Scalar curvature and Isoperimetric collapse in dimension three.

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Let (M, g) be a compact Riemannian three-manifold with scalar curvature $R_g \geq R_0 > 0$. We prove that, fixed any $\delta > 0$, the δ-volume radius $\nu^\delta(q)$ at any point q is controlled from below by $\|Ric\|_{L^p_g}$ $(p > 3/2)$, R_0 and the volume radius $\nu^\delta(o)$ at one point o. This implies in particular that the $(2, p)$-harmonic radius and the $H^{2,p}_g$-standard Sobolev norm of g in (suitable) harmonic coordinates are controlled all over M. The work originated in the study of the problem of the evolution of the volume radius in General relativity in the Maximal (asymptotically flat) gauge.

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1 Introduction.

The scalar curvature of a Riemannian three-manifold is a quantity entangled with the notions of volume and isoperimetry. The relation is well illustrated in an result due to Kazuo [1] that we explain next. Recall that the Yamabe invariant Y_g of a three-manifold (M,g) is defined as the infimum of the Yamabe functional

$$Y(\bar{g})=\frac{\int_M R_{\bar{g}}dv_{\bar{g}}}{Vol_g(M)^{\frac{1}{3}}},$$

where $\bar{g}$ varies in the conformal class $[g]$ of g. A *Yamabe metric* g_Y is one of volume one in $[g]$ and of constant scalar curvature R_{g_Y} equal to Y_g (thus an absolute minimum of the Yamabe functional).

Lemma 1 (Kazuo) *Let g_Y be a Yamabe metric with scalar curvature $R_{g_Y}>0$. Then, for any metric ball $B(p,r)$ with $r\leq \frac{\sqrt{8}}{\sqrt{R_{g_Y}}}$, we have*

$$Vol(B(p,r))\geq 10^{-\frac{3}{2}}R_{g_Y}^{\frac{3}{2}}r^3 \tag{1}$$

The inequality (1) is directly related to the conformal transformation of the scalar curvature which, if $\bar{g}=\phi^4 g$ gives $\bar{R}\phi^5=-8\Delta\phi+R\phi$, and therefore

$$Y(\bar{g})=\frac{\int_M \bar{R}dv_{\bar{g}}}{Vol_{\bar{g}}(M)^{\frac{1}{3}}}=\frac{\int_M(8|\nabla\phi|^2+R\phi^2)dv_g}{(\int_M\phi^6 dv_g)^{\frac{1}{3}}}. \tag{2}$$

By Aubin's estimate $Y_g\leq 6Vol(S^3_1)^{\frac{2}{3}}$ we see that the inequality (1) is valid for any radius r less than the universal radius $2/(\sqrt{3}(Vol(S^3_1)^{\frac{1}{3}})$. It is worth to stress that the estimate (1) does not make any assumption on the Ricci tensor.

The inequality (1) follows by using (2) above with suitable piece-wise linear functions $\phi(r)$ where $r(q)=dist(q,p)$. The formula (2) and Kazuo's inequality establishes a relation between the Yamabe invariant, the diameter of M (denoted by d in what follows) and the L^2-Sobolev constant of metric balls of radius $d/4$ which, recall, is defined by

$$C_S(\Omega)^{\frac{1}{2}}=\inf_{\phi/Supp(\phi)\subset\Omega}\frac{(\int_\Omega|\nabla\phi|^2dv_g)^{\frac{1}{2}}}{(\int_\Omega\phi^6 dv_g)^{\frac{1}{6}}}.$$

Indeed let g_Y be a Yamabe metric of volume one and positive Yamabe invariant $Y_g=R_{g_Y}$. Given p, there is always q with $dist(q,p)\geq d/2$. Picking that $q(p)$ it follows that the balls $B(q,d/4)$ and $B(p,d/4)$ are disjoint. By inequality (1) we have $Vol_{g_Y}(B(q,\inf\{d/4,\sqrt{8}/\sqrt{R_{g_Y}}\}))\geq c_1(\inf\{d/4,\sqrt{8}/\sqrt{R_{g_Y}}\})^3$ and thus $Vol_{g_Y}(B(p,d/4))\leq 1-c(\inf\{d/4,\sqrt{8}/\sqrt{R_{g_Y}}\})^3$. Therefore for any ϕ with $Supp(\phi)\subset B(p,d/4)$ we have

$$R_{g_Y}(1-(1-c(\inf\{d/4,\sqrt{8}/\sqrt{R_{g_Y}}\})^2))(\int_{B(p,d/4)}\phi^6 dv_g)^{\frac{1}{3}}\leq 8\int_{B(p,d/4)}|\nabla\phi|^2dv_g.$$

Thus *when it is positive, the Yamabe invariant of a Yamabe metric g_Y and its diameter, control the Sobolev constant of metric balls of radius $diam(M)/4$*. In turn, the Sobolev constant constant of a region Ω gives the estimate $Vol_g(B(p,r)) \geq cC_S(\Omega)r^3$, for metric balls $B(p,r) \subset \Omega$, in the same way as Y_g gave the estimate (1). On the other hand the isoperimetric constant $\mathfrak{J}(\Omega)$ of a domain $\Omega \subset M$ (see [2]) which is defined by

$$\mathfrak{J}(\Omega) = \inf_{\bar{\Omega}\subset\Omega} \frac{Area(\partial\bar{\Omega})}{Vol(\bar{\Omega})^{\frac{2}{3}}}.$$

is well known [2] to control the Sobolev constant $C_S(\Omega)$ from below through the natural bound provided by the inequality

$$\frac{\int_\Omega 8|\nabla\phi|^2 dv_{g_y}}{(\int_\Omega \phi^6 dv_{g_y})^{\frac{1}{3}}} \geq \frac{1}{8}\mathfrak{J}(\Omega)^2,$$

valid for any non-negative function ϕ with $Supp(\phi) \subset \Omega$. In turn the isoperimetric constant gives the lower bound $Vol_g(B(p,r)) \geq (\mathfrak{J}(\Omega)/3)^3 r^3$ for any metric ball $B(p,r) \subset \Omega$.

From the discussion above it comes to us that a useful quantity to consider (following what is becoming standard in the literature) and that it will be used all through is the *volume radius*. Let (M,g) be a Riemannian-three manifold. We do not necessarily assume that (M,g) is complete, with or without boundary. Given $\delta > 1$ (fixed) define the *δ-volume radius* at a point $\{p\}$ (to be denoted by $\nu^\delta(p)$), as

$$\nu^\delta(p) = \sup\{r < Rad(p)/\forall B(q,s) \subset B(p,r),\ \frac{1}{\delta^3}w_1 s^3 \leq Vol(B(p,s)) \leq \delta^3 w_1 s^3\},$$

where $Rad(p)$ is the radius at p and is defined as the distance from p to the boundary of the metric completion of (M,g). The boundary of M is the metric completion minus the set M, we will denote it as usual by ∂M. w_1 is the volume of the unit three-sphere. The *volume radius of a region* Ω denoted by $\nu^\delta(\Omega)$ is defined as the infimum of $\nu^\delta(p)$ when p varies in Ω.

Before continuing with the introduction let us introduce some terminology and background material that we will use all through.

Let (M,g) be a non-complete Riemannian manifold, *we say that (M,g) is ν^δ-complete*[2] *iff $\nu^\delta(M) = 0$ and $Rad(p_i)/\nu^\delta(p_i) \rightarrow \infty$ when $Rad(p_i) \rightarrow 0$*. The intuition is that these type of manifolds are complete at the scale of ν^δ, namely that when one scales the metric g by $1/\nu^\delta(p_i)^2$ then the boundary of M_i lies further and further away from p_i as $\nu^\delta(p_i) \rightarrow 0$. Note that in the metric $\tilde{g}_{p_i} = \frac{1}{\nu^\delta(p_i)^2} g$ it is $\nu^\delta_{\tilde{g}_{p_i}}(p_i) = 1$

This family of manifolds comprises in particular those which are the limit of compact Riemannian manifolds with uniformly bounded integral curvature. This fact follows from the Yang isoperimetric inequality (with bounded curvature) [3] (for related results see also [4]). To be more precise, let $\{(M_i, g_i, p_i)\}$ be a sequence of pointed Riemannian three-manifolds with

[2] This terminology is ours.

1. $\nu^{\delta}(p_i) \geq \nu_0 > 0$,
2. The integral L^p_g-curvature
$$\|Ric\|_{L^p_g} = (\int_{M_i} |Ric|^p_g dv_{g_i})^{\frac{1}{p}},$$
with p greater than the critical power $3/2$, is uniformly bounded above by Λ
3. There is a sequence of points $\{q_i\}$ such that $dist_{g_i}(q_i, p_i) \leq d_0$ and $\nu^{\delta}_{g_i}(q_i) \to 0$,

then there is a subsequence of $\{(M_i, g_i)\}$ converging in the weak $H^{2,p}$-topology to a ν^{δ}-complete Riemannian manifold $(\bar{M}, \bar{g}, p)$. The notion of weak convergence of Riemannian is standard in the literature and can be stated as follows. *A pointed sequence (M_i, g_i, p_i) converges in the weak $H^{2,p}$ topology to the Riemannian space $(\bar{M}, \bar{g}, \bar{p})$ iff for every $\Gamma > 0$ there is $i_0(\Gamma)$ such that for any $i > i_0$ there is a diffeomorphism $\varphi_i : B_{\bar{g}}(p, \Gamma) \cap (\bar{M} \setminus B_{\bar{g}}(\partial\bar{M}, 1/\Gamma)) \to B_{g_i}(p_i, \Gamma) \cap (M_i \setminus B_{g_i}(\partial M_i, 1/\Gamma))$ such that $\varphi^* g_i$ converges weekly in $H^{2,p}$ (with respect to the inner product defined by $\bar{g}$) to $\bar{g}$ over the region $B_{\bar{g}}(p, \Gamma) \cap (\bar{M} \setminus B_{\bar{g}}(\partial\bar{M}, 1/\Gamma))$.*

Given a domain Ω on a ν^{δ}-complete Riemannian manifold (M, g), we say that *Ω is δ-collapsed in volume radius or simply δ-collapsed iff it is $\nu^{\delta}(\Omega) = 0$.* If the integral L^p_g-curvature is finite then any ν^{δ}-complete Riemannian manifold is also $\nu^{\delta'}$-complete for any $\delta' > \delta$. As we will be concerned with Riemannian manifolds with uniformly bounded L^p_g-curvature we will work with a fixed δ all through.

More in general, we say that *a sequence of domains $\{\Omega_i\}$ on a sequence of compact manifolds $\{(M_i, g_i)\}$ collapses if for any $\delta > 0$ (a fixed δ during this article) $\nu^{\delta}(\Omega_i) \to 0$.* When the integral L^p_g curvature is bounded above then (tubular neighborhoods of a fixed radius) collapsed sets are also *isoperimetrically collapsed*, namely that their isoperimetric constant is zero.

Finally, we will adopt the standard terminology for *ϵ-thin* and *ϵ-thick regions. We will say that $\Omega \subset M$ is ϵ-thin (thick) if $\nu^{\delta}(\Omega) \leq (>)\epsilon$* (strictly speaking we would need to include the parameter δ in the definition, but as it will be not be of interest to us to vary it, we will forget any mention when talking about thick and thin regions).

These definitions can be seen at work in the Examples 1 and 2 that are discussed later.

