ON SOLUTIONS FOR DERIVATIONS OF A NOETHERIAN k-ALGEBRA AND LOCAL SIMPLICITY

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ABSTRACT. We introduce a general notion of solution for a Noetherian differential k-algebra and study its relationship with simplicity, where k is an algebraically closed field; then we analyze conditions under which such solutions may exist and be unique, with special emphasis in the cases of k-algebras of finite type and formal series rings over k. Using that notion we generalize a criterion for simplicity due to Brumatti-Lequain-Levcovitz and give a geometric characterization of that; as an application we give a new proof of a classification theorem for local simplicity due to Hart and obtain a general result for simplicity of formal series rings over k.

1. INTRODUCTION

In the classical theory of complex (or real) ordinary differential equations the Existence and Uniqueness Theorem asserts that if D is an analytic vector field on $\mathbb{C}^n$ and $P \in \mathbb{C}^n$ is not a zero of D, then there exists an analytic map $\gamma : \Delta \to \mathbb{C}^n$, where $\Delta \subset \mathbb{C}$ is an open disk containing 0, such that $\gamma(0) = P = (p_1, \ldots, p_n)$ and $\gamma'(t) = D(\gamma(t)), t \in \Delta$.

Denote by $\mathcal{O}_{n,P}$ the ring of germs of analytic functions at $P \in \mathbb{C}^n$ and think of D as a derivation of that ring; hence $D = \sum_{i=1}^n f_i \partial_{z_i}$, where $f_i \in \mathcal{O}_{n,P}$ and $z_1, \ldots, z_n$ are coordinates for $\mathbb{C}^n$. A solution as above is given by means of n functions $z_1(t), \ldots, z_n(t)$, which are analytic in a neighborhood of 0, and such that

$$z_i'(t) = f_i(z_1(t), \ldots, z_n(t)), z_i(0) = p_i, i = 1, \ldots, n.$$

Since an element in $\mathcal{O}_{n,P}$ may be represented as a power series in $z_1 - p_1, \ldots, z_n - p_n$, with positive convergence radius, we obtain a unique $\mathbb{C}$-homomorphism $\varphi : \mathcal{O}_{n,P} \to \mathcal{O}_{1,0}$ which maps z_i to $z_i(t)$; for an element $g \in \mathcal{O}_{n,P}$ we have $\varphi(g)(0) = 0$ if and only if $g(P) = 0$. Notice that, conversely, such a $\mathbb{C}$-homomorphism that satisfies $\varphi \circ D = \partial_t \circ \varphi$, where $\partial_t : \mathcal{O}_{1,0} \to \mathcal{O}_{1,0}$ is the derivative with respect to t, determines a unique solution. Motivated by this remark one may consider a k-derivation D of an abstract k-algebra and say what a solution of D means (see Definition 1).

In [Sei1967] one finds more or less implicitly, and besides a lot of important results, the essential material to study a more general notion of solution. More explicitly, [Ha1975] and [BLL2003] consider solutions of suitable algebraic differential equations associated to a k-derivation to study simplicity of local differential rings. Furthermore, such a simplicity is characterized in [Ha1974] for an interesting class of local rings. The reader may also consult [No1994, Thm 1.6.1] where there is a version of the Existence and Uniqueness Theorem for formal non autonomic differential equation systems.

In this work, which was inspired by [BLL2003], we introduce and study systematically a general notion of solution associated to a Noetherian differential k-algebra and its relationship with simplicity, for an algebraically closed field k. More precisely, in §2 we analyze conditions under which such solutions may exist and be unique with special emphasis in the cases of k-algebras of finite type and formal series rings over k (Theorem 7). In §3 we first generalize the simplicity criterion given in [BLL2003] and give a geometric characterization of that (Theorem 12 and Corollary 14); next we give a new proof of the classification theorem for local simplicity [Ha1974, Thm. 2] and obtain a general result for simplicity of formal series rings over k (Corollary 16 and Theorem 18).

2. Existence and uniqueness Theorem

Let k be an algebraically closed field of characteristic 0 and let R be a k-algebra. We denote by $\mathrm{Der}_k(R)$ the k-vector space of k-derivations of R. If $R = k[[t]]$ we denote by ∂_t the canonical derivation in $\mathrm{Der}_k(k[[t]])$.

