R). This means that R preserves the *foliation* $\mathcal{F}_\partial$ of $\mathbb{C}^2$ associated to ∂ (or to ω_∂), see Remark 2.1 p. 5.

We denote $Pol(\mathcal{F}_\partial)$ the group consisting of polynomial automorphisms of $\mathbb{C}^2$ which preserve the foliation $\mathcal{F}_\partial$. There is a natural homomorphism

$$\mathrm{Aut}(\partial) \hookrightarrow Pol(\mathcal{F}_\partial).$$

Let us denote $\mathcal{F}$ the singular holomorphic foliation of the projective plane $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$ which is the extension of $\mathcal{F}_\partial$ in $\mathbb{C}^2$. All along the paper, if ∂ is a simple derivation, both $\mathcal{F}_\partial$ in $\mathbb{C}^2$ and its extension $\mathcal{F}$ in $\mathbb{P}^2$ are called *foliations associated to simple derivations.*

But the reader must be warned that, even if $\mathcal{F}_\partial$ has no singularity, some singularities of $\mathcal{F}$ along the line at infinity L_∞ are unavoidable, see [Bru00, Prop. 2.1]. Also beware that the line at infinity L_∞ may be invariant by $\mathcal{F}$.

Denote $Bir(\mathcal{F})$ the group of birational transformations of $\mathbb{P}^2$ which preserve a foliation $\mathcal{F}$; the elements in $Bir(\mathcal{F})$ are sometimes called *birational symmetries* of $\mathcal{F}$. If $\mathcal{F}$ extends a foliation $\mathcal{F}_\partial$ of $\mathbb{C}^2$, then there is a natural homomorphism

$$Pol(\mathcal{F}_\partial) \hookrightarrow Bir(\mathcal{F})$$

whose meaning is that a (non-linear) polynomial automorphism of $\mathbb{C}^2$ extends to a special type of birational map of $\mathbb{P}^2$. Namely, a birational map with a unique (proper) point of indeterminacy $p \in L_\infty$, whose net effect on $\mathbb{P}^2$ is to replace L_∞ by the strict transform of the last exceptional curve introduced in the elimination of the indeterminacy point.

In Section 7, we propose a construction of simple derivations ∂ with arbitrary large finite $Pol(\mathcal{F}_\partial)$. This shows the optimality of the following.

Theorem A. *Let $\mathcal{F}$ be a foliation of $\mathbb{P}^2$ whose restriction $\mathcal{F}|_{\mathbb{C}^2}$ to $\mathbb{C}^2$ has no algebraic invariant curve. Then $Bir(\mathcal{F})$ is finite; in particular, a foliation associated to a simple derivation admits only finitely many birational symmetries.*

Theorem A is actually derived from the next result which determines, in particular, the positions that foliations associated to simple derivations may occupy in the birational classification of foliations. This classification is based on the notion of *Kodaira dimension* of a foliation, denoted $\kappa(\mathcal{F})$, whose range is $\kappa(\mathcal{F}) \in \{-\infty, 0, 1, 2\}$, see Section 2.

For a reduced divisor in a quasiprojective surface, we say it is a *rational* curve if its projective closure has geometric genus zero. By a *Riccati foliation* on $\mathbb{P}^2$ we mean a foliation which, up to a birational modification of $\mathbb{P}^2$, is everywhere transverse to the general fiber of a rational fibration (see §2.13 for more details).

Theorem B. *Let $\mathcal{F}$ be a foliation of the projective plane such that the restriction $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant rational curve.*

  (*i*) *Then $\kappa(\mathcal{F}) \geq 1$;*
 (*ii*) *If $\mathcal{F}|_{\mathbb{C}^2}$ has no invariant algebraic curve, then $\kappa(\mathcal{F}) = 1$ if and only if $\mathcal{F}$ is a Riccati foliation.*
(*iii*) *The cases $\kappa(\mathcal{F}) \in \{1, 2\}$ are realized by foliations associated to simple derivations.*

Note that Theorem B applies to a class of foliations which is larger than the one of foliations associated to simple derivations and that case B-(*ii*) includes the foliations associated to *Shamsuddin derivations cf.* [Sha77].

In fact, we obtain Theorem B-(*i*) as a special case of the following with $(X, D) = (\mathbb{P}^2, L_\infty)$.

Theorem C. *Let X be a smooth projective rational surface and let D be a reduced divisor on X. Suppose that $X \setminus D$ is simply connected. Then any foliation $\mathcal{F}$ on X that possesses no invariant rational curve outside D satisfies $\kappa(\mathcal{F}) \geq 1$.*

In Section 5 we study the foliations associated to examples of simple derivations found throughout the literature and discuss their birational equivalence.

2. Preliminaries on foliations

The paper relies on concepts and results of the theory of singularities and birational geometry of foliations on algebraic complex surfaces. We present some basic facts in this preliminary section but along the paper, when necessary, we refer the reader to the corresponding sections of [Bru00] or [Bru03], where the theory is masterfully explained.

2.1. **First definitions.** On a smooth complex surface X, a foliation $\mathcal{F}$ is given by an open covering (U_i) of X and local vector fields $v_i \in H^0(U_i, TX)$ with *isolated zeroes* such that there exist non vanishing holomorphic functions (g_{ij}) on the intersections $U_i \cap U_j$ satisfying

$$v_i = g_{ij} v_j. \tag{1}$$

The locus defined by the vanishing of the local vector fields (v_i) is called the *singular locus* of $\mathcal{F}$ and denoted $\mathrm{Sing}(\mathcal{F})$.

The cocycle (g_{ij}) defines a line bundle $T^*\mathcal{F}$ on X, its dual is denoted $T\mathcal{F}$. Relation (1) means that the family (v_i) defines a section of $T^*\mathcal{F} \otimes TX$ and hence a sheaf map $T\mathcal{F} \to TX$. Two data $((U_i), (v_i))$, $((U'_j), (v'_j))$ are said to define the same foliation if the images of the associated sheaf maps are the same.

The line bundle $T\mathcal{F}$ is called the *tangent bundle* of the foliation and its dual $T^*\mathcal{F}$ is the *cotangent bundle* of $\mathcal{F}$. As defined, the line bundle $T\mathcal{F}$ is not canonically attached to $\mathcal{F}$, but its isomorphism class in the *Picard group* $\mathrm{Pic}(X)$ of X is.

One may also consider foliations on normal singular complex surfaces. They are defined by the datum of a foliation on the complement of the singular locus of the surface.

2.2. **Rational vector fields and 1-forms.** If X is smooth projective, $T\mathcal{F}$ possesses a non trivial rational section and $\mathcal{F}$ can be given by a rational vector field $\mathcal{X}$, hence in $\mathrm{Pic}(X)$ we have

$$T\mathcal{F} = \mathcal{O}_X(div(\mathcal{X})),$$

where $div(\mathcal{X})$ denotes the divisor of zeroes and poles of $\mathcal{X}$. On a suitable (Zariski) open covering the local vector fields v_i are defined by setting $v_i = h\mathcal{X}_{|U_i}$, for a well chosen rational function h on U_i. This is how we associate a foliation to a simple derivation: we have a *preferred projective compactification of* $\mathbb{C}^2$, namely $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty, (x, y) \mapsto (x : y : 1)$, and a polynomial vector field on $\mathbb{C}^2$ extends to a rational vector field on $\mathbb{P}^2$.

One can also define a foliation by local holomorphic 1-forms with isolated zeroes (ω_i) that vanish on the local vector fields (v_i). If X is projective, such a family (ω_i) is obtained by locally eliminating poles and codimension 1 zeroes of a non trivial rational 1-form. Hence, on a smooth projective surface, a foliation may be defined by either a non trivial rational 1-form or a a non trivial rational vector field.

2.3. **Curves and foliations.** A curve C is termed *invariant by $\mathcal{F}$* or *$\mathcal{F}$-invariant* if it is tangent to the local vector fields defining $\mathcal{F}$. When a compact curve $C \subset X$ is *not* $\mathcal{F}$-invariant we have the very useful *formula*

$$T^*\mathcal{F} \cdot C = \mathrm{tang}(\mathcal{F}, C) - C \cdot C,$$

where $\text{tang}(\mathcal{F}, C)$ is the sum of orders of tangency between $\mathcal{F}$ and C, *cf.* [Bru00, Prop. 2.2].

2.4. **Degree of plane foliations.** If $X = \mathbb{P}^2$ is the projective plane, the *degree* of $\mathcal{F}$ is $\deg(\mathcal{F}) \in \mathbb{Z}_{\geq 0}$ defined as the number of tangencies of $\mathcal{F}$ with a general projective line. In this case, the previous formula gives

$$T^*\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}(\deg(\mathcal{F}) - 1).$$

Moreover, if $\mathcal{F}$ is induced by a polynomial vector field with isolated zeroes $P(x,y)\partial_x + Q(x,y)\partial_y$ on $\mathbb{P}^2$ then, using the usual total degree for bivariate polynomials,

$$\deg(\mathcal{F}) = \begin{cases} max(\deg P, \deg Q) \text{ if the line at infinity is invariant,} \\ max(\deg P, \deg Q) - 1 \text{ otherwise.} \end{cases}$$

2.5. **Camacho-Sad formula.** The self-intersection of a smooth compact invariant curve C can be computed from certain indices of $\mathcal{F}$ along C. For every $p \in C$, consider a local defining 1-form ω for $\mathcal{F}$ around p. If C has a local equation $f = 0$ at p, one has a local decomposition $\omega = hdf + f\eta$. Define the Camacho-Sad index $CS(\mathcal{F}, C, p)$ as $Res_p(-\eta/h)$. The Camacho-Sad *formula* is then

$$C \cdot C = \sum_{p \in C} CS(\mathcal{F}, C, p).$$

2.6. **Birational maps and foliations.** Let X and Y be projective surfaces with at most normal singularities and $\phi : X \dashrightarrow Y$ a birational map. If we have a foliation $\mathcal{F}$ on X given by the rational vector field $\mathcal{X}$, we can define a foliation $\phi_*\mathcal{F}$ on Y as the one defined by the rational vector field $\phi_*\mathcal{X}$. Conversely, from a foliation $\mathcal{G}$ on Y, one defines $\phi^*\mathcal{G} := (\phi^{-1})_*\mathcal{G}$. We say that the foliations $\mathcal{F}$ and $\phi_*\mathcal{F}$ are *birationally equivalent* and that $\phi_*\mathcal{F}$ is a *birational model of $\mathcal{F}$*.

Remark 2.1. If $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is induced by a polynomial automorphism R of $\mathbb{C}^2$ and $\mathcal{F}$ is given by a polynomial vector field $\mathcal{X}$ on $\mathbb{C}^2$ with isolated zeroes, the condition $\phi^*\mathcal{F} = \mathcal{F}$ is tantamount to $R_*\mathcal{X} = h\mathcal{X}$ for a suitable rational function h. However, as R is a polynomial automorphism, the vector $R_*\mathcal{X}$ is a polynomial vector field on $\mathbb{C}^2$, with isolated zeroes. In particular, the factor h is a constant $c \in \mathbb{C}^*$. A similar reasoning works with polynomial 1-forms.

2.7. **Singularities.** Around a singular point $p \in X$, using local centered coordinates z, w the foliation $\mathcal{F}$ is defined by a holomorphic vector field $v = f(z,w)\frac{\partial}{\partial z} + g(z,w)\frac{\partial}{\partial w}$ with $f(0,0) = g(0,0) = 0$ and $\gcd(f,g) = 1$. The *Milnor number* of the singular point is defined as

$$\mu(p, \mathcal{F}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{<f,g>}$$

Denote λ_1, λ_2 the eigenvalues of the linear part (first jet) of $(z,w) \mapsto (f(z,w), g(z,w))$. We say that p is a *reduced* singularity of $\mathcal{F}$ if at least one of them, say λ_2, is not zero and if $\lambda := \lambda_1/\lambda_2 \notin \mathbb{Q}_{>0}$; otherwise the singularity is *non-reduced*. A special case of non-reduced singularity occurs when the linear part is the identity, in this case $\lambda = 1$, and we say that p is a *radial point*.

If $\lambda \neq 0$ we say that the singularity is *non degenerate*; otherwise we call it a *saddle-node*. The separatrix of a saddle-node which is tangent to the non-zero eigenvector is called the *strong separatrix*. If there is a second separatrix, it is called *weak separatrix*.