Let us continue now with the introduction. The relation between the scalar curvature and the volume radius on metrics of non-negative scalar curvature but non-necessarily Yamabe is, in general, difficult to establish. The situation however changes if one assumes a priori information on the L^p-norm of the Ricci tensor. For instance it is straightforward to prove that *the volume radius and the Yamabe invariant on compact three-manifolds with positive scalar curvature control each other (from below and away from zero) if one assumes an a priori upper bound on the L^p ($p > 3/2$) norm of the Ricci tensor.* The main goal of this article is to prove that an upper bound on the p-integral norm of *Ricci* and a positive lower bound on the scalar curvature strongly influences the volume radius and therefore the Yamabe invariant if one assumes an a priori knowledge of the volume radius at one point. We will prove

Theorem 1 *Let M be a compact three-manifold and suppose that $R \geq R_0 > 0$. Let o be an arbitrary point in M. Then, for any $\delta > 1$ and $p > 3/2$ we have $diam(M) \leq diam(\|Ric\|_{L^p_g}, R_0, \nu^\delta(o))$ and $\nu^\delta(q) \geq \nu(\|Ric\|_{L^p_g}, R_0, \nu^\delta(o))$ for any point q in M.*

The a priori control control on the volume radius at one point is indeed a necessary condition. Consider for instance $M = S^2 \times S^1$ with the metrics $g_l = l^2 d\theta^2 + h_1$ where θ is the angle on the S^1 factor, l is its length and h_1 is the round metric in S^2 of Gaussian curvature equal to one. As $l \to 0$ there is uniform collapse, while the integral (L^p_g) curvature remains uniformly bounded above and the scalar curvature is, for any l, equal to one.

Theorem 1 imply in particular the next compactness Corollary

Corollary 1 *The space of compact Riemannian three-manifolds (M, g) with scalar curvature $R \geq R_0 > 0$, $\|Ric\|_{L^p_g} \leq \Lambda$ and $\nu^\delta(o) \geq \nu_0 > 0$ is precompact in the weak H^2-topology.*

It is well known that the volume radius $\nu^\delta(q)$ at a point q is controlled from below by the volume radius $\nu^\delta(o)$ at some other point o, the distance $dist(o, q)$ and a lower bound on the Ricci curvature (and therefore also by an L^∞_g-bound on Ric). This is a direct consequence of the standard Bishop-Gromov volume comparison. Thus the condition $R \geq R_0 > 0$ is not necessary if in Corollary 1 we replace the upper bound on $\|Ric\|_{L^p_g}$ by an upper bound on $\|Ric\|_{L^\infty_g}$. But, of course it is not true in general that $\nu^\delta(q)$ is controlled from below by $\nu^\delta(o)$, $dist(q, o)$ and $\|Ric\|_{L^p_g}$ without assuming a lower bound on the scalar curvature as the classical examples below (due to D.Yang [3]) show[3].

Example 1 Let $g_\epsilon = dx^2 + (\epsilon + x^2)^4 (d\theta_1^2 + d\theta_2^2)$ be a family of metrics on $[-1, 1] \times S^1 \times S^1$. When $\epsilon \to 0$ the volume radius at the central torus $\{x = 0\}$ collapses to zero, while the volume radius at any point in the torus $\{x = 1\}$ remains uniformly bounded below. The integral curvature $\|Ric\|_{L^2_g}$ remains uniformly bounded above but the scalar curvature diverges to minus infinity at the central torus.

Example 2 Let $g_\epsilon = dr^2 + r^2 d\theta_2^2 + (\epsilon + r^2)^4 d\theta_1^2$ be a family of metrics in the manifold $D^2 \times S^1$ where D^2 is the two-dimensional unit disc. As $\epsilon \to 0$ the volume radius at the central circle $\{r = 0\}$ collapses to zero, while the radius at any point in the torus $\{r = 1\}$ remains uniformly bounded below. The integral curvature $\|Ric\|_{L^2_g}$ remains uniformly bounded above but the scalar curvature at the central circle diverges to negative infinity.

[3] One can interpret that, to many effects, the failure of the volume comparison under integral bounds on the curvature can be well remedied by assuming a positive lower bound on the scalar curvature. This fact can be seen as another reflection of the entanglement between volume and scalar curvature.

There are various properties in these two examples that are characteristic to any process of isoperimetric collapse with bounded integral curvature. We discuss some of them in the next two paraphs.

Looked at the central torus (Example 1) or central circle (Exmaple 2), the family of scaled metrics $(1/\epsilon^2)g_\epsilon$ converge to the flat space $S^1 \times S^1 \times \mathbb{R}$ (with the product metric) or $S^1 \times \mathbb{R}^2$ (with the product metric) respectively. As was explained before, a sequence of pointed spaces (M_i, g_i, p_i) with $\|Ric\|_{L^p_{g_i}}$ $(p > 3/2)$ uniformly bounded and $\nu^\delta(p_i) \to 0$, can be rescaled to obtain a non-trivial complete flat limit, having at the limit point p, at least one length minimizing (geodesic) loop of length one. That the limit is flat follows from the fact that the power p we consider is above the critical scaling $p = 3/2$.

In general, as the two previous examples show, the (very) thin regions (with bounded integral curvature) are enclosed by a set of disjoint two-tori. It is true on the other hand that given a set of embedded solid tori on a compact three-manifold one can always find a sequence of metrics collapsing the interior regions of the tori while keeping the integral L^p_g-curvature uniformly bounded above. Thus there are no topological restrictions for the phenomenon of collapse with bounded integral curvature on any manifold.

These observations will be conceptually important in the proof of Theorem 1.

I the next paragraphs we will describe the main ideas underlying the Proof of Theorem 1 and at the same time we will explain the contents of the different partial results that add up to give its proof.

The idea underlying the proof of Theorem 1 is simple but requires some introduction. We will follow an argument by contradiction which will come out of the next two main *steps*:

1. Understand the topology of the very collapsed regions on manifolds having an a priori (uniformly fixed) upper bound on the L^p_g-Ricci curvature and an a priori (uniformly fixed) positive lower bound on the scalar curvature.

2. Use this knowledge to prove that, on these very collapsed regions, there are stable minimal tubes ($\sim S^1 \times [-1, 1]$), with diameter bounded above, joining small and large geodesic loops. Show that such tubes do not exist using *size relations* of stable minimal surfaces (see below).

These two steps are carried out through several Propositions and Corollaries.

(*Step 1*). In Proposition 3 we prove that the thin region can always be decomposed into regions which are *one collapsed* (where the scaled geometry, as in Example 2, has one collapsed (S^1) fiber), *two collapsed* (where the scaled geometry, as in Example 1, has two collapsed (S^1) fibers), or regions lying somewhat in between of the previous two that we shall call *pseudocusps*. In topological terms, Proposition 3 shows explicitly the well known fact that thin regions have the structure of a *graph manifold*, namely they are the union along two-tori of a finite set of S^1 fibrations over two-surfaces. The topology of thin regions is further explored in the central Lemma 2 where information on its π_1 and π_2 homotopy groups is provided. None of Proposition 3 or Lemma 2 indeed depend on the positive lower bound of the scalar curvature. When this condition is considered a further simplification arises on the topology of thin regions (the simplification is possible too when R is non-negative). Indeed in Proposition 4 we are able to rule out some crucial tori joining the different sectors of the graph manifold (the thin regions). This will allow us to classify, in Corollary 2, the three types of collapsed *ends* of a limit manifold $(\bar{M}, \bar{g}, \bar{o})$ (of a sequence of pointed manifolds $\{(M_i, g_i, o_i)\}$ with $R_{g_i} \geq R_0 > 0$, $\|Ric_{g_i}\|_{L^p_{g_i}} \leq \Lambda$ and $\nu^\delta(o_i) \geq \nu_0 > 0)$).

(*Step 2*). The topological properties of these three types of ends are central to prove that, at each one of them, small and large closed curves (indeed almost geodesic loops) can be joined through area minimizing tubes. Thus the Plateau problem with this kind of boundary is solvable through a tube. The existence of such surfaces is then ruled out using general *size relations*[4] on stable minimal surfaces originally due (in full generality) to Castillon [6]. Let us see this argument in the Examples 1 and 2. In Example 1 the limit manifold (as $\epsilon \to 0$) is $\bar{M} = T^2 \times [-1, 0) \cup T^2 \times (0, 1]$ and the limit metric is $\bar{g} = dx^2 + x^8(d\theta_1^2 + d\theta_2^2)$. In this case the stable minimal tube $[\bar{x}, 1] \times \{\bar{\theta}_1\} \times S^1$, where $\bar{x}$ is positive and close to zero and $\bar{\theta}_1$ is a fixed angle (in the first factor S^1 of T^2) joins a small closed curve (almost a geodesic loop) with a large closed curve. Similarly in Example 2 the limit manifold is $\bar{M} = (D^2 - \{(0,0)\}) \times S^1$ and the limit metric is $\bar{g} = dr^2 + r^2 d\theta_1^2 + r^8 d\theta_2^2$. The stable minimal tube $[\bar{r}, 1] \times \{\bar{\theta}_1\} \times S^1$, where $\bar{r}$ is positive and close to zero and $\bar{\theta}_1$ is a fixed polar angle in the two-disc D^2 also joins a small and a large closed curve (almost geodesic loops). For what was said in the paragraph after the Examples 1 and 2 it is clear that when the limit metrics are scaled by one over the length squared of the small boundary component of the tubes then they look more and more like the flat tube $[0, \infty) \times S^1$ (with the flat product metric). The key point is that *size relations* rule out stable surfaces with these characteristics. Let us explain this point in more detail. We will use the following argument (and won't be repeated) in the Proof of Theorem 1. Will be refer it there as the property of *non-collapse at a finite distance* for stable minimal surfaces.

Consider a Riemannian surface $(\mathcal{S}, h)$ diffeomorphic to the tube $[-1, 1] \times S^1$. Suppose that the stability inequality

$$\int_{\mathcal{S}} |\nabla f|^2 + \kappa f^2 dA \geq 0, \tag{3}$$

holds, where f, in H^1_g, vanishes on the boundary of $\mathcal{S}$. κ is the Gaussian curvature of $\mathcal{S}$. Such inequality arises for instance if $\mathcal{S}$ is a stable minimal tube inside a three-manifold with non-negative scalar curvature. Consider now a smooth loop $\mathcal{L}$ embedded in $\mathcal{S}$ and isotopic to any one of the two loops $\mathcal{L}_1$ and $\mathcal{L}_2$ that form the boundary of $\mathcal{S}$. Let L_1 and L_2 be the distances from $\mathcal{L}$ to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively and let A_1 be the area of the set of points at a distance less or equal than L_1 form $\mathcal{L}$

[4]The terminology *Size Relations* is due to us [5].

in the component that contains $\mathcal{L}_1$ (and similarly for A_2). Finally let $l = length(\mathcal{L})$, $l_1 = length(\mathcal{L}_1)$ and $l_2 = length(\mathcal{L}_2)$. It follows from [6] (see also [5] for related results in the context of this article), using the stability inequality (3), that the following size relation holds

$$2l(\frac{1}{L_1} + \frac{1}{L_2}) \geq \frac{A_1}{L_1^2} + \frac{A_2}{L_2^2}. \tag{4}$$

Note that the expression is scale invariant. A direct consequence of this geometric relation is: *there are no sequences $\{\mathcal{S}_i\}$ of stable minimal tubes satisfying*

1. $Area(\mathcal{S}_i) \geq A_0 > 0$,
2. $diam(\mathcal{S}_i) \leq L_0$,
3. $l_{1,i} \to 0$,
4. *The pointed scaled spaces* $(\mathcal{S}_i, \frac{1}{l_{1,i}^2} h_i, p_i)$, *where* $p_i \in \mathcal{L}_{1,i}$, *converge to the flat tube* $[0, \infty) \times S^1$.