Recall that an element $D \in \mathrm{Der}_k(R)$ extends in a unique form to a k-derivation in the total quotient ring of R by the formula $D(a/s) = D(a)s^{-1} - as^{-2}D(s)$; then it also extends to any localization of R.

Definition 1. *Let $D \in \mathrm{Der}_k(R)$ be a derivation and let $\mathfrak{p} \in \mathrm{Spec}(R)$ be a prime ideal; denote by $k(\mathfrak{p})$ the residue field of $\mathfrak{p}$. A k-homomorphism $\varphi : R \to k(\mathfrak{p})[[t]]$ is said to be a solution of D passing through $\mathfrak{p}$ if $\varphi \circ D = \partial_t \circ \varphi$ and $\varphi^{-1}((t)) = \mathfrak{p}$. When $\varphi(R) \not\subset k(\mathfrak{p})$ we say the solution is nontrivial.*

A solution $\varphi : R \to K[[t]]$ as above, $K = k(\mathfrak{p})$, factorizes through the localization map $R \to R_\mathfrak{p}$ to give a solution $\varphi_\mathfrak{p} : R_\mathfrak{p} \to K[[t]]$ passing through $\mathfrak{p}R_\mathfrak{p}$ (of D thought as a derivation $R_\mathfrak{p} \to R_\mathfrak{p}$).

Remark 2. A solution φ as in the definition above is trivial if and only if it induces a monomorphism $R/\ker\varphi \to k(\mathfrak{p})$; in this case $\mathfrak{p} = \ker\varphi$.

If $D \in \mathrm{Der}_k(R)$, as pointed out in [Sei1967] we may extend D to an element in $\mathrm{Der}_k(R[[t]])$ and define the exponential R-automorphism $e^{tD} : R[[t]] \to R[[t]]$ given by

$$\alpha \mapsto \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n(\alpha),$$

where $D^n = D^{n-1} \circ D$, for $n \geq 0$, and $D^0 = Id$. Notice that e^{tD} restricts to R to give a k-homomorphism $R \to R[[t]]$.

In the sequel we think of D as an element in $\mathrm{Der}_k(R)$ or $\mathrm{Der}_k(R[[t]])$ according to our convenience.

Denote by $\epsilon_{\mathfrak{p}} : R \to k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ the canonical map and let $\epsilon_{\mathfrak{p}} \otimes 1 : R \otimes_k k[[t]] = R[[t]] \to k(\mathfrak{p}) \otimes_k k[[t]] = k(\mathfrak{p})[[t]]$ be its natural extension to power series. If $D \in \mathrm{Der}_k(R)$, then $\epsilon_{\mathfrak{p}} \circ D$ is a $\epsilon_{\mathfrak{p}}$-derivation of R, i.e. $(\epsilon_{\mathfrak{p}} \circ D)(ab) = (\epsilon_{\mathfrak{p}} \circ D)(a)\epsilon_{\mathfrak{p}}(b) + \epsilon_{\mathfrak{p}}(a)(\epsilon_{\mathfrak{p}} \circ D)(b)$ for $a, b \in R$. We have:

Lemma 3. *The map $(\epsilon_{\mathfrak{p}} \otimes 1) \circ e^{tD}|_R : R \to k(\mathfrak{p})[[t]]$ defines a solution of D passing through $\mathfrak{p}$. Moreover, that solution is nontrivial if and only if $\epsilon_{\mathfrak{p}} \circ D \neq 0$.*

Proof. For $\alpha \in R$ we have

$$
\begin{aligned}
((\epsilon_{\mathfrak{p}} \otimes 1) \circ e^{tD} \circ D)(\alpha) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \epsilon_{\mathfrak{p}}(D^{n+1}(\alpha)) \\
&= \partial_t \left(\sum_{n=-1}^{\infty} \frac{t^{n+1}}{(n+1)!} \epsilon_{\mathfrak{p}}(D^{n+1}(\alpha)) \right) \\
&= \left(\partial_t \circ (\epsilon_{\mathfrak{p}} \otimes 1) \circ e^{tD} \right)(\alpha).
\end{aligned}
$$