We say that p is a *Morse point* if it is non degenerate and, in suitable coordinates, admits a local holomorphic first integral of the form $\phi(z, w) = z^2 + w^2 + \text{h.o.t.}$; note that for Morse points $\lambda = -1$.

2.8. **Reduced and relatively minimal models of foliations.** A foliation $\mathcal{F}$ on a smooth surface is said to be *reduced* if all its singularities are reduced. After Seidenberg [Sei68], foliations on smooth projective surfaces always admit a *reduction of singularities*: a birational morphism $\Sigma : M \to X$ obtained as a composition of blowing-ups such that $\overline{\mathcal{F}} := \Sigma^*\mathcal{F}$ is a reduced foliation.

Such a *reduced model* $\overline{\mathcal{F}}$ is not unique. Indeed, by performing a blowing-up at either a non-singular point or a reduced singularity the transformed foliation remains reduced. Doing such an "unnecessary" blowing-up creates a *foliated exceptional curve* or $\overline{\mathcal{F}}$-*exceptional curve*: a rational curve of self-intersection -1 whose contraction to a point q yields a foliated surface with at most a reduced singularity at q. A reduced model $\overline{\mathcal{F}}$ is called a *relatively minimal model* when it is free of $\overline{\mathcal{F}}$-exceptional curves.

2.9. **Kodaira dimension.** The *Kodaira dimension* $\kappa(\mathcal{F})$ of $\mathcal{F}$ is defined by

$$\kappa(\mathcal{F}) := \limsup_{n \to +\infty} \frac{1}{\log n} \log h^0((T^*\overline{\mathcal{F}})^{\otimes n}),$$

for any reduced model $\overline{\mathcal{F}}$. This is a birational invariant with values in $\{-\infty, 0, 1, 2\}$.

2.10. **Zariski decomposition.** If $\mathcal{F}$ is not birationally equivalent to a rational fibration, Miyaoka and Fujita's results assure that the cotangent line bundle $T^*\mathcal{F}$ admits a unique so-called *Zariski decomposition* [Băd01, Thm. 14.14 p. 220]

$$T^*\mathcal{F} =_{\mathbb{Q}} \mathbf{N} + \mathbf{P},$$

where

- $=_{\mathbb{Q}}$ means equality in the group of rational divisors $\text{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$,
- the *positive part* $\mathbf{P}$ is a *nef* $\mathbb{Q}$-divisor (i.e. $\mathbf{P} \cdot C \geq 0$ for every curve C),
- the *negative part* $\mathbf{N} = \sum_j \alpha_j \mathbf{N}_j$ is a $\mathbb{Q}^+$-divisor ($\alpha_j \in \mathbb{Q}^+$) and each connected component of $\cup_j \mathbf{N}_j$ is contractible to a normal singularity,
- $\mathbf{P} \cdot \mathbf{N}_j = 0$, $\forall j$.

The numerical Kodaira dimension of $T^*\mathcal{F}$ is then defined depending on the numerical properties of $\mathbf{P}$:

$$\begin{cases} \nu(T^*\mathcal{F}) = 0 & \text{if } \mathbf{P} \equiv 0, \\ \nu(T^*\mathcal{F}) = 1 & \text{if } \mathbf{P} \cdot \mathbf{P} = 0 \text{ and } \mathbf{P} \not\equiv 0, \\ \nu(T^*\mathcal{F}) = 2 & \text{if } \mathbf{P} \cdot \mathbf{P} > 0; \end{cases}$$

where $\mathbf{P} \equiv 0$ means $\mathbf{P}$ intersects any divisor trivially. The numerical Kodaira dimension $\nu(\mathcal{F})$ of a foliation $\mathcal{F}$ is then defined as the numerical Kodaira dimension $\nu(T^*\overline{\mathcal{F}})$, where $\overline{\mathcal{F}}$ is any reduced birational model of $\mathcal{F}$.

2.11. **Birational classification.** The birational classification of foliations is done in the spirit of Enriques-Kodaira classification of surfaces, according to the values of $\nu(\mathcal{F})$ and $\kappa(\mathcal{F})$.

We restrict now to mentioning a striking consequence of the classification: $\nu(\mathcal{F}) = \kappa(\mathcal{F})$ unless $\nu(\mathcal{F}) = 1$ and $\kappa(\mathcal{F}) = -\infty$ in which case $\mathcal{F}$ is a Hilbert modular foliation as they will be described in Section 3.1. A few other aspects will be used in the text with suitable references.

2.12. **Nef models of foliations.** If $\overline{\mathcal{F}}$ is a relatively minimal (reduced) model of $\mathcal{F}$ and if $\overline{\mathcal{F}}$ is not a rational fibration, McQuillan's theorem [Bru00, Thm. 8.1] assures that the support of $\mathbf{N}$ in $T^*\overline{\mathcal{F}} =_{\mathbb{Q}} \mathbf{N} + \mathbf{P}$ is a union of so-called *maximal $\overline{\mathcal{F}}$-chains*.

A $\overline{\mathcal{F}}$-chain is a chain of invariant rational $(-n)$-curves, with $n \geq 2$, which starts with a curve containing just one singularity of $\overline{\mathcal{F}}$ and where the other components, if it has more than one, contain two singularities, all singularities being reduced and non-degenerate. The contraction of a $\overline{\mathcal{F}}$-chain produces a rational surface singularity, more precisely, a cyclic quotient singularity. The induced foliation on the resulting singular surface is called a *nef model* of $\overline{\mathcal{F}}$, denoted along the paper by $\mathcal{F}_{nef}$.

2.13. **Rational fibrations and Riccati foliations.** Consider a rational fibration $p : X \to B$, $x \in B$ a regular value of p and a foliation $\mathcal{F}$ on X. Assume that $\mathcal{F}$ may be defined by a holomorphic vector field v around the fiber $F := p^{-1}(x)$.

2.13.1. Let $U \subset B$ be a coordinate patch containing x such that $p|_U$ is isomorphic to $\pi : F \times U \to U$ and v si holomorphic on U. For a coordinate w centered at x on U and an affine chart z on $F \simeq \mathbb{P}^1$, one has

$$(\star) \ v = [a(w) + b(w)z + c(w)z^2]\partial_z + d(w)\partial_w,$$

where a, b, c, d are holomorphic on U. If there exists an $\mathcal{F}$-invariant section Γ of p over U, up to an automorphism of π we can suppose it is $z = 0$, in which case $a = 0$. We see that the vanishing of v on Γ implies that $d = 0$ and that all the fibers of p are $\mathcal{F}$-invariant.

2.13.2. The condition $d \neq 0$ means that $\mathcal{F}$ is everywhere transverse to the general fiber of p. In this case, one says that $\mathcal{F}$ is a *Riccati foliation* with respect to p and the fibration p is called an *adapted fibration* for $\mathcal{F}$. The fibers of p which are not everywhere transverse to $\mathcal{F}$ are called *special fibers*. Untill the end of Section 2.13.3 we suppose $\mathcal{F}$ is a Riccati foliation with respect to p. We also assume v does not vanish on F, which can be arranged up to dividing v suitably.

The $\mathcal{F}$-invariance of the regular fiber F is then characterized by $d(0) = 0$. It is equivalent to the existence of a singularity of $\mathcal{F}$ in the fiber F.

2.13.3. Restricting $\mathcal{F}$ to the complement of the special fibers $(p^{-1}(x_i))_{i=1,\dots,n}$, one obtains a flat $\mathbb{P}^1$-bundle and its monodromy representation $\pi_1(B \setminus \{x_1, \dots, x_n\}) \to \mathrm{PSL}_2(\mathbb{C})$. If the basis of the fibration is $B = \mathbb{P}^1$ and there is only one special fiber F, the monodromy is trivial and any $\mathcal{F}$-invariant local section at F extends to a global invariant section.

The next paragraph explains the local structure of nef models of Riccati foliations.

2.13.4. Let $\overline{\mathcal{F}}$ be a reduced model of a Riccati foliation $\mathcal{F}$. After [Bru03], up to birational morphism (contraction of curves in $\overline{\mathcal{F}}$-invariant fibers), one reaches a nef model of the Riccati foliation for which each fiber has one of the local types $(a), (b), (c), (d), (e)$ described below. The figure is adapted from [Bru03, p. 20].

(a)   (b)   (c)   (d)   (e)
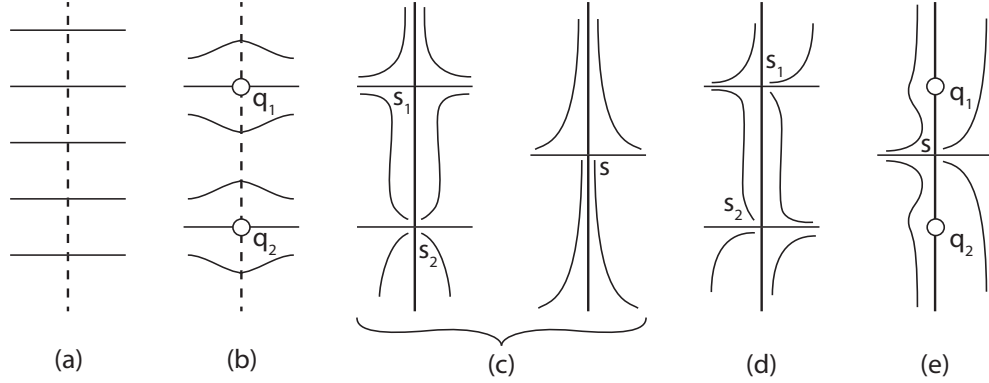
(a) The fiber is regular and transverse to the foliation.
(b) The surface admits two cyclic quotient singularities q_1, q_2 of the same order $k \geq 2$ along the fiber. The fiber has order k and is not invariant by the foliation.
(c) There are two possibilities: the fiber is regular and invariant and either contains two non degenerate singularities s_1, s_2 or a unique saddle-node s with Milnor number $\mu = 2$ whose strong separatrix is transverse to the fiber.
(d) The fiber is regular and contains two saddle-nodes whose strong separatrices are contained in the fiber.
(e) The surface admits two quotient singularities with order equal to 2 along the fiber; there is a saddle node on the fiber whose strong separatrix is in the fiber. The fiber has order 2.

The cotangent line bundle of $\mathcal{F}_{nef}$ is obtained as the pull-back of a $\mathbb{Q}$-divisor in the basis, in particular $\kappa(\mathcal{F})$ is at most 1.

If the basis of the adapted fibration is $B = \mathbb{P}^1$ and there is only one special fiber for $\mathcal{F}_{nef}$, the triviality of the monodromy prevents this fiber from being of type (b) or (c). However, cases (d) and (e) are realized in Examples 5.6 and 5.7, respectively.

2.13.5. A key tool in the birational geometry of Riccati foliations are the *elementary transformations*. These are birational modifications of ruled surfaces consisting of one blowing-up and a successive contraction: after a blow-up at p, the total transform of the fiber of p is the sum of

two (-1)-curves, contraction of the strict transform of the fiber yields a new ruled surface. The self-intersection of any section passing through p decreases of 1 after such a transformation. For other sections the self-intersection increases of one.

3. Proof of Theorem C

In order to prove Theorem C, we will rule out successively $\kappa(\mathcal{F}) = -\infty$ and $\kappa(\mathcal{F}) = 0$.

The birational classification of foliations with $\kappa(\mathcal{F}) = -\infty$ ([Bru03] or [McQ01]) asserts that this class is composed by *rational fibrations* and by foliations birationally equivalent to the so-called *Hilbert modular foliations*. Hence, in order to prove $\kappa(\mathcal{F}) \neq -\infty$, we only need to exclude Hilbert modular foliations.

3.1. **Exclusion of Hilbert modular foliations.** A *Hilbert modular surface* is defined (following [Bru03, p. 25]) as a (possibly singular) projective surface Y containing a (possibly empty) curve $C \subset Y \setminus \mathrm{Sing}(Y)$ such that:

- each connected component of C is a cycle of smooth rational curves, contractible to a normal singularity; if a connected component of C is irreducible, then it is a rational nodal curve.
- $Y \setminus C$ is uniformised by the bidisc $\mathbb{H} \times \mathbb{H}$, i.e. we have an isomorphism of analytic spaces

$$Y \setminus C \simeq Y_\Gamma := (\mathbb{H} \times \mathbb{H})/\Gamma$$

  where Γ is a lattice (discrete finite covolume subgroup) of $\mathrm{PSL}(2,\mathbb{R})^2 \subset \mathrm{Aut}(\mathbb{H} \times \mathbb{H})$;
- Γ is irreducible (i.e. does not contain a finite index subgroup of the form $\Gamma_1 \times \Gamma_2$ with $\Gamma_j \subset \mathrm{PSL}(2,\mathbb{R})$, $j = 1, 2$).