Note that the stable tubes that were considered for the Examples 1 and 2 satisfy (if we let $\bar{x}$ and $\bar{r}$ go to zero) precisely the properties above. To see the claim take any increasing and diverging sequence $\{d_i\}$ and consider the loop $\bar{\mathcal{L}}_i$ at a distance equal to d_i from $\mathcal{L}_{1,i}$ in the metric $\frac{1}{l_{1,i}^2} h_i$. Suppose too that $\{d_i\}$ was chosen in such a way when h_i is scaled as in *item 4* the region enclosed by $\mathcal{L}_{1,i}$ and $\bar{\mathcal{L}}_i$ is closer and closer (globally) to the tube $[0, d_i] \times S^1$. Now if we chose as $\mathcal{L}_i$ the loop at a distance $d_i/2$ from $\mathcal{L}_{1,i}$, it follows from the scale invariance of the expression 4 that $l_i(1/L_{1,i} + 1/L_{2,i}) \to 0$. But the right hand side of (4) is bounded below by a positive number which is a contradiction.

In Propositions 7 and 8 we prove that the area minimizing (stable) tubes considered at each one of the limit ends, would look (when the metric is scaled by one of over the length squared of the smallest) as the flat tube $[0, \infty) \times S^1$, if the smallest end of the tube is chosen sufficiently small. In this way one can construct a sequence of stable loops satisfying *item 4* in the list above. If one consider this sequence then the condition in *item 1* is not difficult to have if the largest loop is set fixed or set to vary in a fixed boundary. The main point where the condition $R \geq R_0 > 0$ is used is to guarantee that the the diameter of the tubes is bounded above and therefore that the condition in *item 3* holds. In fact the diameter can be estimated from above by a numeric constant times $1/\sqrt{R_0}$. This well known estimate has its roots in the work of Fischer-Colbrie [7] and a clear proof can be found in [8].

Once these two propositions are proved we have all what is needed to carry up the two main steps. The proof by contradiction of Theorem 1 is then closed up.

The present article originated in an effort to study the problem of the evolution of the volume radius in the *Constant Mean Curvature* or *Maximal* gauges in General Relativity. Recall that if $\Sigma \subset \mathbf{M}$ is a space-like hypersurface on an Einstein ($\mathbf{Ric} =$

0) Lorentzian four-manifold $(\mathbf{M},\mathbf{g})$ then the energy constraint gives $R_g = |K|_g^2 - k^2$ where R_g is the scalar curvature of the three-metric g in Σ and K its second fundamental form ($k = tr_g K$). Thus as $R \geq R_0 = -k^2$, we have that in the CMC gauge the scalar curvature is a priori bounded below by $-k^2$, while in the Maximal gauge ($k = 0$) it is bounded below by zero. Theorem 1 does not apply directly to any of these two scenarios, however the following conjecture, if valid, would extend the conclusions of Theorem 1 to the cases where R_0 is zero or negative and therefore applicable to General Relativity [5].

Conjecture 1 *Let (M, g) be a compact*[5] *Riemannian three-manifold with $R \geq R_0$ and let p be a number greater than $3/2$. Then*

1. *If $R_0 > 0$ then $\nu^\delta(q)$ is controlled from below by $\|Ric\|_{L^p_g}$ and $\nu^\delta(p)$. Therefore $Vol_g(M)$ is controlled from below and above by them too.*
2. *If $R_0 = 0$ then $\nu^\delta(q)$ is controlled from below by $\|Ric\|_{L^p_g}$, $\nu^\delta(p)$ and $Vol_g(M)$.*
3. *If $R_0 < 0$ then $\nu^\delta(q)$ is controlled from below by $\|Ric\|_{L^p_g}$, $\nu^\delta(p)$, $Vol_g(M)$ and $dist(q, p)$.*

The Einstein CMC or Maximal flow is one example of a geometric flow where one would like to have an a priori relation between the integral curvature and the volume radius. The wish is in essence due to the hyperbolic character of the Einstein equations and thus the necessity of using integral norms to control the flow. The quest of the a priori relation is also present in some elliptic problems, typically when one is interested in minimizing an (integral) action of the curvature.

From an analytic viewpoint Theorem 1 claims that (say $p = 2$) $R_0 (> 0)$, $\|Ric\|_{L^2_g}$ and $\nu^\delta(o)$ control the usual H^2-Sobolev norm of the metric g in harmonic coordinates, whose size, or harmonic radius, is controlled by them too. This is a direct consequence of the well know (local) fact [9] that at any point p in a Riemannian manifold (M, g) the intrinsic quantities $\nu^\delta(p)$ and $\|Ric\|_{L^2_g(B_g(p,\nu^\delta(p)))}$ control the H^2-*harmonic radius* at p, where the harmonic radius is defined as the supremum of the radius $r > 0$ for which there is a harmonic coordinate chart $\{x\}$ covering $B_g(p, r)$ and satisfying

$$\frac{3}{4}\delta_{jk} \leq g_{jk} \leq \frac{4}{3}\delta_{jk}, \tag{5}$$

$$r\Big(\sum_{|I|=2,j,k} \int_{B(p,r)} |\frac{\partial^I}{\partial x^I} g_{jk}|^2 dv_x\Big) \leq 1, \tag{6}$$

where in the sum above I is the multindex $I = (\alpha_1, \alpha_2, \alpha_3)$, and as usual $\partial^I/\partial x^I = (\partial_{x^1})^{\alpha_1}(\partial_{x^2})^{\alpha_2}(\partial_{x^3})^{\alpha_3}$. Both expressions above are invariant under the simultaneous scaling $\tilde{g} = \lambda^2 g$, $\tilde{x}^\mu = \lambda x^\mu$ and $\tilde{r} = \lambda r$. Any harmonic chart $\{x\}$ satisfying the conditions 5 and 6 will be called a *canonical harmonic chart*.

[5] It is not difficult to adapt *item 2* below to the Asymptotically Flat setting.

Thus Theorem 1 claims that the usual coordinate-dependent H^2-Sobolev norms are controlled all over M by the intrinsic quantities R_0, $\|Ric\|_{L^2_g}$ and $\nu^\delta(o)$.

Finally it worth to remark that the crucial upper estimate on the diameter of the stable tubes has been used (for discs instead of tubes) by Schoen and Yau in the context of General Relativity in [10].

Acknowledgments: I would like to thank Lars Andersson and Jan Metzger, for their constant interest and support on the present work.

1.1 Some terminology.

Because we will be dealing with metrics which are scaled from a single metric, it will be important to us to be manifestly clear to which scaling or metric we are referring to. For this reason we are forced to include very often a sub-index indicating the metric from which geometric quantities are calculated. For instance metrics balls with center in a set C and radius r with respect to a metric g will be denoted often by $B_g(C, r)$. Similarly, the distance between two sets (typically points) C and C' in the metric g will be denoted by $dist_g(C, C')$. The L^p_g-norm of a tensor T over a domain Ω is defined and denoted by

$$\|T\|_{L^p_g(\Omega)} = (\int_\Omega |T|^p dv_g)^{\frac{1}{p}}.$$

2 Proof of Theorem 1.

2.1 Flat three-manifolds.

If a compact three-manifold M is flat then it is finitely covered by a flat torus, moreover there are only ten diffeomorphism-types. If instead the manifold M is complete but non-compact then it has a soul, that means a compact, flat and totally geodesic submanifold S such that M is isometric to its normal bundle. If the soul S is a torus (and M is orientable) then M is the metric product of a flat torus (the soul) and $\mathbb{R}$. If the soul is $S = S^1$ (and say of length one) then M is isometric to a flat manifold $F = \mathbb{R}^3/\mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}^3$ by a translation of magnitude one along the z-axis composed with a rotation of angle α (between $-\pi$ and π along the (x, y)-plane. This last category of flat three-manifolds is indeed rich in complexity. We will denote such spaces by $F(\alpha, 1)$ and if the translation along the z-axis is by an amount h then by $F(\alpha, h)$.

Example 3 Consider for instance a flat manifold which is obtained in this way with $\alpha = 2\pi/p$, p a natural number and $h = 1/p$. In this way the soul (the fiber at the origin) has length one over p and the torus at a distance of $p/(2\pi)$ from it gets closer and closer as p goes to infinity to the torus $S^1 \times S^1$ with the factors S^1 orthogonal and of length one. This *pseudo-cusp* (that in this example is the region at a distance $p/(2\pi)$ from the soul) is an interesting example showing that there are flat three-manifolds with some regions very collapsed and some other not at all. When p goes to infinity then the pseudo-cusp approaches the metric product $S^1 \times S^1 \times [0, \infty)$ and its volume (counted from the soul to the torus at distance $d = p/(2\pi)$) increases therefore to infinity. Pseudo-cusps (see later for a precise definition) will play an important role in the elucidation of the global topology of collapsed regions on three-manifolds with bounded integral curvature.

When it comes to deal with $F(\alpha, h)$ ($\alpha > 0$) spaces it will be important to us to understand the relation between the length of length minimizing geodesic loops based at a point p and the distance from p to the soul of F. We present now a first proposition that gives some information on this relation.

Proposition 1 *Consider the set of $F(\alpha, 1)$ spaces. Then for any $d > 0$ there is a natural $n(d)$ and numbers $0 \leq n_1 \leq, \ldots, \leq n_m \leq n(d)$ such that for any α between $-\pi$ and π there is $n(\alpha)$ (a number in the list before) for which the torus lying at a distance d from the soul in the metric $\frac{1}{n(\alpha)^2} g$ has a closed loop of length between one and two. Any other closed loop has length greater or equal than $1/n(\alpha)$.*

Proof:

We identify angles between $-\pi$ and π in the unit circle with the points in it. For any α in $[-\pi, \pi]$ find the least n such that the rotation $\mathcal{R}_{n\alpha}$ of the point $(x, y) = (1, 0)$ lies in the interval $(-1/d, 1/d)$ and choose and interval $(\alpha - \phi(\alpha), \alpha + \phi(\alpha))$ such that the rotation $\mathcal{R}_{n\beta}$ of $(1, 0)$ is also in the interval $(-1/d, 1/d)$ for any $\beta \in (\alpha - \phi(\alpha), \alpha + \phi(\alpha))$. The set of those intervals cover the closed set $[-\pi, \pi]$. Find a finite subcover $\{(\alpha_1 - \phi_1, \alpha_1 + \phi_1), \ldots, (\alpha_m - \phi_m, \alpha_m + \phi_m)\}$. Therefore for each interval there is a number $n(\alpha_j)$, we denote by $n(d)$ the greatest of them. Suppose that the rotation (giving rise to the flat manifold) is of angle α. If α belongs to the interval $\alpha_i + \phi_i, \alpha_i - \phi_i)$ let n_i be the corresponding natural. Then consider the point $(n_i d, 0)$. After rotation by $\mathcal{R}_{n\alpha}$ its z-component is n_i and on the (x, y)-plane it lies at a distance at most n_i from $(n_i d, 0)$ along the circle of radius $n_i d$. The initial and final points (in $\mathbb{R}^3$) therefore join through a segment of magnitude at least n_i and at most $\sqrt{2} n_i$. After scaling the metric by $1/n_i^2$ we get a loop of length between one and two and at distance d from the soul. To see that any other loop must a have a length greater or equal than $1/n_i$ observe that the process of sliding it to the soul is length-decreasing.