Clearly $(\epsilon_{\mathfrak{p}} \otimes 1) \circ e^{tD}(\alpha) \in tk(\mathfrak{p})[[t]]$ if and only if $\epsilon_{\mathfrak{p}}(\alpha) = 0$, that is, if and only if $\alpha \in \mathfrak{p}$. Moreover, if for an element $\alpha \in R$ we get $\epsilon_{\mathfrak{p}}(D(\alpha)) \neq 0$, then the linear term in the power series $\sum_{n=0}^{\infty} \frac{t^n}{n!} \epsilon_{\mathfrak{p}}(D^n(\alpha))$ does not vanish; notice that $\epsilon_{\mathfrak{p}} \circ D = 0$ implies $\epsilon_{\mathfrak{p}} \circ D^n = 0$, $\forall n \geq 0$. Putting all together we obtain the proof. $\square$

Remark 4. All derivations admit at least a solution passing through a given prime ideal by Lemma 3. However, as shown in Example 9, for a given derivation all such solutions may be trivial.

If $f : A \to B$ is a homomorphism of commutative rings, then $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ denotes the map $\mathfrak{p} \mapsto \mathfrak{p}^c := f^{-1}(\mathfrak{p})$. It is continuous with respect to the related Zariski topologies. Recall that the correspondences $A \mapsto \mathrm{Spec}(A)$ and $f \mapsto f^*$ induce an equivalence between the categories of k-algebras of finite type and affine varieties over $\mathrm{Spec}(k)$, since k is algebraically closed (Nullstelensatz).

Definition 5. *Two k-homomorphisms $\varphi, \psi : R \to K[[t]]$ are said to be topologically equal if $\varphi^* = \psi^*$.*

Example 6. For $n \geq 1$ define the k-homomorphism $f_n : k[x] \to k[[t]]$ given by $x \mapsto t^n$. Then $f_n = f_m$ implies $n = m$, but $f_n^* = f_m^*$ for all $m, n \geq 1$.

We denote by $k[[x_1, \ldots, x_n]]$ the power series k-algebra in n indeterminates.

Theorem 7. *Assume that R is Noetherian and let $\mathfrak{p} \in \mathrm{Spec}(R)$. Let $D \in \mathrm{Der}_k(R)$ be a derivation. Then:*

a) D admits a solution passing through $\mathfrak{p}$ which is nontrivial if and only if $\epsilon_{\mathfrak{p}} \circ D \neq 0$.

b) Two solutions of D passing through $\mathfrak{p}$ are topologically equal; in particular, if one of these is trivial, then the other one is too.

c) If in addition $\mathfrak{m} = \mathfrak{p}$ is a maximal ideal and R is either of finite type or a quotient algebra of the form $k[[x_1, \ldots, x_n]]/I$, where I is an ideal of power series, then D admits a unique solution passing through $\mathfrak{m}$.

Proof. The statement (a) follows from Lemma 3.

To prove (b) consider a solution $\varphi : R \to k(\mathfrak{p})[[t]]$ whose existence is assured by part (a).

First of all we note that if $\varphi(\alpha) = 0$, then $\varphi(D(\alpha)) = \partial_t(\varphi(\alpha)) = 0$. Hence $D(\ker \varphi) \subset \ker \varphi$. Notice also that $\ker \varphi \subset \mathfrak{p}$.

On the other hand, since R is Noetherian there exists a unique prime ideal $\mathfrak{q} \subset \mathfrak{p}$ which is maximal among the ideals $\mathfrak{a} \subset R$ satisfying $D(\mathfrak{a}) \subset \mathfrak{a}$ (see [Sei1967, §3]). From $D(\mathfrak{q}) \subset \mathfrak{q}$ we deduce $\partial_t(\varphi(\mathfrak{q})) \subset \varphi(\mathfrak{q})$, which is contained in $tK[[t]]$. Since ∂_t does not stabilize nontrivial ideals then $\mathfrak{q} \subset \ker \varphi$. We conclude that $\mathfrak{q}$ is the kernel of any solution of D passing through $\mathfrak{p}$. Since, by definition, any solution contracts $tk[[t]]$ to $\mathfrak{p}$, the statement (b) follows.

We prove (c) in the case $R = k[[x_1, \ldots, x_n]]/I$; the other case is analogous. By Lemma 3 we only need to prove the uniqueness statement.