The natural singular foliations of the Hilbert modular surface Y which come from the horizontal and vertical foliations by discs of $\mathbb{H} \times \mathbb{H}$ are called *Hilbert modular foliations*. Both foliations leave invariant the curve C and, in the desingularization of the surface, they leave invariant the exceptional divisors.

Proposition 3.1. *Let X be a smooth projective surface and let D be a reduced divisor on X such that $X \setminus D$ is simply connected. Let $\mathcal{F}$ be a foliation of X such that $\mathcal{F}|_{X \setminus D}$ has no invariant rational curve. Then $\mathcal{F}$ is not birationally equivalent to a Hilbert modular foliation.*

Proof. Suppose by contradiction that $\mathcal{F}$ is birationally equivalent to a Hilbert modular foliation. Let $\overline{\mathcal{F}}$ be a relatively minimal model of $\mathcal{F}$ on a suitable smooth projective surface M. Then M is the minimal desingularization of a Hilbert modular surface Y (*cf.* [Bru00, Thm. 5.1]) and $\overline{\mathcal{F}}$ is the transform of a Hilbert modular foliation on Y.

As described above, one has $Y = Y_\Gamma \sqcup C$, for an irreducible lattice Γ of $\mathrm{PSL}(2,\mathbb{R})^2$ and C a contractible cycle of rational curves in Y.

Let

$$U := X \setminus (D \cup \mathrm{Sing}(\mathcal{F})).$$

Then, as it contains no singularity of $\mathcal{F}$ nor invariant rational curves, U is isomorphic to a non-empty Zariski open subset of M and is also isomorphic to a non-empty Zariski open subset of $\dot{Y}_\Gamma$, the complement of the quotient singularities of Y_Γ. The group homomorphism

$$\pi_1(U) \to \pi_1(\dot{Y}_\Gamma)$$

induced by the injection is surjective. As U is simply connected, so is $\dot{Y}_\Gamma$.

By the irreducibility of Γ, the subset $F \subset \mathbb{H} \times \mathbb{H}$ given by the points that have non trivial stabilizer under Γ is discrete (*cf.* [Shi63]). The action of Γ on $\mathbb{H} \times \mathbb{H}$ can be restricted to an action on $(\mathbb{H} \times \mathbb{H}) \setminus F$. The quotient map

$$(\mathbb{H} \times \mathbb{H}) \setminus F \to [(\mathbb{H} \times \mathbb{H}) \setminus F]/\Gamma = \dot{Y}_\Gamma$$

is then a covering, in the strict sense of topologists. As $(\mathbb{H} \times \mathbb{H}) \setminus F$ is simply connected, this proves that the fundamental group of $\dot{Y}_\Gamma$ is isomorphic to Γ. But $\dot{Y}_\Gamma$ is simply connected, so Γ is trivial, a contradiction. $\square$

This result is sharp if $(X, D) = (\mathbb{P}^2, L_\infty)$: a pair of Hilbert modular foliations with exactly one invariant rational quintic curve in $\mathbb{C}^2$ is given in [MP05]. It gives rise to an irreducible nodal curve invariant by the reduced models of the foliations.

3.2. **Exclusion of vanishing Kodaira dimension.**

We procede by contradiction and suppose we have a foliation $\mathcal{F}$ that satisfies the hypotheses of Theorem C and $\kappa(\mathcal{F}) = 0$. Let $\Sigma : M \to X$ be a reduction of singularities of $\mathcal{F}$ and denote the resulting reduced foliation on M by $\overline{\mathcal{F}}$.

Denote by $q : M \to M'$ a finite (possibly trivial) sequence of blowing-downs of foliated exceptional curves, such that the foliation $\overline{\mathcal{F}}' := q_*\overline{\mathcal{F}}$ is a relatively minimal model of $\mathcal{F}$. By a theorem of McQuillan [Bru00, Thm. 9.2], we have $\nu(\mathcal{F}) = 0$. We can then apply the following, which is actually the simplest case of another theorem of McQuillan [Bru00, Thm. 8.2].

Lemma 3.2. *Let X be a simply connected projective variety endowed with a relatively minimal foliation $\mathcal{G}$ such that $\nu(\mathcal{G}) = 0$. Then $\mathcal{G}$ can be defined by a holomorphic vector field with isolated zeroes.*

Proof. Let $T^*\mathcal{G} =_\mathbb{Q} \mathbf{N} + \mathbf{P}$ be the Zariski decomposition of $T^*\mathcal{G}$. Since X is simply connected, we have $h^1(X, \mathbb{C}) = 0$, in particular, $h^1(X, \mathcal{O}_X) = 0$, by Hodge theory. Under these conditions, numerical equivalence of divisors on X is the same as equality modulo torsion. The numerical triviality of $\mathbf{P}$ then implies $T^*\mathcal{G} =_\mathbb{Q} \mathbf{N}$. The non trivial coefficients of $\mathbf{N}$ are not integers [Bru00, Addendum p. 100] hence, $T^*\mathcal{G}$ being an integral divisor, $\mathbf{N} = 0$ and $T^*\mathcal{G}$ is a torsion line bundle. In particular, if n is the order of torsion, from the cohomology exact sequence induced by

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathcal{O}_X^* \xrightarrow{(\cdot)^n} \mathcal{O}_X^* \to 0$$

on sees $T^*\mathcal{G}$ is in the image of the map $H^1(X, \mathbb{Z}/n\mathbb{Z}) \to H^1(X, \mathcal{O}_X^*)$. The simple connectedness of X yields $T^*\mathcal{G} = 0$ and trivial tangent bundle $T\mathcal{G} = 0$. Therefore $\mathcal{G}$ can be induced by a holomorphic vector field with isolated zeroes. $\square$

The foliation $\overline{\mathcal{F}}'$ of M' is hence induced by a holomorphic vector field $\mathcal{X}$ with isolated zeroes. Let $U \subset M'$ be the isomorphic image of $X \setminus (D \cup \operatorname{Sing}(\mathcal{F}))$ under $q \circ \Sigma^{-1}$. Following the argument of Brunella in [Bru00, pp. $77 - 78$], one sees that with an additional birational map $M' \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ which induces an isomorphism between U and its image V, one transforms the pair $(M', \mathcal{X})$ in $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{X}_1 \oplus \mathcal{X}_2)$, where the vector fields $\mathcal{X}_j$ are holomorphic vector fields on $\mathbb{P}^1$.

More precisely, because of its rigidity, every (-1)-curve in M' must be invariant by the holomorphic vector field $\mathcal{X}$. Consequently, contracting such a curve we obtain a surface on which $\mathcal{X}$ induces a still holomorphic vector field. Performing a maximal sequence of such contractions, one reaches a pair $(S, \mathcal{X}_S)$ where $\mathcal{X}_S$ is a holomorphic vector field on the surface S. By construction S is a minimal rational surface, it is hence $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$. The subset $U \subset M'$ does not contain invariant rational curves so that it is mapped isomorphically on its image V_S in S. If $S = \mathbb{P}^2$, the blowing-up of any singularity of the foliation then transforms $\mathbb{P}^2$ in the first Hirzebruch surface, where the pull-back of $\mathcal{X}_S$ is still holomorphic. The blowing-up is an isomorphism over V_S, since V_S contains no singularity. In this way, we obtain a birational map $\varphi : M' \dashrightarrow \mathbb{F}_n$ which maps U isomorphically to its image and such that $\mathcal{R} = \varphi_* \mathcal{F}$ is generated by a global holomorphic vector field $\mathcal{Y}$ on $\mathbb{F}_n$.

If $n = 0$, we have the announced map. Suppose $n > 0$. The negative section Γ is invariant by the vector field (again, due to its rigidity). We cannot have $\mathcal{Y}_{|\Gamma} = 0$, for in this case the fibers of the rational fibration p on $\mathbb{F}_n$ would all be $\mathcal{R}$-invariant (see 2.13.1) and $\mathcal{F}$ would be a pencil of rational curves. Hence, $\mathcal{R}$ is a Riccati foliation with respect to p. One must have an invariant fiber F for $\mathcal{R}$: otherwise, the absence of monodromy forces $\mathcal{R}$ to be a pencil of rational curves. Hence U is mapped by φ to a simply connected Zariski open subset V_n in

$$C_n := \mathbb{F}_n \setminus (\Gamma \cup F).$$

Under a suitable isomorphism $\mathbb{C}^2 \simeq C_n$, the restriction of the fibration $\mathbb{F}_n \to \mathbb{P}^1$ to C_n corresponds to the first projection of $\mathbb{C}^2$. Any fiber $\tilde{F}$ of $\mathbb{F}_n \to \mathbb{P}^1$ which is distinct from F must intersect V_n; otherwise we would have a surjective map $0 = \pi_1(V_n) \to \pi_1(C_n \setminus \tilde{F}) = \mathbb{Z}$. Hence F is the unique $\mathcal{R}$-invariant fiber for this projection, since $\mathcal{R}$ has no invariant rational curve in V_n.

By Remark $A.1$ (Appendix), we have a sequence of elementary transformations (see 2.13.5) centered at singularities of the foliation that transforms $\mathbb{F}_n$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and preserves the holomorphicity of the vector field. As the centers of the elementary transformations are singularities of the foliation, V_n is mapped isomorphically to its image $V \subset \mathbb{P}^1 \times \mathbb{P}^1$. The resulting vector field on $\mathbb{P}^1 \times \mathbb{P}^1$ is holomorphic and hence has the form $\mathcal{X}_1 \oplus \mathcal{X}_2$, where both terms are holomorphic vector fields on $\mathbb{P}^1$.

The zeroes of $\mathcal{X}_j$ $(j = 1, 2)$ give $\mathcal{R}$-invariant fibers of the j-th projection. In particular, as the simply connected Zariski open subset V cannot embed in the complement of two distinct such fibers, the vector fields $\mathcal{X}_j$ both have a unique zero. In suitable affine charts z_j, they take the form $\mathcal{X}_j = \partial_{z_j}$. This is a contradiction, since the vector field $\partial_{z_1} + \partial_{z_2}$ leaves invariant the rational curves given by the levels of $z_1 - z_2$.

4. Proof of Theorem B-(ii)

Let $\mathcal{F}$ be a foliation that satisfies the hypotheses of Theorem B-(ii) and has Kodaira dimension $\kappa(\mathcal{F}) = 1$. We prove here that $\mathcal{F}$ must be a Ricatti foliation. The other implication of Theorem B-(ii) follows from TheoremB-(i) and the fact that Riccati foliations have Kodaira dimension at most 1.

The foliation $\mathcal{F}_{|\mathbb{C}^2}$ has no invariant algebraic curve, therefore is not birationally conjugate to a fibration. According to the birational classification of foliations [Bru00, Thm. 9.1], $\mathcal{F}$ is either a Riccati foliation or a *turbulent foliation*; recall that the definition of turbulent foliation is obtained from the one of Riccati foliation by replacing "rational fibration" with "elliptic fibration" (see page 3).

Therefore, proving Theorem $B - (ii)$ amounts to excluding the turbulent case.

This will be done in Proposition 4.3, using the notion of a *transversely affine* foliation. Consider a foliation $\mathcal{F}$ on a surface X given by $\omega = 0$ where ω is a rational 1-form on X. We say that $\mathcal{F}$ is *transversely affine* if there exists a *closed* rational 1-form η such that

$$d\omega = \omega \wedge \eta.$$

Remark 4.1. If $\tilde{\omega} = g\,\omega$ is another 1-form defining $\mathcal{F}$, then $\tilde{\eta} := \eta - dg/g$ is closed and satisfies $d\tilde{\omega} = \tilde{\omega} \wedge \tilde{\eta}$, so that this definition is independent of the defining 1-form ω.

The following has already been noticed in [Per03, Prop. 22], we give a slightly different proof.