□

The simple proposition below, shows in particular that in any $F(\alpha, h)$ space we have $\lim l(p_i)/d(p_i) \to 0$ for any sequence of points $\{p_i\}$ such that $d(p_i) \to \infty$. Before stating the proposition let us introduce some notation. Let $l(p)$ be the length of the shortest geodesic loop based at p in a $F(\alpha, h)$-space. Then denote by $\tilde{g}_p$ the scaled metric $\tilde{g}_p = \frac{1}{l(p)^2} g_F$. Naturally the shortest geodesic loop considered before has length one in the metric $\tilde{g}_p$. Let $d(p)$ be the distance from p to the soul of F in the metric g_F.

Proposition 2 *Consider a space $(F(\alpha, h), g_F)$. Suppose that for a point p with $d(p) = d$ we have $l(p) = 1$ (which by symmetry implies that any p with $d(p) = d$ has $l(p) = 1$). Then*

1. *There is $h(d)$ such that $0 < h(d) \leq h \leq 1$. Thus the set of spaces $F(\alpha, h)$ with the property above is compact.*

2. *For any $\bar{d}$ there is $\tilde{d}$ such that for any point q with $dist_{g_F}(q, Soul) \geq \tilde{d}$ it is $dist_{\tilde{g}_q}(q, Soul) \geq \bar{d}$.*

3. *For any sequence of points q_i such that $dist_{g_F}(q_i, Soul(F)) \to \infty$ the limit space of the pointed space $\{F(\alpha, h), \tilde{g}_{q_i}, q_i)\}$ is either the metric product $S^1 \times \mathbb{R}^2$ or a product $T^2 \times \mathbb{R}$ where the factor T^2 has a flat metric.*

Proof:

Item 1. For each possible angle α find the least $n(\alpha)$ such that the rotation $\mathcal{R}_{n(\alpha)\alpha}$ of the point $(x, y) = (d, 0)$ lies in the interval between $(-1/4d, 1/4d)$. Also for any α find $\phi(\alpha)$ such that for any $\beta \in (\alpha - \phi(\alpha), \alpha + \phi(\alpha))$ the rotation $\mathcal{R}_{n(\alpha)\beta}$ of the point $(x, y) = (1, 0)$ lies again in the interval $(-1/4d, 1/4d)$. Let $\{(\alpha_1 - \phi_1, \alpha_1 - \phi_1), \ldots, (\alpha_n - \phi_n, \alpha_n + \phi_n)\}$ be cover of $[-\pi, \pi]$. Let $m = max\{n(\alpha_1), \ldots, n(\alpha_n)\}$. Then if $h < 1/m$ the shortest geodesic loop at p must have length less than one which is not possible.

Item 2. Suppose the claim is not true. Then there is $\bar{d}$ such that for any $\tilde{d} = n$ there is q_n having $dist(q_n, Soul) \geq n$ and $dist_{\tilde{g}_{q_n}}(q, Soul) \leq \bar{d}$. But then it must be $l(q_n) \to \infty$ and thus the Soul in the scaled metric $(1/l(q_n)^2)g_F$ must have length less than $1/l(q_n)$ which, for sufficiently high n, contradicts *item 1*.

$\square$

Item 3. To prove this *item* note that the torus which lies at a distance equal to $dist(q_i, Soul)$ from the soul converges into a flat and totally geodesic two manifold in the limit space. If it is $S^1 \times \mathbb{R}$ then the limit space is $S^1 \times \mathbb{R}^2$ while if it is a flat torus T^2 (with a flat metric) then the limit space is the product $T^2 \times \mathbb{R}$.

Example 4 Consider a flat three-manifold which is obtained by a rotation by an angle $2\pi/p$, p a (fixed) positive integer and translation by one along the z-axis. Again we will denote it by $F(2\pi/p, 1)$. Consider the points $(x, 0, 0)$. When $x \sim 0$ the shortest geodesic loop based at $(x, 0, 0)$ is homotopic to the central fiber $(\{(0, 0, z), z \in [0, 1]\})$ and its length is close to one. However when x goes to infinity then the shortest loop homotopic to the central fiber and based at $(x, 0, 0)$ increases its length to infinity. Instead the loop $\{(x, 0, z), z \in [0, p]\}$ keeps its length equal to p as x goes to infinity and is the only simple geodesic loop based at $(x, 0, 0)$ whose length remains bounded. Thus the pointed spaces $\{F(2\pi/p, 1), (1/p^2)g_F, (x, 0, 0)\}$ converge, as $x \to \infty$, to the metric product $S^1 \times \mathbb{R}^2$.

Example 5 We obtain the same convergence as in the previous example for the pointed spaces $\{(F(2\pi/p, 1), (1/2)g_F, (p/(2\pi), 0, 0))\}$ where now the points depend on on the natural number p (and therefore on the metric). This time however the shortest geodesic loop approaching S^1 is the loop
$\{((p/(2\pi) \cos(2\pi/p), (p/2\pi) \sin(2\pi/p), 1) - (p/2\pi, 0, 0))\theta + (p/2\pi, 0, 0)\theta \in [0, 1]\}$.

Examples 4 and 5 show in particular that convergence to a single flat space can be in many different fashions. Note that in these three examples the limit carries some topology. What Proposition 2 shows is that any pointed sequence $\{(F(\alpha_i, 1), g_{F_i}, \{p_i\})\}$ with the distance from p_i to the soul of $F(\alpha_i, 1)$ going to infinity can be rescaled to converge to $T^2 \times \mathbb{R}$ or $S^1 \times \mathbb{R}^2$.

Remark 1 Consider a torus T^2 lying at a definite but arbitrary distance from the soul (S^1) in a $F(\alpha, h)$-space and consider the solid torus D enclosed by T^2. Then the inclusion $i : T^2 \to D$ induces a map on π_1 whose kernel is precisely represented by the circle which is the intersection of T^2 with the plane (x, y). Therefore a closed curve in T^2 in the kernel of the inclusion will have length greater or equal than 2π times the distance from T^2 to the soul. Thus if that distance is large and a loop in T^2 has length one then it cannot lie in the kernel of the inclusion and must project non-trivially into the central fiber. This fact will be of central importance later.

2.2 Sectioning the thin parts of the manifold M.

Let us start by explaining how to break thin regions into parts which are one-collapsed (and that will be called *region (c)*), two-collapsed (that will be called *region (b)*) and those lying somewhat in between of them that we shall call *pseudo-cusps* (or *region (a)*). From now on assume that $\|Ric\|_{L^p_g} \leq \Lambda$.

We are going to work with three parameters d, ϵ and $\bar{\epsilon}$, and we need to impose some conditions on them that we will call *adjustments*. The adjustments are made in order to guarantee that basic properties of flat manifolds that will be useful to decompose and understand the topology of the thin region also hold in balls $B_{\frac{1}{\nu^\delta(p)^2}g}(p, d)$ centered at any point p with $\nu^\delta(p) \leq \bar{\epsilon}$. Namely to most of purposes these balls can be thought as flat.

Adjustment 1 *Given $d > 0$ and $\epsilon > 0$ we can find $\bar{\epsilon}(d, \epsilon, \Lambda) > 0$ (from now on we will forget writing the Λ-dependence) such that for any p with $\nu^\delta(p) \leq \bar{\epsilon}$ there is a complete and flat manifold $(F(p), g_{F(p)})$, that we shall call an associated space, such that $B_{\frac{1}{(\nu^\delta(p))^2}g}(p, d)$ is ϵ-close in the strong H^2-topology to a ball $B_{g_F}(q, d)$ in F.*

In what follows we will work not with the scaled metric $\frac{1}{(\nu^\delta(p))^2}g$ but instead with the metric $\tilde{g}_p(q) = \frac{1}{l_1^2}\frac{1}{(\nu^\delta(p))^2}g(q)$ where l_1 is the length of the shortest geodesic loop $\mathcal{L}_1$ at p in the metric $\frac{1}{(\nu^\delta(p))^2}g$. Of course in the metric $\tilde{g}$ it is $length(\mathcal{L}_1) = 1$. Note that $\tilde{g}_p = (1/length_g(\mathcal{L}_1))^2 g$. We will assume therefore that $\bar{\epsilon}$ was chosen in such a way that $B_{\tilde{g}_p}(p, d)$ is ϵ-close in the strong H^2-topology to $B_{g_F}(q, d)$ in F and recall that this means that there exists a diffeomorphism $\varphi_p : B_{g_F}(q, d) \to B_{\tilde{g}_p}(p, d)$ with $\|\varphi_p^*\tilde{g}_p - g_F\|_{H^2_{g_F}} \leq \epsilon$. We will keep the notation $(F(p), g_{F(p)})$ for the associated spaces and φ_p for the diffeomorphism between the metric balls.

As said if a point is very collapsed then there is, at the point, a length minimizing loop $\mathcal{L}_1$. The sets of points in M for which $l(p) = length_g(\mathcal{L}_1) \leq \bar{\epsilon}$ will be called the

$\bar{\epsilon}$-thin part and denoted by $M_{\bar{\epsilon}}$ (we will keep the notation $l(p)$). Defined like this, it may happen that the boundary of $M_{\bar{\epsilon}}$ is somehow rough as we have control on $length(\mathcal{L}_1)$ only in C^0. To avoid this slight inconvenience we will find a substitute for $\partial M_{\bar{\epsilon}}$ with more desirable properties. Still, we will denote the new region as $M_{\bar{\epsilon}}$.

Adjustment 2 *Fix a number K much greater than one (say for instance $K = 10$ but the actual value is not important). Now, given the ϵ in the Adjustment 1, find $d(K, \epsilon)$ and $\bar{\epsilon}(d, \epsilon, K)$ such that for any p in $M_{\bar{\epsilon}}$ for which:*

1. *Any $(F(p), g_{F(p)})$ associated to $\tilde{g}_p$ at p does not have a soul S^1 at a distance less or equal than $d/2$ and,*

2. *there is a second closed loop $\mathcal{L}_2$ at p not homotopic to $\mathcal{L}_1$ (the shortest) with $length(\mathcal{L}_2) \leq K$,*

then the metric $\tilde{g}_p$ is, in $B_{\tilde{g}_p}(p, d/4)$, ϵ-close in the strong H^2-topology to the flat (product) metric $g_F = dx^2 + h$ (h flat in T^2) in a ball of radius $d/4$ in the flat manifold $F = T^2 \times \mathbb{R}$.

In the following we will make use of a convenient metric $g^*(p) = (1/\nu^{\delta}(p))^2 g(p)$. Note that $g^*(q)$ is C^0-close to $(1/\nu^{\delta}(p))^2 g(q)$ if q varies in a neighborhood of p in the metric $(1/\nu^{\delta}(p))^2 g(q)$.

Let p be a point as in the Adjustment 2. A torus near p in the metric g^* will be called a *K-torus* if it is the image under φ_p of a torus C^1-close to the torus that passes through $\varphi_p^{-1}(p)$ in $F(p)$ and that lies at a constant distance from the soul, if the soul is S^1, or simply C^1-close to the leaf T^2 in $F = T^2 \times \mathbb{R}$ that passes through $\varphi^{-1}(p)$, if the soul is T^2. How close in C^1 is not important. We include this condition to provide some flexibility in the choice of the torus but as we said it has not particular relevance. The terminology K-torus naturally arises from the fact that one such torus has one shortest geodesic loop of length close to one and a second geodesic loop (following the first in length) whose length lies in a range close to the range $[1, K]$. Note that if a flat torus has one shortest geodesic loop of length one and another geodesic loop (following the first in length) of length in the range $[1, K]$ then the angle that they form cannot be arbitrarily small and there is a lower bound (depending on K) for it. In this sense K-torus are non-*degenerate*.