By assumption D comes from an element $D_1 \in \mathrm{Der}_k(k[[x_1, \ldots, x_n]])$ such that $D_1(I) \subset I$. We have $D_1 = \sum_{i=1}^n f_i \partial/\partial x_i$, for some $f_i \in k[[x_1, \ldots, x_n]]$, $i = 1, \ldots, n$. Let $\varphi : k[[x_1, \ldots, x_n]] \to k[[t]]$ be a solution of D_1 passing through a maximal ideal M with $M/I = \mathfrak{m}$, and put $x_i(t) := \varphi(x_i)$, $i = 1, \ldots, n$. Hence $M = (x_1 - p_1, \ldots, x_n - p_n)$ where $p_i = x_i(0)$, $i = 1, \ldots, n$.

Consider the r truncation k-homomorphism $[\ ]_r : k[[t]] \to k[t]$, $r = 0, 1, 2, \ldots$, which to a power series $\sum_{i=0}^{\infty} a_i t^i$ associates $\sum_{i=0}^{r} a_i t^i$.

Recall that a power series $f \in k[[x_1, \ldots, x_n]]$ admits a Taylor development, around $p = (p_1, \ldots, p_n)$, as

$$f(x) = \sum_{j=0}^{\infty} \lambda^j (x - p),$$

where $x = (x_1, \ldots, x_n), \lambda^0(x-p) = f(p)$ and $\lambda^j \in k[x_1, \ldots, x_n]$ is a suitable homogeneous polynomial of degree j, for $j \geq 1$. Therefore

$$f(x(t)) \equiv \sum_{j=0}^{r} \lambda^j([x_1(t) - p_1]_r, \ldots, [x_n(t) - p_n]_r) \bmod(t^{r+1}). \tag{1}$$

Now, from $\varphi \circ D_1 = \partial_t \circ \varphi$ it follows

$$f_i(x(t)) = \partial_t x_i(t), \ i = 1, \ldots n.$$

By applying (1) to $f_1, \ldots, f_n$ we deduce that the coefficient of the degree r term of $\partial_t x_i(t)$ is determined by a finite number of coefficients of f_i and the coefficients of $[x_1(t) - p_1]_r, \ldots, [x_n(t) - p_n]_r$, for all $i = 1, \ldots, n$. This proves that φ is uniquely determined by the $f_j's$ and p. Since D_1 stabilizes I we easily deduce that φ factorizes through $k[[x_1, \ldots, x_n]]/I$ to give a (unique) solution of D passing through $\mathfrak{m} = M/I$. This completes the proof.

$\square$

Let i be a nonnegative integer number. Following [Ha1974] we say R is i-singular at $\mathfrak{p} \in \operatorname{Spec}(R)$ if $\mathfrak{p}R_{\mathfrak{p}}$ can not be generated by $i + \dim R_{\mathfrak{p}}$ elements.

Corollary 8. *Let $(R, \mathfrak{m})$ be a local ring with $R/\mathfrak{m} = k$ and let $D \in \operatorname{Der_k}(R)$. Suppose that R is either complete or a localization of a k-algebra of finite type. If $\mathfrak{p} \in \operatorname{Spec}(R)$ is a minimal i-singular prime, then there exist only trivial solutions of D passing through $\mathfrak{p}$ if and only if $D(R \setminus \mathfrak{p}) \subset \mathfrak{p}$. In particular, if $\mathfrak{m}$ is a minimal 0-singular prime, then any solution of D passing through $\mathfrak{m}$ is trivial.*

Proof. By [Ha1974, Thm. 1] we know that D stabilizes all minimal i-singular primes, $i \geq 0$. Hence $D(\mathfrak{p}) \subset \mathfrak{p}$. Theorem 7(a) implies there is a solution passing through $\mathfrak{p}$ which is trivial when $D(R) \subset \mathfrak{p}$. The "if and only if" part of the corollary then follows from Theorem 7(b).

For the last assertion we note that R coincides with $k + \mathfrak{m}$ as k-vector space, from which it follows $D(R) \subset \mathfrak{m}$.

$\square$

Example 9. Consider $R = k[x, y, z]$, $\mathfrak{p} = (x, y)$ and the derivation $D = y\partial_x + xz\partial_y$. Hence $R/\mathfrak{p}$ may be identified with $k[z]$ in such a way that $k(\mathfrak{p}) = k(z)$ is the field of rational functions in one indeterminate.