Proposition 4.2. *Every turbulent foliation is a transversely affine foliation.*

Proof. As transversely affine structures may be transported by birational transformations, stable reduction [Bru00, Prop. 4.6] and [CLNL$^+$07, Thm. 2.21 p. 37] reduce the proof to the case where the foliation $\mathcal{F}$ is transverse to the general fiber of an elliptic fiber bundle $\pi : X \to C$.

Let X^* be the complement of the set of invariant fibers, and $F = \pi^{-1}(b) \subset X^*$ a fiber. Using the foliation to identify nearby fibers, we obtain a multiform submersion $\widetilde{X^*} \to F$ which defines the foliation; it lifts to a submersion $f : \widetilde{X^*} \to \mathbb{C}$ to the universal cover of $F \simeq \mathbb{C}/\Lambda$. By construction the monodromy group of f fixes the lattice Λ and must lie in the group $\mathrm{Aff}(\mathbb{C})$ of affine morphisms. Hence the monodromy of df is linear (contained in $\mathbb{C}^*$). In particular, if v is a rational vector field on X which is not tangent to $\mathcal{F}$, the meromorphic function

$$g = df(v) : \widetilde{X^*} \to \mathbb{C}$$

has the same monodromy as df and

$$\omega := \frac{df}{g}$$

is a well defined meromorphic 1-form on X^*, tangent to $\mathcal{F}$. We have

$$d\omega = -\frac{df \wedge dg}{g^2} = \omega \wedge \eta,$$

with $\eta = -\frac{dg}{g}$ a well defined closed meromorphic one form on X^*.

It remains to show that the pair (ω, η) extends meromorphically in the neighborhood of any $\mathcal{F}$-invariant fiber of π. Let $U \simeq \mathbb{D} \times F$ be such a neighborhood, $\mathbb{D}$ a disc. Let $(z, w) \in \mathbb{D} \times \mathbb{C}$ represent the elements of U, $z = 0$ corresponding to the invariant fiber. We have a local equation of the form

$$dw = \frac{dz}{A(z)},$$

for $\mathcal{F}$, with $A(z)$ holomorphic in $\mathbb{D}$. Let b be a point in $\mathbb{D}$. If the coordinate w is well chosen, in $\mathbb{D}^* \times F$, the submersion f expresses as

$$f(z, w) = w - \int_b^z \frac{ds}{A(s)}$$

and

$$df = dw - \frac{dz}{A(z)}$$

is meromorphic at $z = 0$, and so is g; we have the required extension property. $\qquad\square$

Proposition 4.3. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2 \simeq \mathbb{C}^2 \cup L_\infty$. If $\mathcal{F}$ is a turbulent foliation with $\kappa(\mathcal{F}) = 1$, then it possesses an invariant algebraic curve outside L_∞.*

Proof. By contradiction, suppose $\mathcal{F}$ possesses no invariant algebraic curve in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$.

By Proposition 4.2, $\mathcal{F}$ is transversely affine. The assumption on non existence of invariant algebraic curves allows to use [CP14, Corollary B] to infer that $\mathcal{F}$ is given by the pullback ω of a 1-form

$$\omega_0 = dy + (a(x) + b(x)\, y)\, dx, \quad a, b \in \mathbb{C}[x]$$

under a polynomial map $\mathbb{C}^2 \to \mathbb{C}^2$, which extends as a rational map $H_0 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Denote $\mathcal{G}$ the foliation of $\mathbb{P}^2$ induced by ω_0.

There exist sequences of blowing-ups $\Sigma_X : X \to \mathbb{P}^2$, $\Sigma_Y : Y \to \mathbb{P}^2$ in the source and the target of H_0 such that the following conditions are met.

- The foliations $\overline{\mathcal{F}} := \Sigma_X^* \mathcal{F}$ and $\mathcal{R} := \Sigma_Y^* \mathcal{G}$ have at most reduced singularities.
- There exists an elliptic fibration $f_X : X \to \mathbb{P}^1$ adapted to the turbulent foliation $\overline{\mathcal{F}}$.
- There exists a rational fibration $f_Y : Y \to \mathbb{P}^1$ adapted to the Riccati foliation $\mathcal{R}$.
- The rational map $H : X \dashrightarrow Y$ such that $\Sigma_Y \circ H = H_0 \circ \Sigma_X$ is actually a morphism (*i.e.* holomorphic).

As $\mathcal{F}|_{\mathbb{C}^2}$, $\mathcal{G}|_{\mathbb{C}^2}$ possesses no algebraic invariant curves. Hence, by the already proved item (i) of Theorem B, we must have $\kappa(\mathcal{R}) = \kappa(\mathcal{G}) \geq 1$. As for every Riccati foliation, we have $\kappa(\mathcal{R}) \leq 1$. Consequently

$$\kappa(\mathcal{R}) = 1.$$

By Lemma 4.4 below, f_Y is the Iitaka fibration ([Bru00, p. 107]) of the cotangent divisor $T^*\mathcal{R}$. Similarly f_X is the Iitaka fibration of the cotangent divisor $T^*\overline{\mathcal{F}}$.

From the remark in [Bru00, p. 20] it follows that $T^*\overline{\mathcal{F}} = H^*(T^*\mathcal{R}) \otimes \mathcal{O}_X(D)$ for an effective divisor D on X (see also the proof of [Tou03, Lemme 3.2.8])

Then, Lemma 4.5 below yields that H maps the fibers of f_X in the fibers of f_Y: for general $c \in \mathbb{P}^1$, there exists $r(c) \in \mathbb{P}^1$ such that $H\left(f_X^{-1}(c)\right) \subset f_Y^{-1}(r(c))$.

Consider, for general c, the following restriction of H,

$$H_c : f_X^{-1}(c) \to f_Y^{-1}(r(c)).$$

Denote $R \subset X$ and $B \subset Y$ the ramification and branching curves of H, namely

$$R := \{x \in X; \mathrm{rk}(d_x H) \neq 2\}, \quad B := H(R).$$

The map H_c is étale outside R.

Let $\overline{L}_\infty$ be the strict transform of L_∞ in the sequence of blowing-ups Σ_Y. If B has $\mathcal{R}$-invariant components, they must be contained in $\overline{L}_\infty$ or in the exceptional divisor of Σ_Y, because $\mathcal{G}$ possesses no invariant curve in $\mathbb{C}^2$. Denote B_{inv} the union of these components. Notice that the general fiber of f_Y intersects B_{inv} at most once, because f_Y is induced by the coordinate fibration x on $\mathbb{C}^2$.

We assert (**A**): *for a general c, the curve $f_Y^{-1}(r(c))$ does not intersect $B \setminus B_{inv}$.*

From (**A**), we obtain that, for general c, the map H_c ramifies at most over one point of $f_Y^{-1}(r(c)) \simeq \mathbb{P}^1$, contradicting that $f_X^{-1}(c)$ is elliptic.

We conclude by proving Assertion (**A**). First, remark that any non $\mathcal{R}$-invariant component of B is a curve transverse to the general fiber of f_Y and has a finite number of tangencies with $\mathcal{R}$. Therefore we have, for general c:

(1) For any point $p' \in \left(f_Y^{-1}(r(c)) \setminus B_{inv}\right) \cap B$, B is transverse to both $f_Y^{-1}(r(c))$ and $\mathcal{R}$ at p';
(2) for any point $p \in f_X^{-1}(c) \cap R$, H writes as $(s,t) \mapsto (S,T) = (s^\ell, t)$, with $\ell > 1$, in suitable local coordinates (s,t) and (S,T) centered at p and p' respectively.

Take c such that we have (1) and (2). Suppose we have a point $p' = H(p)$ in $f_Y^{-1}(r(c)) \cap B \setminus B_{inv}$. In the adapted coordinates (S,T) of (2), $S = 0$ is a local equation for B. By (1), $\mathcal{R}$ is transverse to B at p'. Hence we have a local graph $T = \lambda_1 S + o(S), \lambda_1 \in \mathbb{C}$, which is tangent to $\mathcal{R}$. Similarly, the fiber $f_Y^{-1}(r(c))$ passing through p' expresses locally as $T = \lambda_2 S + o(S), \lambda_2 \in \mathbb{C}$. In the neighborhood of p, the pulled-back graphs have equations $t = \lambda_i s^\ell + o(s^\ell)$ and are tangent at $(s,t) = (0,0)$ because $\ell > 1$. One is tangent to $\overline{\mathcal{F}}$ and the other is tangent to $f_X^{-1}(c)$. This implies that p is a tangency point between $\overline{\mathcal{F}}$ and $f_X^{-1}(c)$. As f_X is an adapted fibration for $\overline{\mathcal{F}}$, such an intersection point p' cannot exist for c general enough. $\qquad\square$

For the reader's convenience, we prove two facts that belong to the birational theory of foliations and varieties.

Lemma 4.4. *Let $\overline{\mathcal{F}}$ be a reduced foliation on a projective manifold X, with $\kappa(\overline{\mathcal{F}}) = 1$. Suppose $\overline{\mathcal{F}}$ is a Riccati or a turbulent foliation, with adapted fibration $f : X \to C$. Then f is the Iitaka fibration of $T^*\overline{\mathcal{F}}$.*

Proof. Let F be the general fiber for a fibration f adapted to $\overline{\mathcal{F}}$. Lemma 4.5 shows that the Iitaka fibration associated to F is the fibration f. The proof of [Bru00, Thm. 9.1] shows

$T^*\overline{\mathcal{F}}^{\otimes m} = \mathcal{O}(nF + D)$ for an effective divisor D and suitable integers $m, n > 0$. Lemma 4.5 (with $\mathcal{L} = id_X$) allows to deduce that both divisors F and $T^*\overline{\mathcal{F}}$ have the same Iitaka fibration, yielding the conclusion. $\qquad\square$

In our context, the next lemma should be applied in the case of (foliated) Kodaira dimension 1.

Lemma 4.5. *Let $\mathcal{L} : X_1 \to X_2$ be a morphism between projective manifolds. Let D_1, D_2 be divisors on X_1 and X_2, respectively. Suppose these divisors have equal positive Iitaka dimension. Take $k > 0$ big enough so that $p_i : X_i \dashrightarrow \mathbb{P}\Gamma(X_i, \mathcal{O}(D_i)^{\otimes k})^*$ is the Iitaka fibration of D_i, $i = 1, 2$. Suppose $D_1 = \mathcal{L}^* D_2 + D$ with D effective. Let $r \circ q$ be the Stein factorization of $p_2 \circ \mathcal{L}$. Then q is the Iitaka fibration of D_1.*

Proof. Choosing a nontrivial global section $s \in \Gamma(X_1, \mathcal{O}(D))$ we have an injection

$$\phi_k : \Gamma(X_2, \mathcal{O}(D_2)^{\otimes k}) \to \Gamma(X_1, \mathcal{O}(D_1)^{\otimes k})$$

$$\sigma \mapsto (\mathcal{L}^* \sigma) \otimes s^{\otimes k}$$

and the following diagram commutes, with ϕ_k^* onto.

$$
\begin{array}{ccc}
X_1 & \overset{p_1}{\dashrightarrow} & \mathbb{P}\Gamma(X_1, \mathcal{O}(D_1)^{\otimes k})^* \\
{\scriptstyle \mathcal{L}} \downarrow & & \downarrow {\scriptstyle \phi_k^*} \\
X_2 & \underset{p_2}{\dashrightarrow} & \mathbb{P}\Gamma(X_2, \mathcal{O}(D_2)^{\otimes k})^*
\end{array}
$$

Restricting the maps, with $S = p_1(X_1)$, $T = p_2(X_2)$, we get the following.

$$
\begin{array}{ccc}
X_1 & \overset{p_1}{\dashrightarrow} & S \\
{\scriptstyle \mathcal{L}} \downarrow & & \downarrow \\
X_2 & \underset{p_2}{\dashrightarrow} & T
\end{array}
$$

The map $S \to T$ is onto. As $\dim S = \dim T$, it must be a generically finite map. The uniqueness of the Stein factorization yields $q = g \circ p_1$ for some birational map $g : S \to S'$. This yields the conclusion, because the Iitaka fibration is defined only up to birational transformations in the target. $\qquad\square$

To explain the limits of Proposition 4.3, we present a turbulent foliation with $\kappa(\mathcal{F}) = 1$ having exactly one rational invariant curve in $\mathbb{C}^2$.