A further adjustment is required. Observe first the following elementary property of area minimizing (connected) surfaces $\mathcal{S}$ on a manifold $F = T^2 \times \mathbb{R}$: if $\partial \mathcal{S} \in T^2 \times [d/4, \infty)$ then $\mathcal{S}$ does not enter $T^2 \times (-\infty, 1]$. With this in mind we make the following adjustment.

Adjustment 3 *Lower ϵ (and for each value of ϵ find $d(\epsilon)$ and $\bar{\epsilon}(d, \epsilon)$ satisfying Adjustments 1 and 2) such that if p be a point as in the Adjustment 2 and T^2 a K-torus through p, then for any component of $B_{\tilde{g}_p}(p, d/2) \setminus T^2$ one can deform (see below) g in $B_{g^*}(T^2, 1)$ in such a way that T^2 is mean convex and for any area minimizing (connected) surface $\mathcal{S}$ with $\mathcal{S} \subset B_{g^*}(T^2, d/2)$, it is $\mathcal{S} \cap B_{g^*}(T^2, 1) = \emptyset$. In particular $\mathcal{S}$ does not enter where g was modified.*

We explain now a way in which the metric g can be modified. Consider $\tilde{g}_p$ and as in *Adjustment 2* a diffeomorphism $\zeta : B_{g^*}(p, d/4) \to T^2 \times \mathbb{R}$ such that $\zeta_* \tilde{g}_p$ is ϵ-close in the H^2-Topology to the product metric g_F in the flat space $F = T^2 \times \mathbb{R}$. Say x is the coordinate in the $\mathbb{R}$ factor and consider a function $\xi(x)$ such that $\xi(x) = 1$ if $x \in [0, 1/2)$, ξ is decreasing and $\xi(x) = 0$ if $x > 3/4$. Then, a first modification of g will be $(1/l(p)^2)\zeta^*(\xi g_F + (1-\xi)\zeta_* \tilde{g}_p)$. The desired modification of g to be mean convex is then easily obtained.

Adjustment 4 *Given the ϵ and d from the previous Adjustments we lower now $\bar{\epsilon}$ (we do not need to alter ϵ and d again) until the following is true. For any p such that the associated space is $F(p) = F(\alpha, h)$ with $\alpha > 1/2d$ then for any point q at a distance less or equal than one in the metric g^* from*

$$\varphi_p(B_{\tilde{g}_p}(Soul(F(p)), dist(Soul(F(p)), \varphi_p^{-1}(p))),$$

it is

$$\varphi_p(B_{\tilde{g}_p}(Soul(F(p)), dist(Soul(F(p)), \varphi_p^{-1}(p))) \subset \varphi_q(B_{\tilde{g}_q}(Soul(F(q)), dist(Soul(F(q)), \varphi_q^{-1}(q)) + 1)).$$

We are ready to define the three types of elementary sectors for the thin parts.

Definition 1 *Assume M is a compact three-manifold with $\|Ric\|_{L^p_g} \leq \Lambda$ and d, $\bar{\epsilon}$ and ϵ are chosen as above.*

1. *A point $p \in M_{\bar{\epsilon}}$ is a point of type (a) iff there is an associated space $(F(p), g_{F(p)})$ of the form $F(p) = F(\alpha, h)$, with the soul S^1 at a distance less or equal than $d/2$ from p and $\alpha > 1/(2d)$. If the associated $F(\alpha, h)$ spaces have their souls at a distance less or equal than $d/2$ but instead $\alpha \leq 1/(2d)$ then the point p is of type (c).*

2. *A point $p \in M_{\bar{\epsilon}}$ is a point of type (b) iff it is not of type (a) or (c) as defined above and there is a second closed length minimizing loop $\mathcal{L}_2$ at p, non-homotopic with the shortest $\mathcal{L}_1$ and having length$(\mathcal{L}_2) \leq K$.*

3. *A point $p \in M_{\bar{\epsilon}}$ is a point of type (c) iff it is not of type (a) or type (b).*

In practice we will denote points of type (a) as defined above as (d, a)-points. Similarly points of type (b) as defined above will be denoted as (K, d, b)-points and equally for (K, d, c)-points.

The following proposition explains how the $\bar{\epsilon}$-thin parts of M gets divided into regions of types (a), (b) or (c) along embedded tori (see Figure 1). Just to note, the points of type (a) will be $(d+1, a)$-points and those of type (b) will be $(K+1, d, b)$.

Proposition 3 *Given a compact three-manifold M with $\|Ric\|_{L^p_g} \leq \Lambda$ and d, $\bar{\epsilon}$ and ϵ fixed following Adjustments 1-4, then there is a set of embedded tori (that we shall call elementary tori) separating $M_{\bar{\epsilon}}$ into a set of connected regions (that* *we shall call elmentary regions) made of points of types (a), (b) and (c) only. The elementary tori are either components of $\partial M_{\bar{\epsilon}}$ (which may be K-tori) or K-tori. Two different regions of the same type do not share a boundary component.*

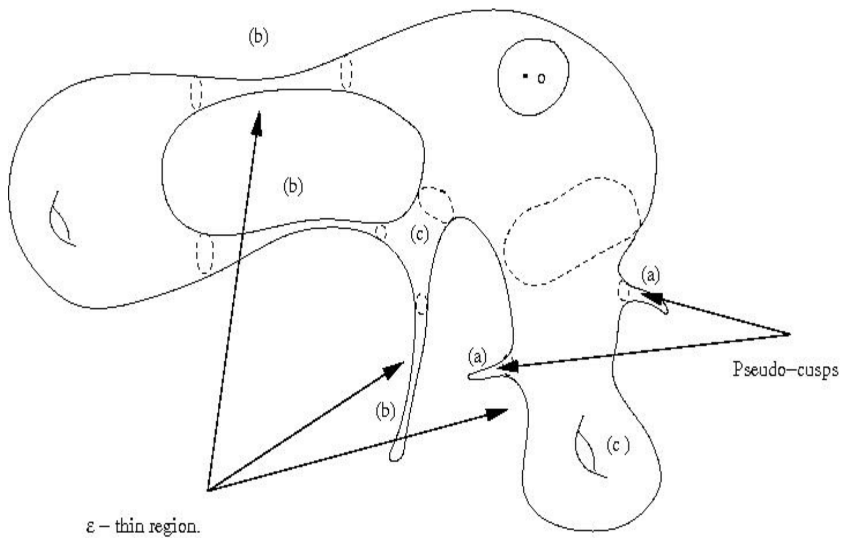

Figure 1: Separation of the ϵ-thin part of M. The regions of type (b) are represented as thin tubes, the regions of type (a) (the pseudo-cusps) are represented by small cusps and the regions of type (c) by thicker regions.

Proof: (of Proposition 3)

Take for now $M_{\bar{\epsilon}}$ to be the region consisting of those points p where $l(p) < \bar{\epsilon}$. At the end of the present construction we will explain (sketchily) how to redefine the boundary of $M_{\bar{\epsilon}}$ to make it somehow more clearly-defined.

The first task is to delimit the region that will consist of points of type $(K+1, d, b)$. Consider the region $M_{\bar{\epsilon}}$ intersected with the 1-neighborhood, in the metric g^*, of the set of points which are of type (K, d, b). Consider its connected components, there are a finite number of them, say $U_1, \ldots, U_n$. Consider U_1 and enclose the set of points of type (K, d) inside it with two tori which are K-well sized (it is straightforward to see that this is possible to do). The resulting region is homemorphic $T^2 \times [-1, 1]$, call it $\tilde{U}_1$. Consider now U_2. If $\tilde{U}_1 \cap U_2 \neq \emptyset$ then enclose the set of points of type (K, d, b) inside $\tilde{U}_1 \cup U_2$ with two K-well sized tori. If not just enclose the set of points of type (K, d, b) inside U_2 by two K-well sized tori in the same way that was done for U_1 and making sure one does not intersects $\bar{U}_1$. Proceed like this for the other components. In this way we obtain a set of open regions, each one homeomorphic to $T^2 \times [-1, 1]$ and (obviously) foliated by K-well sized tori. Note that inside these components there can be points which are of type $(K+1, d, b)$ (the value $K+1$ is somehow arbitrary as some definite value above K would be enough). Denote such region as (b).

We now define the region (a) or the set of pseudo-cusps. Consider the set of points of type (d, a) in $M_{\bar{\epsilon}}$ and not in the region (b). Pick one, say p, and consider the flat space $(F(p), g_{F(p)})$ associated to $\tilde{g}_p$. Using the same notation φ_p as was used before for the diffeomorphism between the spaces, consider the set $\varphi_p(B_{g_{F(p)}}(Soul(F(p)), dist(Soul(F(p)), p))))$ in M. Now consider the set of points at a distance less or equal than one from from it in the metric g^*.

(Case 1) If the set intersects the (b)-region then there is a K-torus (one of the boundaries of a component of (b)) which encloses a region (a pseudo-cusp) containing the original (d, a)-point. Fix that region and leave the torus as its boundary (or, following the terminology, as an elementary K-torus).

(Case 2) Suppose we are not in Case 1. If in the set there are points, say a point q, whose associated space is $F(\alpha, h)$ with $dist_{g_{F(q)}}(q, Soul) \leq d/2$ and $\alpha \leq 1/2d$ then forget analyzing again the (d, a) points lying in the region $\varphi_p(B_{g_{F(p)}}(Soul(F(p)), dist_{g_{F(p)}}(Soul(F(p), p))))$.

(Case 3) Suppose we are not in Cases 1 and 2. If in the set there are no points of type (d, a) then fix $\varphi_p(B_{g_{F(p)}}(Soul(F(p)), dist_{g_F}(Soul(F(p), p)))$ as a pseudo-cusp.

(Case 4) Suppose we are not in Cases 1, 2 and 3. If in the set there are instead points of type (d, a) then start the analysis again from Case 1 to the set $\varphi_q(B_{g_{F(q)}}(Soul(F(q)), dist_{g_{F(q)}}(Soul(F(q)), q) + 1))$ (this is where Adjustment 4 is used).

If one follows the analysis in Cases 1-4 (for the point p) then at the end of it one obtains either: (i) a pseudo-cup if the analysis stops in Cases 1 or 3 or (ii) a region containing the point p if the analysis stops in Case 2. In whatever alternative the (d, a)-points belonging either to the pseudo-cusps (Cases 1 and 3), or the region containing p (Case 2) are not be analyzed again and we proceed with the analysis for the other (d, a)-points in $M_{\bar{\epsilon}}$ and not in (b). It is important to note that the pseudo-cusps defined in this way are disjoint from each other. To see this note first that the pseudo-cusps arising from Case 1 are separated by the (b)-regions from the rest. The pseudo-cusps arising from Case 3 are seen to be disjoint from the rest (of those coming out from Case 3) for the following reason. Suppose P is a pseudo-cusp arising in Case 3. That means that there are no points from the region (b) and (d, a)-points in the 1-neighborhood in the metric g^* of P. But if another pseudo-cusp $\bar{P}$ (arising from Case 3) intersects P, that means that there is a point, say q, not in P, such that the set $\varphi_q(B_{g_{F(q)}}(q, dist_{g_{F(q)}}(Soul(F(q)), q)))$ intersects P. Thus it must also intersect the set of points (not in P) at a distance less or equal than one from P in the metric g^*. If $\bar{\epsilon}$ is small enough (as we will assume) that would imply that in this set there are points of type (d, a) which is not possible.