First of all note that the solution of D (passing through $\mathfrak{p}$) given by Lemma 3 is defined by

$$x \mapsto 0, y \mapsto 0, z \mapsto z.$$

More generally, if $f \in k(z)$, a k homomorphism given by

$$x \mapsto 0, y \mapsto 0, z \mapsto f$$

defines a solution of D passing through $\mathfrak{p}$. All these solutions are trivial (to compare with Theorem 7(b)).

On the other hand, a maximal ideal of R is an ideal of the form $\mathfrak{m}_p = (x - a, y - b, z - c)$ for some $p = (a, b, c) \in k^3$. The unique solution of D passing through $\mathfrak{m}_p$ is a k-homomorphism $\varphi_p : R \to k[[t]]$ such that $\varphi_p \circ D = \partial_t \circ \varphi_p$. In other words, we have

$$y(t) = \partial_t x(t), x(t) z(t) = \partial_t y(t), 0 = \partial_t z(t), \tag{2}$$

where $x(t) := \varphi_p(x), y(t) := \varphi_p(y), z(t) := \varphi_p(z)$.

If $\mathfrak{m}_p \supset \mathfrak{p}$, then $p = (0, 0, c)$ and $x(t) = 0, y(t) = 0, z(t) = c$ satisfy (2). Otherwise, $p = (a, b, c)$ with $a \neq 0$ or $b \neq 0$. Since $D^{2m+1}(x) = yz^m$, $D^{2m}(x) = xz^m$, $D^{2m+1}(y) = xz^{m+1}$ and $D^{2m}(x) = yz^m$, we deduce that the unique solution passing through $\mathfrak{m}_p \not\supset \mathfrak{p}$ is given by $z(t) = c$ and

$$x(t) = \sum_{m=0}^{\infty} \left\{ \frac{ac^m}{(2m)!} t^{2m} + \frac{bc^m}{(2m+1)!} t^{2m+1} \right\}, y(t) = \sum_{m=0}^{\infty} \left\{ \frac{bc^m}{(2m)!} t^{2m} + \frac{ac^m}{(2m+1)!} t^{2m+1} \right\}.$$

Example 10. Let R, $\mathfrak{p}$ and $\mathfrak{m}_p$ be as in the precedent example and let $D = \partial_x + \partial_y + \partial_z$. In this case the solutions passing through $\mathfrak{p}$ are of the form

$$x \mapsto t, y \mapsto t, z \mapsto t + f$$

where $f \in k(z)$. Analogously $\varphi_p(x) = a + t, \varphi_p(y) = b + t, \varphi_p(z) = c + t$ is the unique solution passing through $\mathfrak{m}_p$.

3. On simplicity for local Noetherian k-algebras

A (Noetherian) *differential ring* is a pair (R, D), where R is a Noetherian k-algebra and $D \in \mathrm{Der}_k(R)$. Given two differential rings $(R_1, D_1), (R_2, D_2)$ a k-homomorphism $\psi : R_1 \to R_2$ is said to be a *morphism* of differential rings if $D_2 \circ \psi = \psi \circ D_1$; we also write $\psi : (R_1, D_1) \to (R_2, D_2)$. When ψ is an isomorphism we say (R_1, D_1) and (R_2, D_2) are isomorphic.

Notice that a solution $\varphi : R \to k(\mathfrak{p})[[t]]$ of a derivation $D \in \mathrm{Der}_k(R)$, passing through a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$, is a morphism $(R, D) \to (k(\mathfrak{p})[[t]], \partial_t)$, where one specifies the contraction of the maximal ideal. In particular, if $\psi : (R', D') \to (R, D)$ is a morphism, then $\varphi \circ \psi$ is a solution of D' passing through $\psi^{-1}(\mathfrak{p})$.

Definition 11. *A differential ring (R, D) is said to be simple if the unique ideals stable under D are (0) and R.*

Recall that a commutative ring is said to be *reduced* if it has no nontrivial nilpotent elements.