Example 4.6. Consider the degree 4 foliation $\mathcal{F}$ on $\mathbb{P}^2$ associated to the 1-form on $\mathbb{C}^2$ given by

$$\omega = d(y^2 + x^3) + (y^2 + x^3) \cdot (3y\,dx - 2x\,dy).$$

The foliation $\mathcal{F}$ leaves invariant the cuspidal rational cubic $C : y^2 + x^3 = 0$ and the line at infinity $L = L_\infty$. From [Lor94, Lemme IV.2] it follows that these are the unique algebraic $\mathcal{F}$-invariant curves. The pencil of cubics $\mathcal{E}$ generated by C and the line at infinity L (taken with multiplicity 3) shall give rise to the adapted elliptic fibration for the turbulent foliation $\overline{\mathcal{F}}$, as obtained from $\mathcal{F}$ after reduction of singularities.

To see this, remark that

$$\omega \wedge d(y^2 + x^3) = 6(y^2 + x^3)^2 \, dx \wedge dy$$

which means that the contact locus of $\mathcal{F}$ and $\mathcal{E}$ is exactly $C \cup L$. A reduced model $\overline{\mathcal{F}}$ and the adapted elliptic fibration are obtained after 9 blowing-ups at $L \cap C$ (and infinitely near points), besides 3 extra blowing-ups at the cuspidal point of C (and infinitely near points). The elliptic fibration has just two singular fibers: one of type II^*, in Kodaira's notation, and one which is a blown up fiber of type II, as shown in the next figure, where E_9 is a section. Except E_9 and E_3, all the components of the exceptional divisor are $\overline{\mathcal{F}}$-invariant.
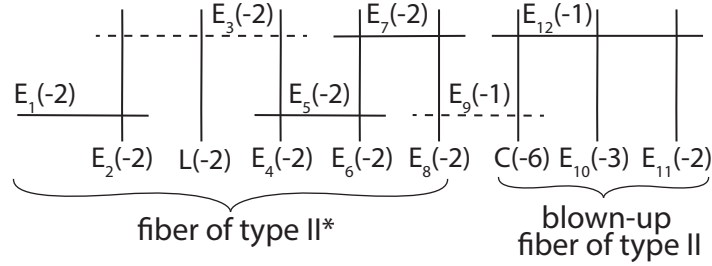
The possibilities for turbulent foliation are $\kappa(\mathcal{F}) = 0, 1, -\infty$. The third option is trivially excluded, because it corresponds to foliations given by rational fibrations. We will show $\kappa(\mathcal{F}) \neq 0$ by contradiction.

Since $\overline{\mathcal{F}}$ is free of foliated exceptional curves, if $\kappa(\mathcal{F}) = 0$ then $T^*\overline{\mathcal{F}} =_{\mathbb{Q}} \mathbf{N}$. The support of the negative part $\mathbf{N}$ is contained in the six $\overline{\mathcal{F}}$-chains, which are given by: i) $\overline{L}$, ii) $\overline{E}_1, \overline{E}_2$, iii) $\overline{E}_4, \overline{E}_5, \overline{E}_6, \overline{E}_7, \overline{E}_8$, iv) $\overline{C}$, v) $\overline{E}_{10}$ and vi) $\overline{E}_{11}$.

Now we assert that there exists a rational fibration whose generic fiber F has tangency with $\overline{\mathcal{F}}$ along the curve $\overline{E}_3$. Based on this, we obtain

$$T^*\overline{\mathcal{F}} \cdot F = tang(\overline{\mathcal{F}}, F) - F \cdot F = tang(\overline{\mathcal{F}}, F) > 0,$$

although F does not intersect any $\overline{\mathcal{F}}$-chain, a contradiction.

To justify the assertion, consider the pencil of rational cuspidal cubics in the projective plane $\mathcal{C}_\lambda : y^2 = \lambda x^3$. There are two base points in the plane at $(1 : 0 : 0)$ and $(0 : 1 : 0)$. Three blowing-ups at $(1 : 0 : 0)$ (and infinitely near points) and three blowing ups at $(0 : 1 : 0)$ (and infinitely near points) are enough to separate the cuspidal cubics of $(\mathcal{C}_\lambda)$ and produce a rational fibration. The third blowing-up at infinity introduces the exceptional line E_3 (figure above). An explicit computation shows that, in local coordinates where $E_3 : (x = 0)$ and the rational fibration is associated to the 1-form dy, the transformed foliation of $\mathcal{F}$ is associated to

$$(6x^4y^4 + 6x^4y^3) \, dx + (2x^5y^3 + 3x^5y^2 + y + 1) \, dy$$

So there is tangency along E_3, as asserted.

5. BIRATIONAL GEOMETRY OF EXAMPLES AND PROOF OF THEOREM B-(iii)

In this section we give examples of foliations $\mathcal{F}$ of $\mathbb{P}^2$ which are associated to simple derivations and whose Kodaira dimension satisfies $\kappa(\mathcal{F}) \in \{1, 2\}$.

In every case we describe the corresponding reduction of singularities and give a nef model. Moreover, we show some birational (non-)equivalences between examples.

We provide diagrams to illustrate the reduction of singularities and nef models. The following conventions are used in the diagrams.

- The affine coordinates $(x, y) \in \mathbb{C}^2$ correspond to $(x : y : 1) = (x_0 : x_1 : x_2) \in \mathbb{P}^2$.
- We denote as (σ_i) the blowing-ups of points composing a reduction of singularities of a given foliation $\mathcal{F}$. The blowing-downs composing the morphism $M \to S$ to a nef model $\mathcal{F}_{nef}$ will be denoted as (q_j), when contracting (-1)-curves, and as (ρ_i) otherwise.
- In the figures, we use nd, sn, m, r, nil for non-$degenerate$, $saddle$-$node$, $Morse$, $radial$ and $nilpotent$ singularities, respectively ($cf.$ Section 2).
- The line at infinity is denoted L and its strict transforms denoted as $\overline{L}$, except in the figures (see the next point).
- In the figures, we use the same symbol for a curve and its strict transforms under extra blowing-ups, but the self-intersection numbers indicated in parentheses (n) will help to avoid confusions.
- The bracket $[m]$ denotes the polar order of the fiber $(z = 0)$ of the Riccati foliation locally defined by $z^m dw + (a(z)w^2 + b(z)w + c(z))\, dz$, with a, b, c holomorphic at $z = 0$.

We start with examples having $\kappa(\mathcal{F}) = 1$. Part of these examples are associated to $Shamsuddin$ $derivations$, see [Sha77]. These are derivations of the form

$$\partial = \partial_x + (a(x)y + b(x))\partial_y, \ a, b \in \mathbb{C}[x].$$

The associated foliation is given by $\omega = dy - (a(x)y + b(x))\, dx$ and is a special Riccati foliation.

$Remark$ 5.1. For a Riccati foliation $\mathcal{F}$, the Kodaira dimension always satisfies $\kappa(\mathcal{F}) \leq 1$. In view of our Theorem B, if such a foliation is associated to a simple derivation of $\mathbb{C}[x, y]$, it must satisfy $\kappa(\mathcal{F}) = 1$. However, for illustrative purposes we present direct computations of the Kodaira dimension for such examples. The fibrations adapted to the examples we propose correspond to pencils of parallel lines in $\mathbb{C}^2$. Notice however that, applying polynomial automorphisms of $\mathbb{C}^2$, we can transform these lines in curves of arbitrary degree, the transformed derivation remaining simple.

$Example$ 5.2. Consider the foliation $\mathcal{F}$ of the projective plane associated to

$$\omega = (xy + 1)\, dx - dy,$$

called Bergman's example in [Cou03]. The extended foliation $\mathcal{F}$ of $\mathbb{P}^2$ has degree 2. The point at infinity $(1 : 0 : 0)$ is a saddle-node with Milnor number $\mu = 3$ whose strong separatrix is the line at infinity $L : x_2 = 0$; in particular L is $\mathcal{F}$-invariant. At $(0 : 1 : 0)$ there is a a quadratic singularity: the blowing-up at this point produces a Riccati foliation on $\mathbb{F}_1$, leaving invariant the exceptional curve E_1. There is just one singular point along E_1, a saddle-node with Milnor number $\mu = 3$, strong separatrix $\overline{L}$ and weak separatrix E_1. This already gives a nef model in for $\mathcal{F}$ in $\mathbb{F}_1$. Taking into account the multiplicity of $\overline{L}$ (thought of) as an invariant fiber, and the formula in [Bru00, p. 48] for the cotangent bundle we deduce

$$T^*\mathcal{F}_{nef} = \mathcal{O}(-2\overline{L}) \otimes \mathcal{O}(3\overline{L}) = \mathcal{O}(\overline{L}) \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$
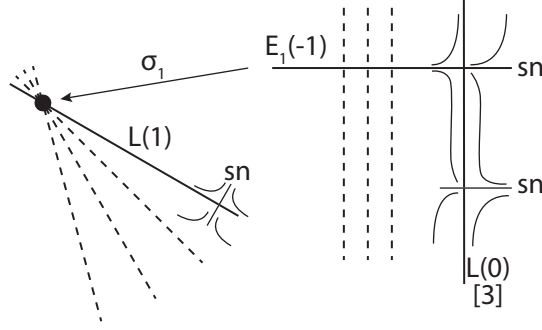
Example 5.3. From [GL12] we consider the foliation of degree 2 in the projective plane associated to

$$\omega = (1 + x\,(2x + y))dy + 2x(2x + y)dx = 0$$

The singularities along L are a saddle-node at $(0 : 1 : 0)$ and a non-reduced (quadratic) singularity at $(-1 : 2 : 0)$. Any affine line $2x + y = c$, $c \in \mathbb{C}$, is completely transverse to the foliation. One blowing-up at $(-1 : 2 : 0)$ is enough to reduce the singularity and produces a Riccati foliation. The exceptional line is invariant and has a saddle-node. $\overline{L}$ is the unique invariant fiber. This is already a nef model. The multiplicity of $\overline{L}$ as an invariant fiber is 3 and we obtain

$$T^*\mathcal{F}_{nef} = \mathcal{O}(-2\overline{L}) \otimes \mathcal{O}(3\overline{L}) = \mathcal{O}(\overline{L}) \quad \text{and} \quad \kappa(\mathcal{F}) = 1$$

Example 5.4. We consider the foliation of degree 3 in the projective plane of Shamsuddin type given by (see [Now94, Exa. 13.3.5])

$$\omega = (yx^2 + xy + x^2)dx - dy = 0.$$

At $(1 : 0 : 0)$ there is a saddle-node with Milnor number $\mu = 4$, whose strong separatrix is L. At the vertical infinity there is a cubic singularity. The blowing-up at this point produces a Riccati foliation relatively to the vertical lines. The exceptional line E_1 is invariant and the unique singularity along E_1 is a saddle-node with Milnor number $\mu = 4$, weak separatrix E_1 and strong separatrix $\overline{L}$. This curve is the unique invariant fiber whose corresponding multiplicity is 4. The foliation on $\mathbb{F}_1$ is already a nef model. The cotangent line bundle can be computed as above,

$$T^*\mathcal{F}_{nef} = \mathcal{O}(-2\overline{L}) \otimes \mathcal{O}(4\overline{L}) = \mathcal{O}(2\overline{L}) \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$

Example 5.5. Consider the Shamsuddin type foliation of degree 4 in the plane associated to

$$\omega = ((x^3 + 1)y + 5x^4 - x^3 - 2x^2 + 4x)dx - dy = 0.$$

At $(1 : -5 : 0)$ there is a saddle-node with Milnor number $\mu = 5$. At $(0 : 1 : 0)$ there is a quartic singular point (algebraic multiplicity = 4). The foliation obtained after blowing up this point is Riccati, having just one singular point along E_1 which is a saddle-node, with Milnor number $\mu = 5$, weak separatrix E_1 and strong separatrix $\overline{L}$.