It remains still to explain how to redefine the boundary of $M_{\bar{\epsilon}}$. In general, even at the scale of the volume radius, $\partial M_{\bar{\epsilon}}$ can be rather irregular set for the fact that, even in canonical harmonic coordinates with respect to $\tilde{g}_p$ (at every p), we have a priori control on $l(q)$ (q close to p) only in C^α. Later on we would have to modify the metric g in a 1-neighborhood of $M_{\bar{\epsilon}}$, in such a way that the new metric has some well defined properties (these properties are written down as *Conditions 1* and *2* right after Proposition 5). It is thus necessary to get rid of the irregularity of $\partial M_{\bar{\epsilon}}$ and take instead of $\partial M_{\bar{\epsilon}}$ another smooth two-manifold, somehow close to it and in such a way that its geometry is controlled at the scale of the volume radius. Although it is more or less clear that this can be done, it is still necessary to explain sketchily the construction. But because it has few to do with the rest of the article and it is essentially a technical aspect of it, we will give only a rapid description in the next paragraphs (this is the only part of the article where a non thorough discussion is given).

A new boundary for the thin region. Denote $C_{\bar{\epsilon}} = (c) \cap M_{\bar{\epsilon}}$ and consider $B_{g^*}(C_{\bar{\epsilon}}, 5)$. For any p in $C_{\bar{\epsilon}}$ and using canonical harmonic coordinates $\{x\}_p$ with respect to $\tilde{g}_p$, it is possible to construct a smooth kernel $\varphi_p(x, y)$ (with support in the chart) and smooth out the function $l(q)$ (to get $l_S(q)$) and the metric $\tilde{g}_p(q)$ (to get $\tilde{g}_{p,S}(q)$). Near p, and with respect to the harmonic coordinates $\{x\}_p$, the metric $(1/l_S(q)^2)\tilde{g}_{p,S}(q)$ is controlled in C^2. One can do this procedure systematically to get a smoothed out metric in $B_{g^*}(C_{\bar{\epsilon}}, 5)$ that we will denote by g^*_S.

Using the length minimizing geodesic loops at every point in $B_{g^*}(C_{\bar{\epsilon}}, 4)$, it is possible to define a S^1-fibration and using the metric g^*_S one can define, in a natural way, a $U(1)$-action over such fibration. Use then the $U(1)$-bundle to average g^*_S and get a $U(1)$-symmetric smooth metric g^*_U.

If we quotient the fibration by the action of $U(1)$ we get a surface Σ, and by projecting g^*_U we get a smooth metric h on Σ (this construction can be done, if d is big enough and $\bar{\epsilon}$ small enough, in such a way that h gets close to flat). The projection will be denoted by Π. Consider in Σ the set $B_h(\Pi(C_{\bar{\epsilon}}), 4)$. Now consider a maximal set of disjoint balls $\{B_1, \ldots, B_m\}$, each of radius one with respect to the metric h and lying inside $B_h(\Pi(C_{\bar{\epsilon}}), 4)$ (namely there exists no ball B disjoint from B_i, $i = 1, \ldots, m$ and inside $B_h(\Pi(C_{\bar{\epsilon}}), 4)$. Let $\{c_1, \ldots, c_m\}$ be their centers. Now, two different centers c_i and c_j will be joined through a length minimizing segment (geodesic) iff: (i) the distance between them in the metric h is less or equal than $9/2$ and (ii) the segment does not intersects any ball other than B_i and B_j. Now, (if h is sufficiently flat) it is not difficult to see that the set of segments satisfy that

1. A segment cannot intersect more than two other segments,
2. Because every ball B_i lies inside a ball $B_h(c, 4)$ (with c in $\Pi(C_{\bar{\epsilon}})$) then there are always c_j and c_k such that the pairs $\{c_i, c_k\}$, $\{c_i, c_j\}$ and $\{c_j, c_k\}$ are joined through a geodesic segment and thus forming a geodesic triangle $\{c_i, c_j, c_k\}$ inside $B_h(\Pi(C_{\bar{\epsilon}}), 4)$.

We will consider the union of all these (solid) geodesic triangles. It may be that this set touches a boundary component of the regions (a) and (b) but without covering it. In this case eliminate from the union all the triangles that touch that component. The union of the solid triangles that are left will be denoted by Δ (see the triangles with dashed sides in Figure 2).

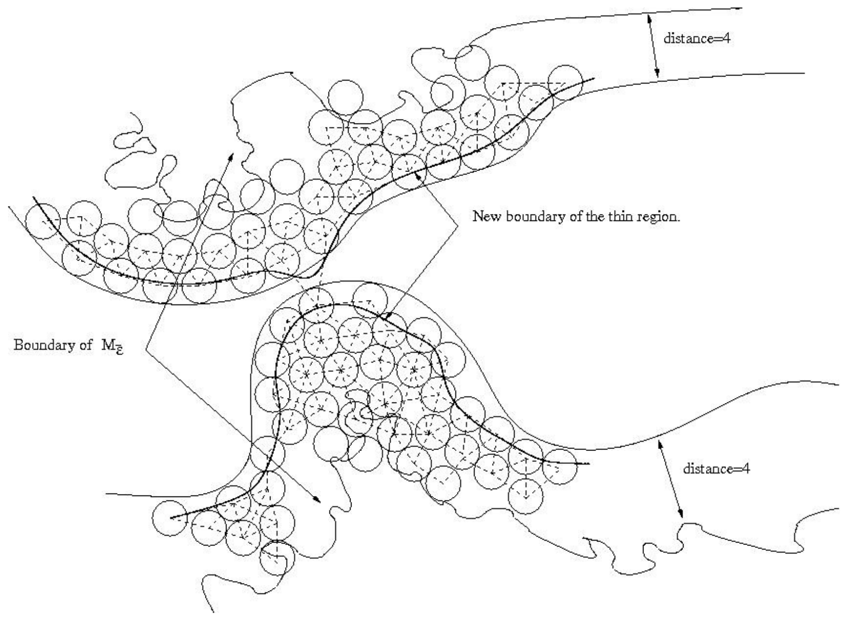

Figure 2: An illustration on how to modify the boundary of the thin region.

Note that if a point c in $\Pi(C_{\bar{\epsilon}})$ has $dist_h(p, \partial B_h(\Pi(C_{\bar{\epsilon}}), 4)) \geq 8$ then $B_h(c, 4) \subset \Delta$. This implies in particular that $M_{\bar{\epsilon}/2} \subset \Delta \cup (a) \cup (b)$.

Let c_i be a center that is in $\partial\Delta$. Let α_i be a solid angle that is delimited by two segments (say s_{i_1} and s_{i_2}). α_i is bounded below by an angle not far from $\pi/6$. Now join the middle points of s_{i_1} and s_{i_2} by a smooth curve tangent to s_{i_1} and s_{i_2} at the their middle points and that rounds the vertex at c_i (see the bold curves in Figure 2). In this way we are modifying $\partial\Delta$ to a set of smooth curves, each one with a curvature (with respect to h) very much controlled. Eliminate all those curves that may lie inside one of the (a) or (b) regions. The pre-image under Π of the resulting set of curves is the new boundary of the thin region that we were looking for.

□

Remark 2 It is now direct to check that the pseudo-cusps constructed in this way are isolated and have the following essential property. *Suppose P is a pseudo-cusp enclosed by a torus $T^2 = \partial P$. Let p be a point in T^2 and consider the metric $\tilde{g}_p$. Then there is an associated flat space (F_P, g_{F_P}) and a diffeomorphism $\varphi_P : B_{g_{F_P}}(Soul(F_P), d') \rightarrow P$ such that $\|\varphi^* \tilde{g}_p - g_{F_P}\|_{H^2_{g_{F_p}}} \leq 2\epsilon$ where, most importantly, $d' \sim d/2$.* In particular, according to the Remark 1, if the distance $d/2$ is big enough (as we will assume) then the shortest loop based at any point in the boundary T^2 is not contractible inside the pseudo-cusp and must wrap around the central fiber a non-zero number of times.

2.3 Positive scalar curvature and the topological classification of the thin regions.

The next proposition explains the nature of the topology of $M_{\bar{\epsilon}}$ (see Figure 3).

Proposition 4 *Let M be a compact three-manifold. Assume that $\|Ric\|_{L^p_g} \leq \Lambda$ and that $R \geq R_0 > 0$. Then there are $\bar{\epsilon}(\Lambda, R_0, \nu^\delta(o))$ and $\bar{\epsilon}'(\Lambda, R_0, \nu^\delta(o))$, with $\bar{\epsilon}' < \bar{\epsilon}$, such that*

1. *If two elementary K-tori in $M_{\bar{\epsilon}'}$ can be joined through a curve that intersects them only once and that starts at one of them and ends in $\partial M_{\bar{\epsilon}}$ then the two tori enclose a region diffeomorphic to $T^2 \times [-1, 1]$.*

2. *An elementary K-torus in $M_{\bar{\epsilon}'}$ cannot be joined through a curve that intersects it only once and which starts and ends at $\partial M_{\bar{\epsilon}}$.*

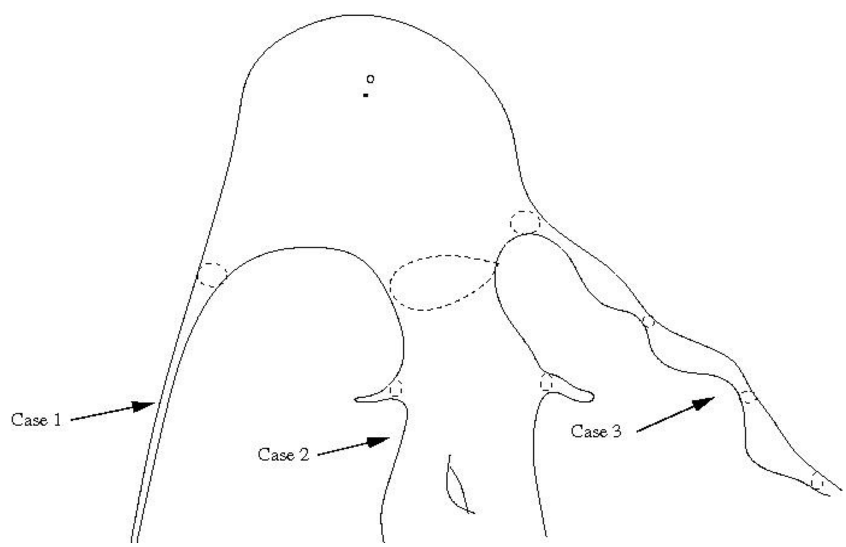

Figure 3: An illustration of the three types of ends (Case 1,2, and 3 in Corollary 2) in a (hypothetical) limit where there is collapse with bounded curvature and $R \geq R_0 > 0$.

A consequence of this Proposition we have the Corollary.