Let A be a commutative ring. For an ideal $\mathfrak{a} \subset A$ we put $V(\mathfrak{a}) := \{\mathfrak{q} \in \mathrm{Spec}(A); \mathfrak{q} \supset \mathfrak{a}\}$: it is the Zariski closed set associated to $\mathfrak{a}$; note that for a prime ideal $\mathfrak{a} = \mathfrak{q}$, $V(\mathfrak{q})$ is the Zariski closure in $\mathrm{Spec}(A)$ of the single set $\{\mathfrak{q}\}$. The dimension $\dim V(\mathfrak{a})$ of $V(\mathfrak{a})$ is $\dim A/\mathfrak{a}$. If $\mathfrak{b} \subset \mathfrak{a}$ one obtains $V(\mathfrak{a}) \subset V(\mathfrak{b})$ and $\dim V(\mathfrak{b}) - \dim V(\mathfrak{a})$ is said to be the *codimension* of $V(\mathfrak{a})$ in $V(\mathfrak{b})$, denoted by $\mathrm{codim}(V(\mathfrak{a}), V(\mathfrak{b}))$.

If in addition A is a K-algebra, where K is a field, we denote by $\mathrm{trdeg}_K A$ the transcendence degree of A over K.

Theorem 12. *Let (R, D) be a differential ring, let $\mathfrak{p} \in \mathrm{Spec}(R)$ be a prime ideal in R. Assume that there exists a nontrivial solution $\varphi : R \to K[[t]]$ of D, $K = k(\mathfrak{p})$, passing through $\mathfrak{p}$. The following assertions are equivalent:*

(a) $(R_\mathfrak{p}, D)$ is simple

(b) $\varphi_\mathfrak{p}$ is one to one.

(c) $R_\mathfrak{p}$ is reduced and the image of $\varphi_\mathfrak{p}^*$ is dense in $\mathrm{Spec}(R_\mathfrak{p})$.

(d) $\ker \varphi$ is the unique minimal prime contained in $\mathfrak{p}$ and there is $u \in R \backslash \mathfrak{p}$ such that $u \ker \varphi = 0$.

If, in addition, R is of finite type, then $(a), (b)$ and (c) are equivalent to

(e) $R_\mathfrak{p}$ is reduced and there is a unique irreducible component X of $\mathrm{Spec}(R)$ containing $V(\mathfrak{p})$ such that $\mathrm{trdeg}_K \varphi(R) = \mathrm{codim}(V(\mathfrak{p}), X)$.
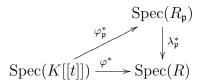
Proof. Since φ is nontrivial, then $\varphi_\mathfrak{p}$ is too and hence $\ker \varphi_\mathfrak{p} \subsetneq R_\mathfrak{p}$. As we noticed in the proof of Theorem 7, $\ker \varphi_\mathfrak{p}$ is the biggest ideal in $R_\mathfrak{p}$ which is stable under D. We deduce that (a) and (b) are equivalent.

On the other hand the image of $\varphi_\mathfrak{p}^*$ is dense in $\mathrm{Spec}(R_\mathfrak{p})$ if and only if all element in $\ker \varphi_\mathfrak{p}$ is nilpotent (see [AM1969, Chap. 1, Exercise. 21]), hence (b) and (c) are equivalent.

Notice that the canonical map $\lambda_\mathfrak{p} : R \to R_\mathfrak{p}$ induces a homeomorphism

$$\mathrm{Spec}(R_\mathfrak{p}) \simeq \{\mathfrak{q} \in \mathrm{Spec}(R); \mathfrak{q} \subset \mathfrak{p}\}, \tag{3}$$

and consider the commutative diagram

$$\begin{array}{ccc}
 & & \mathrm{Spec}(R_{\mathfrak{p}}) \\
 & \overset{\varphi_{\mathfrak{p}}^*}{\nearrow} & \downarrow \lambda_{\mathfrak{p}}^* \\
\mathrm{Spec}(K[[t]]) & \xrightarrow{\ \varphi^*\ } & \mathrm{Spec}(R)
\end{array}$$

Since the images of φ^* and $\varphi_{\mathfrak{p}}^*$ are dense in $V(\ker\varphi)$ and $V(\ker\varphi_{\mathfrak{p}})$, respectively, we deduce that the image of $\varphi_{\mathfrak{p}}^*$ is dense in $\mathrm{Spec}(R_{\mathfrak{p}})$ if and only if $V(\ker\varphi_{\mathfrak{p}}) = \mathrm{Spec}(R_{\mathfrak{p}})$ if and only if the right-hand side in (3) is contained and dense in $V(\ker\varphi)$. This is equivalent to say that $\ker\varphi$ is the unique minimal prime contained in $\mathfrak{p}$. Moreover, since the extension of $\ker\varphi$ in $R_{\mathfrak{p}}$ is $\ker\varphi_{\mathfrak{p}}$ we easily deduce that (c) and (d) are equivalent. This completes the first part of the proof.