The foliation on $\mathbb{F}_1$ is already a nef model. The multiplicity of $\overline{L}$ as an invariant fiber is 5 and as before we obtain

$$T^*\mathcal{F}_{nef} = \mathcal{O}(-2\overline{L}) \otimes \mathcal{O}(5\overline{L}) = \mathcal{O}(3\overline{L}) \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$

Example 5.6. From [Now94, Exa. 13.3.7] we have a foliation of degree 8 of Shamsuddin type given by

$$\omega = ((x^3 + 1)y + x^8 + 3x^5 + 1)dx - dy = 0;$$

on $\mathbb{C}^2$ it has neither algebraic invariant curve nor singularities. At $(0:1:0)$ there is a highly degenerate singularity (with algebraic multiplicity $= 8$). The blowing-up at this point produces a Riccati foliation, which is not reduced yet. It needs four additional blowing-ups. From the second blown up point to the fifth, the algebraic multiplicity is $= 2$. Along the fifth exceptional line E_5 there are three singular points: two saddle-nodes with Milnor number $\mu = 5$ and one Morse point. The foliation obtained is reduced but not a relatively minimal model.

To obtain a relatively minimal model we contract $\overline{L}$, E_2, E_3 and E_4, in this order.
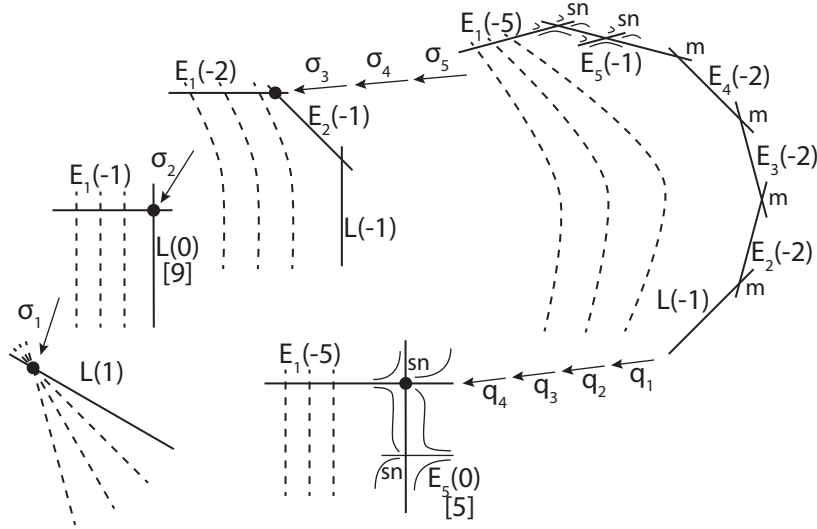
As we contract foliated exceptional curves with Morse points, the strict transform of E_5 contains just two saddle-nodes. In this example the relatively minimal model is already a nef model (on the Hirzebruch surface $\mathbb{F}_5$).

Arguing as in the precedent example we deduce

$$T^*\mathcal{F}_{nef} = \mathcal{O}(-2\overline{E}_5) \otimes \mathcal{O}(5\overline{E}_5) = \mathcal{O}(3\overline{E}_5) \quad \text{and} \quad \kappa(\mathcal{F}) = 1.$$

Example 5.7. This example is not of Shamsuddin type but is of Riccati type. From [MMON01] (see also [Cou08]) we consider the foliation of degree 2 in the projective plane associated to

$$\omega = dx - (x^2 + y)dy = 0.$$

At the horizontal infinity point there is a radial point (Milnor number $\mu = 1$), whose blowing-up produces a Riccati foliation completely transverse to the exceptional line E_1. There is a nilpotent singularity at the vertical infinity point whose Milnor number is $\mu = 6$ (thanks to Darboux's formula in the plane, *cf.* [Bru00, p. 19]). The blowing-up at the nilpotent singular point produces an invariant exceptional curve E_2 having just one quadratic singularity, at the intersection with $\overline{L}$. The blowing-up at this quadratic singularity produces three singularities along E_3: two of them being non-degenerate reduced singularities, placed at the intersections of E_3 with $\overline{L}$ and E_2, and a third one being a saddle-node, with strong separatrix E_3. We assert

that the Milnor number of this saddle-node is $\mu = 4$: indeed, it follows from the diagram on the top of [Bru00, p. 47] and the fact that a nilpotent singular point in $\mathbb{P}^2$ has Milnor number $\mu = 6$.

The foliation obtained is not a nef model for the original Riccati foliation; denote as π the corresponding adapted fibration. The morphism $\rho = \rho_2 \circ \rho_1$ contracts two (-2)-curves and produces a singular surface with two quotient singularities q_1, q_2 along the strict transform of E_3 (where there is also a saddle-node with Milnor number $\mu = 4$). The resulting foliation of this surface is nef.
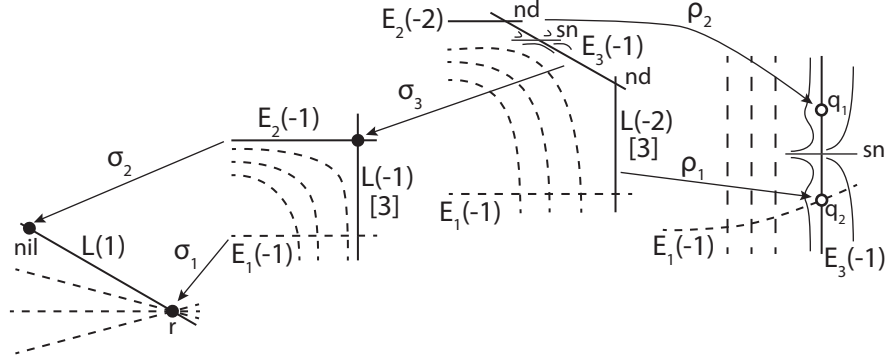
We deduce (see [Bru03, p. 20])

$$deg(\pi_*(T^*\mathcal{F}_{nef})) = -2 + \frac{4+1}{2} = \frac{1}{2} > 0 \quad \text{and} \quad \kappa(\mathcal{F}) = 1,$$

where the leftmost term is the degree of the rational divisor $\pi_*(T^*\mathcal{F}) \in \mathrm{Pic}(\mathbb{P}^1) \otimes \mathbb{Q}$.

All the remaining examples have $\kappa(\mathcal{F}) = 2$.

Example 5.8. Consider the foliation $\mathcal{F}$ of degree $\deg(\mathcal{F}) = 2$ in the projective plane associated to the following equation in $\mathbb{C}^2$:

$$\omega = x\,(1 + xy)\,dx - (1 + xy + x^3)\,dy = 0$$

which was taken from [CDGBM10, Prop. 1.3]. The line at infinity is *not* invariant and there is just one saddle-node with Milnor number $\mu = 7$ at infinity. So the singularity of $\mathcal{F}$ is reduced. The cotangent line bundle is $T^*\mathcal{F} = \mathcal{O}(1)$ and $\kappa(\mathcal{F}) = 2$. As there is no curve with negative self intersection in $\mathbb{P}^2$, $\mathcal{F}$ is its own nef model. The group $Pol(\mathcal{F}|_{\mathbb{C}^2})$ contains the linear automorphism $L_j(x, y) \mapsto (j \cdot x, j^2 \cdot y)$ where j is a primitive cubic root of the unity and $L_j^*(\omega) = j^2 \cdot \omega$. The affine line $x = 0$ is transverse to the foliation and, for $c \neq 0$, the affine line $x = c$ has one movable tangency.

Example 5.9. From [Now08], we obtain a family of foliations $\mathcal{F}_k$ of degree $k \geq 2$ in the plane associated to simple derivations. It is defined by the family of 1-forms

$$\omega_k = (y^k + x)dx - dy, \ k \in \mathbb{N}^*.$$

For $k = 2$ this coincides with Example 5.7, up to permutation of (x, y).

We assert that $\kappa(\mathcal{F}_k) = 2$ for all $k \geq 3$. At the vertical infinity point, each $\mathcal{F}_k$ has a radial point p. The exceptional line of the blowing-up at p belongs to the contact divisor between

the transformed foliation and the rational fibration corresponding to the pencil of lines by p. For simplicity, let's focus on the case $k = 3$. The reduction of singularities of $\mathcal{F}_3$ is made up of 4 blowing-ups at quadratic singularities of the foliation. The fourth blowing-up introduces E_4 having a saddle-node and 2 extra non-degenerated reduced points (at the intersections of E_4 with the strict transforms of E_2 and E_3). The Zariski decomposition of the cotangent line bundle is

$$T^*\overline{\mathcal{F}_3} =_{\mathbb{Q}} \mathbf{P} + \frac{2}{3}\overline{L} + \frac{1}{3}\overline{E_3} + \frac{1}{3}\overline{E_2}.$$

The nef model is obtained after contraction of the support of $\mathbf{N}$ and introduces two quotient singularities of the surface q_1, q_2.
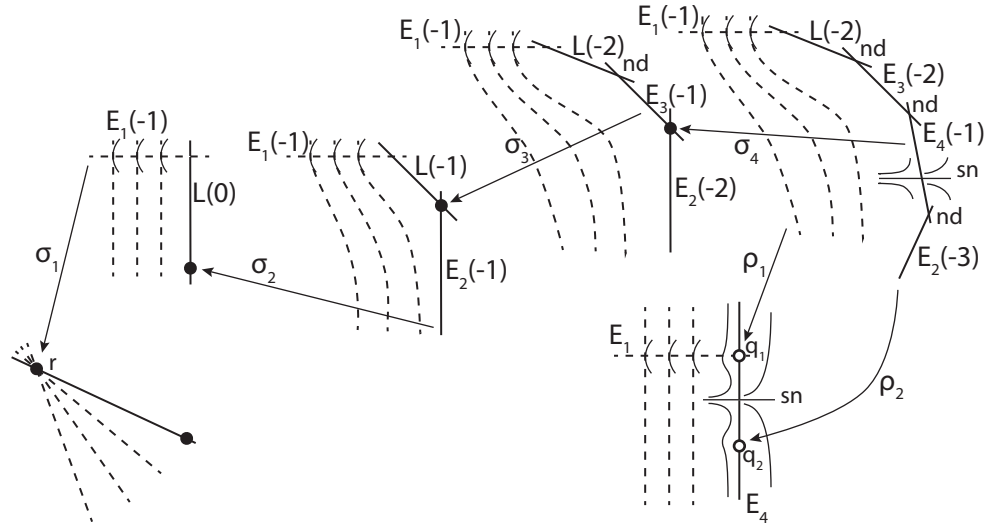
Since the 4 blown up singularities were quadratic $(l(\mathcal{F}) = 2)$, we can compute

$$T^*\overline{\mathcal{F}_3} \cdot T^*\overline{\mathcal{F}_3} = (deg(\mathcal{F}_3) - 1)^2 - \sum_{i=1}^{4}(l_{p_i}(\mathcal{F}_3) - 1)^2 = 4 - 4 = 0$$

Combining this with

$$\mathbf{N} \cdot \mathbf{N} = (\frac{2}{3}\overline{L} + \frac{1}{3}\overline{E_3} + \frac{1}{3}\overline{E_2})^2 = -1$$

and $\mathbf{P} \cdot \mathbf{N} = 0$, the conclusion is $\mathbf{P} \cdot \mathbf{P} = 1$. Therefore the numerical Kodaira dimension is 2 and also $\kappa(\mathcal{F}_3) = 2$.

Example 5.10. From [Jor81] we have a foliation $\mathcal{F}$ of degree 3 in the projective plane associated to

$$\omega = y^3 \, dy - (1 - xy) \, dx.$$

At $(0 : 1 : 0)$, there is a quadratic singularity whose reduction is composed by 3 blow ups. The foliation has $\kappa(\mathcal{F}) = 2$ and its nef model is not much different than the one of Example 5.9. At last, we remark that the affine lines $y = c$, $c \neq 0$, exhibit one movable tangency point with the foliation (at the intersection of $y = c$ and $y = \frac{1}{x}$). The affine line $y = 0$ is completely transverse to the foliation, a property that will be useful in Section 7.

Example 5.11. From [Kou12] we have examples of foliations $\mathcal{F}_{r,s,g}$ in the projective plane with degrees $\deg(\mathcal{F}_{r,s,g}) = s + 1$, defined for $r, s \in \mathbb{N}$ such that $r + 2 \leq s$ and $g \in \mathbb{C}^*$ by

$$\omega_{r,s,g} := (xy^s + g)\, dx - y^r\, dy.$$

We have a singular point at $(0 : 1 : 0)$ with algebraic multiplicity s. At $(0 : 0 : 1)$, there is a singular point with algebraic multiplicity 3. Except for $y = 0$, all horizontal affine lines exhibit one movable contact point with the restricted foliation $\mathcal{F}_{r,s,g}|_{\mathbb{C}^2}$. However, the affine line $y = 0$ is completely transverse to the foliation.