Corollary 2 *Let $\bar{M}$ be a limit of a sequence of pointed spaces $\{(M_i, g_i, o_i)\}$ such that for each i we have the uniform bounds $R_{g_i} \geq R_0 > 0$, $\|Ric\|_{g_i} \leq \Lambda$ and $\nu_{g_i}(o) \geq \nu_0 > 0$. Then, with the same procedure as in Proposition 3, we can cut $\bar{M}_{\bar{\epsilon}'}$ along elementary tori such that each connected component with zero (global) volume radius is either,*

1. *A two-collapsed region foliated by K-torus.*

2. *A one-collapsed region, possibly containing pseudo-cusps and whose boundaries consist of K-tori or components of $\partial M_{\bar{\epsilon}'}$.*

3. *A hybrid region, which alternates two-collapsed regions diffeomorphic to $T^2 \times [-1,1]$ (as in item 1) and one-collapsed regions also diffemorphic to $T^2 \times [-1,1]$ which have K-tori at each one of their ends.*

Proof (of Corollary 2):

Note first that for each i, the decomposition of $M_{i,\bar{\epsilon}'}$ is constructed by first defining $M_{i,\bar{\epsilon}'}$ (in the same way as in $M_{\bar{\epsilon}}$ was defined in Proposition 3) and then intersecting it with the elementary regions of the decomposition of $M_{\bar{\epsilon}}$. Consider the decomposition of $\bar{M}_{\bar{\epsilon}'}$ which is the limit of the decompositions of $M_{i,\bar{\epsilon}'}$. The set of elementary K-tori of that decomposition is partitioned as follows: *two tori are equivalent iff they can be joined to $\partial \bar{M}_{\bar{\epsilon}}$ through a curve that intersects them only once*. It is direct from Proposition 4 that this is actually an equivalence relation. Note that in each one of such classes, the curve that joins them (only once) gives them a natural order. Note too that according to Proposition 4 any elementary K-torus separates $\bar{M}_{\bar{\epsilon}'}$ into two connected components. Now, if an equivalence class has an infinite number of elements then we have two possibilities: either (I) the set of consecutive tori in the class which enclose a region with $\nu^{\delta} = 0$ is finite or (II) such set of pairs of tori is infinite. In case (I) we get finite number of ends of type as in *item 2* and one end of type as in *item 3*. In case (II) we get only an infinite number of ends of type as in *item 2*. Remove now the regions enclosed by the K-tori lying in everyone of the equivalence classes having an infinite number of elements and consider all the equivalence classes of K-tori (having a finite number of K-tori) which are left.

Given one of such classes consider the first of the tori (where the curve joining them starts), remove it and consider the component not containing the point $\bar{o}$. If it has $\nu^{\delta} = 0$ then it is of type (b) or (c), possibly, in this last case, with components of type (a) (pseudo-cusps) (it cannot be only of type (a) otherwise it would not be $\nu^{\delta} = 0$). If it is of type (b) then it defines an end as in *item 1*, if it is of type (c) (possibly with components of type (a)) then it defines an end as in *item 2*. Remove now all these regions, namely all the regions enclosed by the K-tori. There may be still left some tori of the decomposition of $\bar{M}_{\bar{\epsilon}'}$ which are not K-tori. Any connected component enclosed by them and not containing the point $\{o\}$ which has $\nu^{\delta} = 0$ cannot be or contain a (b)-region and therefore must define an end as in *item 2*.

□

To Prove Proposition 4 we need to stablish first a fundamental topogical Lemma and a Proposition on the regularity of area minimizing (minimal) surfaces on manifolds where there is a priori control on the L^p_g-norm of Ricci.

Lemma 2 (Main topological Lemma) *Let $\Pi : M^* \to S^*$ be a S^1-fibration over a compact surface S^* with boundary consisting of the curves $\{\ell_1, \ldots, \ell_m\}$, $m \geq 2$ (thus M^* has toric boundary). Consider a set of solid tori $\{D_1, \ldots, D_n\}$, $n \leq m-2$*

$(D_i \sim D^2 \times S^1,\ i = 1, \ldots, n)$ and a set of homeomorphisms $\varphi_i : \Pi^{-1}(\ell_i) \to \partial D_i$, $i = 1, \ldots, n$, such that the S^1 fibers in $\Pi^{-1}(\ell_i)$ represent (under φ) a non-trivial element in $\pi_1(D_i)$. Denote by M the manifold that follows under this indentification. Then,

1. *For $i = n+1, \ldots, m$, the inclusion $i : \Pi^{-1}(\ell_i) \to M$ induces a one-to-one map in π_1.*

2. *Any embedded two-sphere in M bounds a three-ball.*

3. *And embedded projective two-space lies in a three-ball.*

4. *Let N be an oriented three-manifold made out of a collection of manifolds $\{M_1, \ldots, M_j\}$ of the type considered in item 1, by identifying some of the boundaries of the M_i by toral automorphisms. We will assume ∂N has at least two connected components. Let T^2 be one of the components of ∂M_i for some $i = 1, \ldots, j$. Then the inclusion $i : T^2 \to N$ induces a one-to one map in π_1.*

Proof:

Item 1. Let ℓ be a non-trivial loop in $\Pi^{-1}(\ell_i)$ where $i \in \{n+1, \ldots, m\}$. Let $\varphi : D^2 \to M$ be such that $\varphi : \partial D^2 \to \ell$ homeomorphically. Deform φ if necessary to intersect ∂M^* transversely. Then, $\varphi^{-1}(\partial M^*)$ is a set of closed curves in D^2. Consider the set of those curves $C_1, \ldots, C_k$ that are not enclosed by any other curve except ∂D^2. The region that $\partial D^2, C_1, \ldots, C_k$ enclose in D^2 is mapped into S^*. By collapsing every component of ∂S^* except ℓ_i into a point we get a map from D^2 into a compact surface with a connected boundary and such that its image avoids at least one point (we can always make this because $n \leq m-2$). This image is always retractable (in the quotient $S^*/_\sim$ of S^*) into ℓ_i. This already shows that the curve $\Pi(\partial D^2)$ has to be retractable in ℓ_i, and thus if a loop ℓ in $\varphi^{-1}(\ell_i)$ is contractible in M it must homotopic to the fiber S^1. We start below the proof that the fiber S^1 is a non-trivial element in $\pi_1(M)$.

A usefull retracted space. Consider $S^1 \times [-1, 1]$. Consider a g_f-folded map f from $S^1 \times \{-1\}$ into S^1. Now two points A and B in $S^1 \times [-1, 1]$ are identified if A and B belong to $S^1 \times \{-1\}$ and have the same image under f. Denote such space by T_{g_f}. Now consider k of them $T_{g_{f_1}}, \ldots, T_{g_{f_k}}$ and identify the boundaries $S^1 \times \{1\}$ of each of them to each other in the natural way. We will call such space an $R_{f_1,\ldots,f_n}$*-retracted space* (or simply an *R-retraction*). Note that the curve $S^1 \times \{1\}$ is non-contractible.

Consider now a domain $\Pi^{-1}(\Omega)$ in M containing $D_1, \ldots, D_n$ where Ω is a domain in $S^*/_\sim$ homeomorphic to the open disc D^2. We claim that Ω is retractable into an R-retracted space. To see this observe that the solid tori $D_1, \ldots, D_n$ can be seen as topological spaces in the following way. Let ℓ be a closed geodesic in the torus ∂D_i homotopic to the S^1 fiber in $\Pi^{-1}(\ell_i)$. By parallel translation we can foliate ∂D_i by closed geodesics isotopic to each other, and by contracting them towards the central fiber ($(x, y, z) \to (\lambda x, \lambda y, z)$, $\lambda < 1$) $\{o\} \times S^1$ (o is the center of D^2) we get a foliation of $D_i \setminus (\{o\} \times S^1)$ by curves which are closed geodesic of the tori lying at a constant distance from the central fiber. Note that any two of such curves, one of which is a contraction towards the central fiber of the other, have the same projection into the central fiber $\{o\} \times S^1$. Thus we can think $D_i \setminus (\{o\} \times S^1)$ as $[-1, 0) \times \partial D_i$. If we add a last slice $\{0\} \times \partial D_i$ to $[-1, 0) \times \partial D_i$ then we can think the solid torus D_i as $[-1, 0] \times \partial D_i/_\sim$ where two points are identified iff they belong to $\{0\} \times \partial D_i$ and they have the same projection (as points in ∂D_i) into the central fiber $\{o\} \times S^1$. It is now direct that each solid torus D_i is retractable into a $T_{g_{f_i}}$ space where g_{f_i} is the number of times the S^1 fiber of ∂D_i in M wraps into the central fiber of the solid torus $\{o\} \times S^1$. Therefore $\Pi^{-1}(\Omega)$ retracts into an R-retracted space.

Consider now a map $\varphi : D^2 \to M$ such that $\varphi(\partial D^2)$ is homotopic in $\Pi^{-1}(\ell_i)$ to the fiber S^1. Then, as we proved above, $\Omega = \Pi(\varphi(Int(D^2)))$ is homeomorphic to an open disc in $S^*/_\sim$. $\Pi^{-1}(\Omega)$ is retractable into a R-retractable space with the fiber S^1 retracted into a non-trivial element of the π_1 of the retracted space, which is a contradiction. This finishes *item 1.*

Item (2). Consider an embedding $\varphi : S^2 \to M$. Denote the central fibers of the solid tori $D_1, \ldots, D_n$ by $C_1, \ldots, C_n$.

By deforming φ if necessary assume that $\varphi(S^2)$ intersects the central fibers transversely. Let $\{o\}$ be a point in S^2 whose image under φ does not lie in any of the central fibers. Let $\mathcal{L}_0$ denote the constant map from S^1 into $\{o\}$. Now find and embedded curve $\mathcal{L}_1$ in S^2 based at $\{o\}$ such that it encloses (in one of the components that it separates in S^2) the points $\varphi^{-1}(C_1)$ but not (in the same component) the points $\varphi^{-1}(C_i)$, $i = 2, \ldots, n$. Denote such component by $[\mathcal{L}_0, \mathcal{L}_1]$. In $S^2 \setminus [\mathcal{L}_0, \mathcal{L}_1]$ consider an embedded curve, based at $\{o\}$ such that it encloses (in one of the components that it separates) in $S^2 \setminus [\mathcal{L}_0, \mathcal{L}_1]$ the points $\varphi^{-1}(C_2)$ but not (in the same component) the points $\varphi^{-1}(C_i)$, $i = 3, \ldots, n$. Proceed like this to find a set of curves $\{\mathcal{L}_0, \ldots, \mathcal{L}_1\}$ based at $\{o\}$ and such that the region $[\mathcal{L}_{i-1}, \mathcal{L}_i]$ contains the points $\varphi^{-1}(C_i)$ but not the points $\varphi^{-1}(C_j)$, $i \neq j$. One can take the last curve $\mathcal{L}_n$ not an embedded curve but actually the constant curve $\{o\}$.

The image of $[\mathcal{L}_0, \mathcal{L}_1)$ lies in a region in M: (i) containing C_1, but not containing C_i, $i = 2, \ldots, n$, (ii) not containing the image of $[\mathcal{L}_1, \mathcal{L}_n)$, and (iii) topologically a solid torus. By passing to the universal cover of that region we deduce that there is an *isotopy*

$$\bar{\varphi}_1 : S^2 \times [0, \frac{1}{n}] \to M$$

with $\bar{\varphi}_1(p, t) = \varphi(p)$ if $p \in [\mathcal{L}_1, \mathcal{L}_n)$, $\bar{\varphi}_1([\mathcal{L}_0, \mathcal{L}_1) \times \{1/n\}) \cap C_1 = \emptyset$ and $\bar{\varphi}_1(\mathcal{L}_1 \times \{1/n\})$ contractible in $M \setminus (C_1 \cup \ldots \cup C_n)$. Similarly, the image of $[\mathcal{L}_1, \mathcal{L}_2)$ under φ lies in a region in M: (i) containing C_2, but not containing C_i, $i \neq 2$, (ii) not containg $\bar{\varphi}_1([\mathcal{L}_0, \mathcal{L}_1) \cup [\mathcal{L}_2, \mathcal{L}_n) \times \{1/n\})$, (iii) topologically the solid torus. By passing to the universal cover of that region we deduce that there is an *isotopy*

$$\bar{\varphi}_2 : S^2 \times [\frac{1}{n}, \frac{2}{n}] \to M$$

such that $\bar{\varphi}_2(p,t)=\bar{\varphi}_1(p,1/n)$ if $p\in[\mathcal{L}_0,\mathcal{L}_1)\cup[\mathcal{L}_2,\mathcal{L}_n)$, $\bar{\varphi}_2([\mathcal{L}_1,\mathcal{L}_2]\times\{2/n\})\cap C_2=\emptyset$ and $\bar{\varphi}_2(\mathcal{L}_2\times\{2/n\})$ contractible in $M\setminus(C_1\cup\ldots\cup C_n)$. Continuing in this way we get an isotopy $\bar{\varphi}_n: S^2\times[0,1]\to M$ such that $\bar{\varphi}_n(S^2\times\{1\})\cap(C_1\cup\ldots C_n)=\emptyset$ and such that $\bar{\varphi}_n(S^2\times\{1\})$ is contained in a region topologically $D^2\times S^1$. It follows that $\bar{\varphi}_n(S^2\times\{1\})$ must enclose a ball in M (pass to the universal cover of the region homemorphic to $D^2\times S^1$ and contract in the vertical direction).