Now suppose that R is of finite type. Hence $\dim R/\ker\varphi = \mathrm{trdeg}_k\varphi(R) = \mathrm{trdeg}_K\varphi(R) + \dim R/\mathfrak{p}$. In other words

$$\mathrm{trdeg}_K\varphi(R) = \mathrm{codim}(V(\mathfrak{p}), V(\ker\varphi)). \tag{4}$$

If we assume that assertions (a) to (d) hold, then $X = V(\ker\varphi)$ is the unique component of $\mathrm{Spec}(R)$ containing $V(\mathfrak{p})$. From (4) we know that component has the correct dimension, then (e) holds.

Conversely, assume that (e) holds and notice that $V(\ker\varphi)$ is an irreducible closed set in $\mathrm{Spec}(R)$, which contains $V(\mathfrak{p})$. Then (4) implies $X = V(\ker\varphi)$, from which it follows that $\ker\varphi$ is the unique minimal prime contained in $\mathfrak{p}$. From (3) we deduce, as before, that the image of $\varphi_{\mathfrak{p}}^*$ is dense in $\mathrm{Spec}(R_{\mathfrak{p}})$, i.e. all element in $\ker\varphi_{\mathfrak{p}}$ is nilpotent. Thus $\varphi_{\mathfrak{p}}$ is one to one which completes the proof. $\qquad\square$

Example 13. *Locally nilpotent derivations.* Let (R, D) be a differential ring where D is locally nilpotent, i.e. for each $a \in R$ there exists a positive integer n such that $D^n(a) = 0$. Suppose that R is a (Noetherian) local ring with maximal ideal $\mathfrak{m}$. Let ℓ be the minimum positive integer such that $D^\ell(\mathfrak{m}) \subset \mathfrak{m}$.

Suppose $\ell > 1$ and consider a nontrivial solution of D passing through $\mathfrak{m}$. From Theorem 7 and its proof we know that the kernel of such a solution coincides with the biggest ideal $\mathfrak{q} \subset \mathfrak{m}$ which is stable under D. Using the solution given by Lemma 3 we deduce

$$\mathfrak{q} = \{a \in R; D^i(a) \in \mathfrak{m}, i = 0, \ldots, \ell - 1\}.$$

By using theorem 12 we conclude that (R, D) is simple if and only if the following condition holds:

- $a \in \mathfrak{m}, D(a) \in \mathfrak{m}, \ldots, D^{\ell-1}(a) \in \mathfrak{m}$ imply $a = 0$.

The following corollary generalizes the results in [BLL2003, §1] (see Remark 15).

Corollary 14. *Let R be a k-algebra of finite type without zero divisors and let $\mathfrak{p} \in$ Spec(R). If $D \in \mathrm{Der}_k(R)$ is a derivation such that $D(\mathfrak{p}) \not\subset \mathfrak{p}$, then there is at least a nontrivial solution $\varphi : R \to K[[t]]$ passing through $\mathfrak{p}$ and the following assertions are equivalent:*

(a) $(R_\mathfrak{p}, D)$ is simple.

(b) $\varphi_\mathfrak{p}$ is one to one.

(c) the image of $\varphi_\mathfrak{p}^*$ is dense in $\mathrm{Spec}(R_\mathfrak{p})$.

(d) $\mathrm{trdeg}_K \varphi(R) = \mathrm{codim}(V(\mathfrak{p}), \mathrm{Spec}(R))$.