Example 5.12. According to [Oda95], the foliations in the plane associated to the Liénard equations

$$\omega := (f(x) \cdot y + g(x))\, dx + y\, dy = 0, \quad f, g \in \mathbb{C}[x]$$

do *not* have algebraic solutions in $\mathbb{C}^2$ if three conditions are satisfied: *i*) $f, g \neq 0$, *ii*) $\deg(f) \geq \deg(g)$ and *iii*) $\frac{f}{g}$ is not constant. These foliations do not have singularities in $\mathbb{C}^2$ exactly when $g(x) = c \in \mathbb{C}^*$. Therefore, to produce foliations associated to simple derivations it suffices to take a constant g and a non-constant polynomial $f(x)$. In this case, the affine line $y = 0$ is everywhere transverse to the foliation while the horizontal line $y = c$ exhibit $\deg(f)$ tangencies with the foliation. The line at infinity is invariant by the extended foliation.

Now we establish some birational (non)-equivalences among the Examples.

Proposition 5.13.
  (*i*) *Example* 5.3 *is isomorphic by a linear transformation to a foliation of Shamsuddin type.*
  (*ii*) *Examples* 5.5 *and* 5.6 *are equivalent by a polynomial automorphism of degree five.*
(*iii*) *The foliations of Example* 5.2 *and Example* 5.4 *are* not *birationally equivalent.*

Proof.
(*i*). With the linear change of variables $y = u - 2v$, $x = v$, from the equation of Example 5.3 we obtain the Shamsuddin type

$$\eta = 2dv - (1 + vu)du$$

(*ii*).We start with the nef model of Example 5.6 in the Hirzebruch surface $\mathbb{F}_5$. After an elementary transformation we pass to $\mathbb{F}_4$ keeping the saddle-nodes and the multiplicity [5] of the unique invariant fiber. This is shown in the next figure.
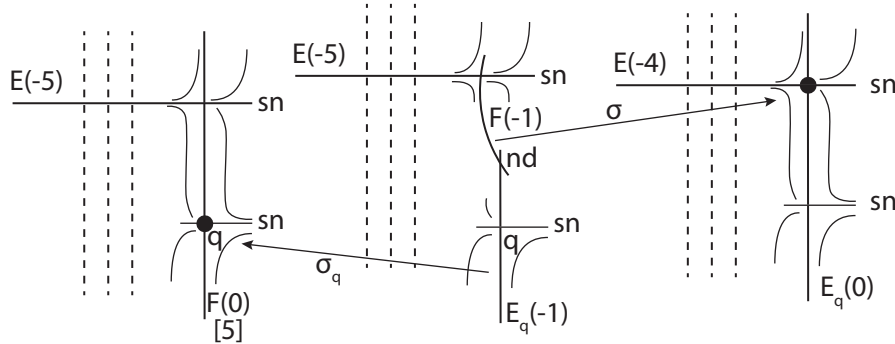
After this we perform three more elementary transformations to obtain a foliation of $\mathbb{F}_1$ and then, after contraction of the (-1)-section, we arrive at the projective plane. The pair of saddle-nodes on the invariant vertical fiber of $\mathbb{F}_5$ is transformed into a pair of singularities along an invariant projective line of a degree 4 foliation. The net effect in the plane can be described concretely by means of a polynomial isomorphism of $\mathbb{C}^2$. Indeed,

$$\eta := ((x^3 + 1)y + 5x^4 - x^3 - 2x^2 + 4x)dx - dy$$

defines the foliation of Example 5.5 and, if $R : \mathbb{C}^2 \to \mathbb{C}^2$ is

$$(x, y) \mapsto (x \,,\, y + x^5 + 2x^2 + 1)$$

then $R^*(\eta)$ is the 1-form of Example 5.6.

(iii). We use the birational invariant $g(\mathcal{F})$ defined in [Men00, p. 139]. In Example 5.2 and Example 5.4 the invariants are $g(\mathcal{F}) = 2$ and $g(\mathcal{F}) = 3$, respectively.   □

Proposition 5.13-(ii) illustrates the general issue of finding the simplest (of least degree) plane birational model for foliations of the plane or derivations. Algorithmic procedures toward this objective would be of great utility.

6. Proof of Theorem A

In order to prove Theorem A we proceed by contradiction: suppose $Bir(\mathcal{F})$ is infinite. From our assumption on algebraic invariant curves, $\mathcal{F}$ cannot have a rational first integral. We have two cases: either

   (1) there exists a birational model $(X, \tilde{\mathcal{F}})$ of $(\mathbb{P}^2, \mathcal{F})$ such that $\mathrm{Aut}(\tilde{\mathcal{F}}) = Bir(\tilde{\mathcal{F}})$ or
   (2) for every birational model $(X, \tilde{\mathcal{F}})$, the inclusion $\mathrm{Aut}(\tilde{\mathcal{F}}) \subset Bir(\tilde{\mathcal{F}})$ is strict.

In case (1), we may apply [CF03, Thm. 1.1] to $(X, \tilde{\mathcal{F}})$, as fibrations are excluded we have one of the following situations.

   (i) There exists a non trivial holomorphic vector field ∂ on $\tilde{X}$ defining a one parameter subgroup of $\mathrm{Aut}(\tilde{\mathcal{F}})$ or
   (ii) the surface X is a generalized Kummer surface (see [CF03, Exa. 1.1]), X is a quotient of an abelian surface A and $\tilde{\mathcal{F}}$ lifts to A as a linear foliation $\mathcal{G}$.

In case (2), we may apply [CF03, Thm. 1.2]. Since fibrations are excluded, we are in the situation of [CF03, Exa. 1.3], in particular up to passing to a birational model $(X, \tilde{\mathcal{F}})$,

   (iii) the surface X is a finite quotient of $S = \mathbb{P}^1 \times \mathbb{P}^1$ and $\tilde{\mathcal{F}}$ lifts to a foliation $\mathcal{G}$ of S given by a differential form $\alpha w dz + \beta z dw$, for some $\alpha, \beta \in \mathbb{C}$.

In cases (ii) and (iii) the foliation $\mathcal{G}$ has Kodaira dimension 0. By the remark in [Bru00, p. 20] or the proof of [Tou03, Lemme 3.2.8], this forces $\kappa(\tilde{\mathcal{F}}) \leq 0$ and contradicts Theorem B.

So we only need to derive a contradiction from situation (i) to complete the proof. Moreover, if ∂ is tangent to $\tilde{\mathcal{F}}$, we may use [Bru00, Prop. 6.6 (iii)] to see that $\tilde{\mathcal{F}}$ is a Riccati foliation with two distinct adapted fibrations and consequently $\kappa(\mathcal{F}) = 0$, contradicting Theorem B.

The conclusion is given by the following argument, proposed by Jorge Pereira. If ∂ is not tangent to $\tilde{\mathcal{F}}$ consider its image $\tilde{\partial}$ in $\mathbb{P}^2$ and take a rational 1-form ω defining $\mathcal{F}$. The form $\eta = \frac{\omega}{\omega(\tilde{\partial})}$ is closed, by the computation [PS02, Proof of Cor. 2] inherited from [CM82]. The poles of η give $\mathcal{F}$-invariant algebraic curves, so that η has no poles in $\mathbb{C}^2$. Subsequently, the first integral $\int_\star^x \eta$ has no monodromy and gives a rational first integral for $\mathcal{F}$, contradiction. $\qquad\square$

7. Polynomial symmetries of foliations associated to simple derivations

Recall that for a foliation $\mathcal{H}$ of $\mathbb{C}^2$, the group $Pol(\mathcal{H})$ is defined as the group of polynomial automorphisms of $\mathbb{C}^2$ that preserve $\mathcal{H}$. If ω is a polynomial 1-form with isolated zeroes that defines $\mathcal{H}$ and ϕ is a polynomial automorphism of $\mathbb{C}^2$, then $\phi \in Pol(\mathcal{H})$ if and only if there exists $c \in \mathbb{C}^*$ such that $\phi^*\omega = c\omega$, see Remark 2.1.

Proposition 7.1. *Given $n \geq 2$ and $B > 0$, there exists a foliation $\mathcal{G}$ associated to a simple derivation and an element in $Pol(\mathcal{G}|_{\mathbb{C}^2})$ of order n and degree greater than B.*

Proof. Let $\mathcal{F}$ be a foliation of the projective plane $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$ with the following two properties:

(a) $\mathcal{F}$ is associated to a simple derivation,
(b) there exists some affine straight line completely transversal to the foliation $\mathcal{F}|_{\mathbb{C}^2}$.

Up to an affine transformation, we can suppose that the line in (b) is $x = 0$. Consider the n to 1 rational map $\phi_n : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\phi_n(x, y) = (x^n, y)$. The foliation $\mathcal{F}$ is defined by a polynomial 1-form $\omega = a(x, y)dx + b(x, y)dy$.

The map ϕ_n extends to a birational map $\widehat{\phi}_n : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and let $\mathcal{G}_0 = \widehat{\phi}_n^* \mathcal{F}$. It is defined by the 1-form $\phi_n^*(\omega) = a(x^n, y)d(x^n) + b(x^n, y)dy$.

Since the affine line $x = 0$ is supposed to be completely transverse to $\mathcal{F}$, then $\mathcal{G}$ has no singularity along $x = 0$. Note also that in $\mathbb{C}^2 \setminus \{x = 0\}$ the map $\phi_n(x, y) = (x^n, y)$ is a local isomorphism, so the pull-back does not introduce any singularity. Any algebraic $\mathcal{G}_0|_{\mathbb{C}^2}$-invariant curve would descend to an $\mathcal{F}|_{\mathbb{C}^2}$-invariant curve, so that no such curve exists. Hence $\mathcal{G}_0$ is associated to a simple derivation.

On the other hand, let ξ be a primitive n-th root of unity. The linear automorphism $T_\xi(x, y) = (\xi \cdot x, y)$ preserves $\mathcal{G}_0$ and has order n. Note that if ∂_n is the derivation dual to the form $\phi_n^*(\omega)$, then $T_\xi \notin \mathrm{Aut}(\partial_n)$ because $T_\xi^*(\phi_n^*(\omega)) = \phi_n^*(\omega)$ and $Jac(T_\xi) = \xi \neq 1$.

Now, for any polynomial $\tau \in \mathbb{C}[x]$, consider the automorphism of $\mathbb{C}^2$ defined by

$$P_\tau(x, y) = (x, y + \tau(x))$$

and set $\mathcal{G} := P_\tau^* \mathcal{G}_0$. If the polynomial $\tau(\xi \cdot x) - \tau(x)$ has degree $d \geq 1$, (e.g. if $\deg_x(\tau) = d$ and $(d, n) = 1$) the map

$$\Gamma_{\xi,\tau}(x, y) := P_\tau^{-1} \circ T_\xi \circ P_\tau(x, y) = (\xi \cdot x,\, y + \tau(\xi \cdot x) - \tau(x))$$

defines a polynomial automorphism of degree d and order n in $Pol(\mathcal{G}_0)$, which completes the proof. $\qquad\square$

The properties $a)$ and $b)$ used in the proof of Proposition 7.1 are verified in all examples of Section 5, so that we have plenty of examples.

For instance, start with Example 5.2 induced by $\omega = (xy + 1)dx - dy$. Assume $n = 2$, so $\phi_2(x, y) = (x^2, y)$. We obtain a foliation $\mathcal{G}_0$ defined by the 1-form

$$\phi_2^*(\omega) = (2x^3 y + 2x)\, dx - dy$$

and $\mathrm{Aut}(\mathcal{G})$ contains the linear automorphism $T(x, y) = (-x, y)$. Now, we consider the automorphism $P_\tau(x, y) = (x, y + x^3)$ and define

$$\Omega := P_\tau^*(\phi_2^*(\omega)) = (2x^3 y - 2x^6 + 3x^2 + 2x)dx - dy.$$

Then the involution $\Gamma_{\xi,\tau} = \Gamma_{-1,x^3} : (x, y) \mapsto (-x, y - 2x^3)$ preserves the 1-form Ω.

At last, note that the foliations that we have constructed in the proof of Proposition 7.1 do not have minimal degree in their birational classes, due to the fact that the automorphism P_τ has degree greater that one. In other terms, such foliations are not *primitive* in the sense of [CD15]. On the other side, the foliation of Example 5.8 is primitive but the exhibited automorphism is linear.

This raises the following question: *Are there non-linear polynomial automorphisms of primitive foliations associated to simple derivations ?*

By [Bru99, Cor. p 293], such automorphisms would necessarily be conjugated to automorphisms of the form $(x, y) \mapsto (ax + P(y), by + c)$ with $a, b \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $P \in \mathbb{C}[y]$.

Appendix A.

This appendix explains and corrects a slight error in the proof of [Bru00, Prop. 6.6], case of rational surfaces.

Suppose $p : \mathbb{F}_n \to \mathbb{P}^1, n > 0$, is the fibration of a Hirzebruch surface and one has a Riccati foliation induced by a global holomorphic vector field v on $\mathbb{F}_n$. One can cover $\mathbb{P}^1$ by two open subsets U_0 and U_∞, both isomorphic to $\mathbb{C}$, in such a way that one has trivializations $p|_{U_i} \simeq U_i \times \mathbb{P}^1 \to U_i$. One can describe $\mathbb{F}_n$ by the gluing of the two products $U_i \times \mathbb{P}^1$ by the rule $(w, z) \sim (t, y) \Leftrightarrow [wt \neq 0$ and $w = 1/t$ and $z = t^n y]$ where z and y are affine charts of $\mathbb{P}^1$ and w and t are coordinates on U_0 and U_∞, respectively. The projection p then corresponds to the first projections of these products and the negative section Γ is the closure of $z = 0$. The holomorphic vector field $v_0 = v|_{p^{-1}(U_0)}$ takes the form $(\star)$ of Section 2.13.1. The section Γ is rigid and hence invariant by v, so that $a = 0$. The transform v_∞ of v_0 in the second chart must have the same form. A straightforward calculation then yields the following necessary and

sufficient condition for the holomorphicity of v_∞ on $U_\infty \times \mathbb{P}^1$, where $\mathbb{C}_k[w]$ are the polynomials of degree $\leq k$.

$$\begin{cases} d \in \mathbb{C}_2[w], d = d_0 + d_1 w + d_2 w^2, \\ b = b_0 - n d_2 w, \\ c \in \mathbb{C}_n[w]. \end{cases}$$

So that v_0 takes the form $v_0 = d(w)\partial_w + z(b_0 - nd_2 w + zc(w))\partial_z$. The restriction $v|_\Gamma$ is holomorphic, hence, choosing well the coordinates w, t on $\mathbb{P}^1$ in the beginning, one may suppose $d(w) = w^2$ or $d(w) = \lambda w$, $\lambda \in \mathbb{C}^*$ or $d = 0$. The option $d = 0$ is excluded, since we have a Riccati foliation with respect to p.

If $b_0 \neq 0$ the foliation has two distinct singularities in the fiber $w = 0$. Otherwise $b_0 = 0$ and $v_0 = d(w)\partial_w + z(-nd_2 w + zc(w))\partial_z$. In these conditions the vector field vanishes on $w = 0$ or $c_0 := c(0) \neq 0$.

If $b_0 = 0$, $c_0 \neq 0$ and $d = \lambda w$ then $d_2 = 0$ and $v_\infty = -\lambda t\partial_t + y(n + yk(t))\partial_y$ where $k(t) = t^n c(1/t)$ has degree n. One sees that v vanishes in two distinct points of the fiber $t = 0$.

If $b_0 = 0$, $c_0 \neq 0$ and $d = w^2$, then v vanishes only at $(w, z) = (0, 0)$ in $\mathbb{F}_n$ and this contradicts Brunella's assertion concerning existence of a zero of v outside Γ in [Bru00, p. 78].

In conclusion, the argument of [Bru00, p. 78] works fine except if the situation $b_0 = 0$, $c_0 \neq 0$ and $d = w^2$ appears in the sequence of elementary transformations. This corresponds to the vector fields of the form $v_0 = w^2 \partial_w + z(-nw + zc(w))\partial_z$.

However, in this case, one can describe an additional branch of the algorithm that allows to reach a holomorphic vector field on $\mathbb{P}^1 \times \mathbb{P}^1$. Perform an elementary transformation centered at the unique singularity of the vector field. From the foliation perspective, this yields a simple pole for the Riccati equation, with trivial monodromy. In these conditions, there is a sequence of elementary transformations centered at singularities of the foliation that eliminates the pole, the pole staying of order at most 1 after each step. On the other hand, the restriction of the vector field to the negative section is unaltered outside $w = 0$, so that its vanishing order at this point remains 2 and that the vector field vanishes on the whole fiber once the Riccati foliation has a simple pole. One concludes that the proposed sequence of elementary transformations preserves the holomorphicity of the vector field. The disappearance of the foliation's singularities shows the obtained bundle is $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, due to Camacho-Sad formula.

Remark A.1. In the above discussion, if $d(w) = \lambda w$, $\lambda \in \mathbb{C}^*$, and the Riccati foliation has a unique invariant fiber, then this fiber corresponds to a simple pole and has trivial monodromy so that it can be eliminated by a sequence of elementary transformations centered at singularities of the foliation. We conclude that in the presence of a unique invariant fiber, one always passes from a "Riccati" holomorphic vector field on $\mathbb{F}_n$ to a holomorphic vector field on $\mathbb{P}^1 \times \mathbb{P}^1$ by a sequence of elementary transformations centered at singularities of the foliation.

REFERENCES

[Băd01]   L. Bădescu, *Algebraic surfaces*, Universitext, Springer-Verlag, New York, 2001, Translated from the 1981 Romanian original by Vladimir Maşek and revised by the author.

[BLL03] P. Brumatti, Y. Lequain, and D. Levcovitz, *Differential simplicity in polynomial rings and algebraic independence of power series*, J. London Math. Soc. (2) **68** (2003), no. 3, 615–630.

[Bru99] M. Brunella, *Minimal models of foliated algebraic surfaces*, Bull. Soc. Math. France **127** (1999), no. 2, 289–305.

[Bru00] ———, *Birational geometry of foliations*, IMPA Monographs, vol. 1, Springer, 2015, Initially published by IMPA, 2000.

[Bru03] ———, *Foliations on complex projective surfaces*, Dynamical systems. Part II, Pubbl. Cent. Ric. Mat. Ennio Giorgi, Scuola Norm. Sup., Pisa, 2003, pp. 49–77.

[CD15] D. Cerveau and J. Déserti, *Action of the Cremona group on foliations on $\mathbb{P}^2_\mathbb{C}$: some curious facts*, Forum Math. **27** (2015), no. 6, 3697–3715.

[CDGBM10] D. Cerveau, J. Déserti, D. Garba Belko, and R. Meziani, *Géométrie classique de certains feuilletages de degré deux*, Bull. Braz. Math. Soc. (N.S.) **41** (2010), no. 2, 161–198.

[CF03] S. Cantat and C. Favre, *Symétries birationnelles des surfaces feuilletées*, J. Reine Angew. Math. **561** (2003), 199–235.

[CLNL$^+$07] D. Cerveau, A. Lins Neto, F. Loray, J. V. Pereira, and F. Touzet, *Complex codimension one singular foliations and Godbillon-Vey sequences*, Mosc. Math. J. **7** (2007), no. 1, 21–54, 166.

[CLNP16] G. Cousin, A. Lins Neto, and J.V. Pereira, *Toward effective liouvillian integration*, arXiv preprint arXiv:1604.05276 (2016).

[CLPT] B. Claudon, F. Loray, J.V. Pereira, and F. Touzet, *Compact leaves of codimension one holomorphic foliations on projective manifolds*, Ann. Sci. Éc. Norm. Supér. (4), To appear.

[CM82] D. Cerveau and J.-F. Mattei, *Formes intégrables holomorphes singulières*, Astérisque, vol. 97, Société Mathématique de France, Paris, 1982, With an English summary.

[Cou03] S. C. Coutinho, *On the differential simplicity of polynomial rings*, J. Algebra **264** (2003), no. 2, 442–468.

[Cou08] ———, *On the classification of simple quadratic derivations over the affine plane*, J. Algebra **319** (2008), no. 10, 4249–4274.

[CP14] G. Cousin and Jorge V. Pereira, *Transversely affine foliations on projective manifolds*, Math. Res. Lett. **21** (2014), no. 5, 985–1014.

[GL12] J. Giné and J. Llibre, *A note on Liouvillian integrability*, J. Math. Anal. Appl. **387** (2012), no. 2, 1044–1049.

[Jor81] D. A. Jordan, *Differentially simple rings with no invertible derivatives*, Quart. J. Math. Oxford Ser. (2) **32** (1981), no. 128, 417–424.

[Jou79] J. P. Jouanolou, *Équations de Pfaff algébriques*, Lecture Notes in Mathematics, vol. 708, Springer, 1979.

[Kou12] S. Kour, *Some simple derivations of $k[x,y]$*, Comm. Algebra **40** (2012), no. 11, 4100–4110.

[Lor94] F. Loray, *Feuilletages holomorphes à holonomie résoluble*, Ph.D. thesis, Rennes 1, 1994.

[LR03] F. Loray and J.C. Rebelo, *Minimal, rigid foliations by curves on $\mathbb{CP}^n$*, J. Eur. Math. Soc. (JEMS) **5** (2003), no. 2, 147–201.

[McQ01] M. McQuillan, *Non-commutative Mori theory*, preprint IHES M/00/15 (2000) (revised:M/01/42 (2001)) (2001), 1–142.

[Men00] L.G. Mendes, *Kodaira dimension of holomorphic singular foliations*, Bol. Soc. Brasil. Mat. (N.S.) **31** (2000), no. 2, 127–143.

[MMON01] A. Maciejewski, J. Moulin-Ollagnier, and A. Nowicki, *Simple quadratic derivations in two variables*, Comm. Algebra **29** (2001), no. 11, 5095–5113.

[MP05] L. G. Mendes and J. V. Pereira, *Hilbert modular foliations on the projective plane*, Comment. Math. Helv. **80** (2005), no. 2, 243–291.

[MP17] L.G. Mendes and I. Pan, *On plane polynomial automorphisms commuting with simple derivations*, J. Pure Appl. Algebra **221** (2017), no. 4, 875–882.

[Now94] A. Nowicki, *Polynomial derivations and their rings of constants*, Uniwersytet Mikołaja Kopernika, Toruń, 1994.

[Now08] ———, *An example of a simple derivation in two variables*, Colloq. Math. **113** (2008), no. 1, 25–31.

[Oda95] K. Odani, *The limit cycle of the van der Pol equation is not algebraic*, J. Differential Equations **115** (1995), no. 1, 146–152.

[Per03]   J.V. Pereira, *Integrabilidade de folheações holomorfas*, Coloóquios Brasileiros de Matemática, vol. 708, IMPA, 2003.

[Poi91]   H. Poincaré, *Sur l'intégration algébrique des équations différentielles*, Comptes Rendus Hebdomadaires de l'Académie des Sciences de Paris **112** (1891), 761–764.

[PS02]   J.V. Pereira and P.F. Sánchez, *Transformation groups of holomorphic foliations*, Comm. Anal. Geom. **10** (2002), no. 5, 1115–1123.

[Sei68]   A. Seidenberg, *Reduction of singularities of the differential equation $A\,dy = B\,dx$*, Amer. J. Math. **90** (1968), 248–269.

[Sha77]   A. Shamsuddin, *Automorphisms and skew polynomial rings*, Ph.D. thesis, University of Leeds, 1977.

[Shi63]   H. Shimizu, *On discontinuous groups operating on the product of the upper half planes*, Ann. of Math. (2) **77** (1963), 33–71.

[Tou03]   F. Touzet, *Sur les feuilletages holomorphes transversalement projectifs*, Ann. Inst. Fourier (Grenoble) **53** (2003), no. 3, 815–846.

GMA-IME-UFF, Campus do Gragoatá, Niterói, RJ, Brazil

E-mail address: gcousin@id.uff.br

Instituto de Matemática, UFRGS, Porto Alegre, Brazil

E-mail address: mendes@mat.ufrgs.br

Centro de Matemática, Facultad de Ciencias, UdelaR, Montevideo, Uruguay

E-mail address: ivan@cmat.edu.uy