Proof. Since $D(\mathfrak{p}) \not\subset \mathfrak{p}$ implies $\epsilon_\mathfrak{p} \circ D \neq 0$ the existence of a nontrivial solution passing through $\mathfrak{p}$ is assured by Theorem 7. Taking into account that in the present case R, and then $R_\mathfrak{p}$, have no zero divisors, the corollary follows readily from Theorem 12. $\qquad\square$

Remark 15. If $R = k[x, y_1, \ldots, y_r]$, $\mathfrak{p}$ is the maximal ideal $\mathfrak{m} = (x - \alpha, y_1 - \beta_1, \ldots, y_r - \beta_r)$, $(\alpha, \beta_1, \ldots, \beta_r) \in k^{r+1}$, and $D \in \mathrm{Der}_k(R)$ is a derivation, then $D(\mathfrak{m}) \not\subset \mathfrak{m}$ if and only if $D = g\partial_x + \sum_{i=1}^r f_i \partial_{y_i}$, where $g, f_1, \ldots, f_r \in k[x, y_1, \ldots, y_r]$ are polynomials not all of them belonging to $\mathfrak{m}$. From the unique nontrivial solution for (R, D), passing through $\mathfrak{m}$, we obtain a solution $\varphi_\mathfrak{m}$ of $(R_\mathfrak{m}, D)$. If we put $x(t) := \varphi_\mathfrak{m}(x), y_i(t) := \varphi_\mathfrak{m}(y_i), i = 1, \ldots, r$, then we may read Corollary 14 as saying $(R_\mathfrak{m}, D)$ is simple if and only if $x(t), y_1(t), \ldots, y_r(t)$ are transcendent over k.

The second part of the next result is essentially [Ha1974, Thm. 2].

Corollary 16. *Let S be a Noetherian local k-algebra with maximal ideal $\mathfrak{m}$ and let $D \in \mathrm{Der}_k(S)$. Then (S, D) is simple if and only if there is a one to one solution passing through $\mathfrak{m}$. In particular, (S, D) is isomorphic to a differential ring (S_0, D_0), where S_0 is a k-subalgebra of $K[[t]]$, $K = k(\mathfrak{m})$, which is stable under ∂_t and $D_0 := \partial_t|_{S_0}$.*

Proof. Note that simplicity implies $D(\mathfrak{m}) \not\subset \mathfrak{m}$, from which we know D admits nontrivial solutions (Theorem 7(a)). By Theorem 12 we deduce S is D-simple if and only if there exists a one to one solution, say $\varphi : S \to K[[t]]$. For the rest of the proof we take $S_0 = \varphi(S)$, and the rest of the assertion is essentially trivial. $\qquad\square$

Remark 17. $D(\mathfrak{m}) \not\subset \mathfrak{m}$ corresponds to saying that D maps a non-unite of S into a unite of S. In the case where S is, in addition, a complete ring, the Zariski Lemma ([Zar1965, Lemma 4], or [Sei1967, page 29]) states that $S = B[[x]]$, where B is a subring of S on which D vanishes; this result gives a sort of algebraic version of the so-called Rectification Theorem for ordinary differential equations which says that in a neighborhood of a non-singular point the flow of such an equation is equivalent to a tubular flow. In [Ha1974, Thm. 2], the author proves the second part of Corollary 16 as a consequence of the Zariski Lemma.

For the completion of a k-algebra of finite type, with respect to a maximal ideal, simplicity is quite rare. In fact, suppose $R = k[[x_1, \ldots, x_n]]$ and let $\varphi : R \to k[[t]]$ be the unique solution passing through $\mathfrak{m} = (x_1, \ldots, x_n)$, associated to a derivation D. Corollary 16 implies $\ker \varphi \neq 0$ unless $n = 1$ and $D = f\partial_{x_1}$ for some $f \in k[[x_1]]$ with $f(0) \neq 0$. More generally, we have:

Theorem 18. *Let R be the quotient of $k[[x_1, \ldots, x_n]]$ by an ideal I. If $D \in \mathrm{Der}_k(R)$, then (R, D) is simple if and only if D lifts to a derivation $\widehat{D} \in \mathrm{Der}_k(k[[x_1, \ldots, x_n]])$ which admits a unique solution $\widehat{\varphi} : k[[x_1, \ldots, x_n]] \to k[[t]]$, passing through the maximal ideal $(x_1, \ldots, x_n)$, such that $\ker \widehat{\varphi} = I$.*

Proof. Note that each element in $\mathrm{Der}_k(R)$ comes from an element in $\mathrm{Der}_k(k[[x_1, \ldots, x_n]])$ which stabilizes I and that under this correspondence we obtain compatible solutions. Recalling that two solutions passing through the maximal ideal have the same kernel (Theorem 7b) the result follows from Corollary 16. $\qquad\square$

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