Item 3. Let $\varphi: P^2\to M$ be an embedding form the two-projective space P^2 into M. The proof of this item follows the same idea as the proof of the previous item. The procedure differs in the fact that now we enclose the points $\varphi^{-1}(C_i)$, $i=1,\ldots,n$ by a set of closed curves $\{\xi_1,\ldots,\xi_j\}$ based at a same point $\{o\}$ whose image under φ does not lie in any of the central fibers. Note that there are closed curves in P^2 which do not separate it into two connected components. The curves ξ_i are chosen in such a way that they do separate P^2 into two connected components and one of them, the one containing $\varphi^{-1}(C_i)$ but not $\varphi^{-1}(C_h),\ h\neq i$, is homeomorphic to the two dimensional disc. Following now the same procedure as in *item 2* we can find an isotopy

$$\bar{\varphi}: P^2\times[0,1]\to M$$

with $\bar{\varphi}(p,0)=\varphi(p)$ and $\bar{\varphi}(P^2\times\{1\})\cap(C_1\cup\ldots\cup C_n)=\emptyset$. The claim follows.

Item 4 Let $i: T^2\to N$ be the inclusion as in the statement of *item 4*. Let ℓ be a non-trivial (in $\pi_1(T^2)$) closed curve and consider a map $\varphi: D^2\to N$ such that $\varphi:\partial D^2\to\ell$ homeomorphically. Deform φ if necessary to have its image intersecting ∂M_i, $i=1,\ldots,j$ transversely. Consider the closed curves $\varphi^{-1}(\partial M_i)$, $i=1,\ldots,j$ in D^2. From them pick one that does not enclose any other of these curves. Observe that the image under φ of the region enclosed by it lies in one and only one of the pieces M_i, say M_f. By *item 1* one can deform φ in such a way that the image of a small neighborhood of the region (not touching the other curves) does not intersects M_f. Continue like this until getting a map from D^2 into N whose image lies in only one of the pieces M_i and apply *item 1.*

□

We need to understand now the regularity of area minimizing minimal surfaces inside the thin regions. Standard regularity [11] results do not apply directly as we are assuming only control in $\|Ric\|_{L^p_g}$. However enough information can be obtained through a standard blow up procedure. We begin with a preliminary definition. Let $\{x\}$ be a canonical harmonic chart (see Conditions 5 and 6 given in the introduction), then the $W(\alpha)$-set of $\{x\}$ is defined as the set of $\{x,y,z\}$ for which $z/(\sqrt{x^2+y^2}\leq\alpha$ $(\alpha>0)$. The α-Lipschitz radius at a point p in a minimal surface S embedded in M (to be denoted by $L^\alpha(p)$) is defined as the supremum of the radius less than the harmonic radius $r_h(p)$ of the manifold M at the point p and such that there is a canonic harmonic coordinate chart for which the connected component of $\mathcal{S}\cap B(p,r)$ containing p lies in $B(p,r)\cap W(\alpha)$.

Proposition 5 *Let N be a compact three-manifold with $\|Ric\|_{L^p_g}\leq\Lambda$. Let $\mathcal{S}$ be a embedded compact minimal surfaces which is area minimizing in its isotopy class.*

Then there are $\bar{\epsilon}(\Lambda, \delta, \alpha)$ and $C(\Lambda, \delta, \alpha)$ such that $L^\alpha(p)/\nu^\delta(p) \geq C$ for an p in $\mathcal{S}$ with $\nu^\delta(p) \leq \bar{\epsilon}$.

Proof:

Assume on the contrary that there is a sequence $\{(N_i, g_i), \mathcal{S}_i, p_i\}$, such that

$$L^\alpha(p_i)/r_h(p_i) \to 0,$$

as $i \to \infty$. Blow up the metrics g_i by $1/L^\alpha(p)^2$ and find a convergent sub-sequence of $\mathcal{S}_i$ into an immersed surface $\mathcal{S}$ in $\mathbb{R}^3$ with $L^\alpha(p) = 1$ where $p = \lim p_i$. Note that because each $\mathcal{S}_i$ is area minimizing in its isotopy class so is $\mathcal{S}$ at least for isotopies with compact support. The standard regularity theory [12] shows that $\mathcal{S}$ is smooth and stable. Schoen's intrinsic estimate [13] for the second fundamental form of stable surfaces show that $\mathcal{S}$ is flat and therefore a plane (note that S cannot be compact otherwise an shrinking homothety would decrease its area). This contradicts the fact that the α-Lipschitz radius is one at p.

□

In practice we will need the estimates in Proposition 5 but for manifolds with boundary. These manifolds will be, in fact, regions N of the manifold M where the metric h that we will consider on N is, near ∂N, a suitable deformation of the metric g. Let us consider now the metric properties that we will require of (N, h). Consider the metrics $\tilde{h}(q) = (1/\nu^\delta(p))^2 h(q)$ and $h^*(p) = (1/\nu^\delta(p))^2 h(p)$ (note the evaluations). We will assume that

1. *(Condition 1)* For every p in the 1-neighborhood of ∂N, with respect to the metric h^*, we have

$$\|Ric_{h_p}\|^2_{C^0_{\tilde{h}_p}(B_{\tilde{h}_p}(p,1))} + \|\nabla^{\tilde{h}_p} Ric_{h_p}\|^2_{C^0_{\tilde{h}_p}(B_{h^*}(p,1))} \leq \Lambda_1. \tag{7}$$

2. *(Condition 2)* For every $r \in [0, 1/3]$ the set $\partial B_{h^*}(\partial N, r)$ is mean convex.

Under this assumptions we can follow the same argument as in Proposition 5 to prove that, given $\alpha > 0$ then for any minimal surface $\mathcal{S} \subset N$ which is area minimizing in its isotopy class, there exist $\epsilon(\Lambda, \Lambda_1, \delta, \alpha)$ and $C(\Lambda, \Lambda_1, \delta, \alpha)$ such that $L^\alpha(p)/\nu^\delta(p) \geq C$ for any p in $\mathcal{S}$ with $\nu^\delta(p) \leq \epsilon$.

Remark 3 *Let $\mathcal{S} \subset N$ be a manifold and minimal surface as considered above. Then there is $A_0(\bar{\epsilon})$ such that for any p in $\mathcal{S}$ with $\nu^\delta(p) \leq \bar{\epsilon}$ then $Area(\mathcal{S}) \geq A_0(\nu^\delta(p))^2$.*

Later in the article we will deal with stable minimal tubes ($\sim S^1 \times [-1, 1]$) which are surfaces with boundary. Therefore we need to get similar Lipschitz estimates as in Proposition 5 but for these type of surfaces. The particular properties that the stable minimal tubes will have will be subordinated to their construction. Therefore let us explain first the construction that will give us the stable minimal surfaces with boundary and after that let us explain how to obtain the required Lipschitz estimates. Let (N, h) be a three-manifold with boundary with the properties 1 and 2 above. Assume that we have B_1 and B_2 two different components of ∂N. Let also C_1 and C_2 be two embedded closed curves in B_1 and B_2 respectively which are not null homotopic in N and which can be joined by an embedded tube $S^1 \times [-1, 1]$.

It is well know that the free boundary Plateau problem can be solved in the class of all embedded tubes joining C_1 to C_2 (if, as we assumed, C_1 and C_2 are not null homotopic inside N, see [14][6] Theorem 1). Let $\mathcal{S}$ be one of such embedded stable minimal surfaces. The condition 1 above for the metric h together with standard estimates for the second fundamental form for stable surfaces show that if p is a point in $\mathcal{S}$ at a distance between 1/3 and 2/3 from ∂N with respect to the metric h^*, then the norm of the second fundamental form H of $\mathcal{S}$ at p and with respect to the metric $\tilde{h}_p$ is bounded above by a numeric constant. These estimates for the second fundamental form on $B_{h^*}(\partial N, 2/3) \setminus B_{h^*}(\partial N, 1/3)$ give already Lipschitz estimates inside this region. Once these Lipschitz estimates are guaranteed one gets Lipschitz estimates by just reproducing the same argument as in Proposition 5 but to the region $N \setminus B_{h^*}(\partial N, 1/2)$. Thus one gets

Proposition 6 *Let (N, h) be a three-manifold with boundary. Assume that (N, h) satisfy conditions 1 and 2 above. Assume that $\|Ric\|_{L^p_g} \leq \Lambda$. Let $\mathcal{S}$ be a minimal surface in N which is area minimizing in its isotopy class and with $\partial\mathcal{S} \subset \partial N$. Then for any $\alpha > 0$ there are $\bar{\epsilon}(\Lambda_1, \Lambda, \alpha, \delta) > 0$ and $C(\Lambda, \Lambda_1, \alpha, \delta) > 0$ such that for any p in $\mathcal{S} \cap (N \setminus B_{h^*}(\partial N, 1/2))$ and with $\nu^\delta(p) \leq \bar{\epsilon}$ it is $L^\alpha(p)/\nu^\delta(p) \geq C$.*

Remark 4 *Let $\mathcal{S}$ be a surface as in the Proposition above. Then, there are $A_0(\bar{\epsilon})$ such that for any p in $\mathcal{S} \cap (N \setminus B_{h^*}(\partial N, 1/2))$ with $\nu^\delta(p) \leq \bar{\epsilon}$ it is $Area(\mathcal{S}) \geq A_0(\nu^\delta(p))^2$.*

Proof (of Proposition 4):

Assume that $\bar{\epsilon}(\Lambda, \alpha)$ has been chosen to satisfy Proposition 5 and less than the $\bar{\epsilon}$ that was fixed before. Let $\bar{\epsilon}'$ be such that any K-torus in $M_{\bar{\epsilon}'}$ has area less than $(A_0\bar{\epsilon}^2)/2$. Consider two elementary tori T^2_1 and T^2_2 in $M_{\bar{\epsilon}'}$ that are joined by a curve starting at T^2_1 passing through T^2_2 and ending at $\partial M_{\bar{\epsilon}}$, and, intersecting also the tori only once. Consider the connected region formed by all the elementary sectors touching the curve, denote it by $\mathcal{R}$. The boundary of $\mathcal{R}$ consists of elementary tori some of which may be K-tori and some of which may be not (and therefore lying in the boundary of $M_{\bar{\epsilon}}$).

We deform now the metric g at the boundary of the region. If a boundary component is a K-torus then deform it following *Adjustment 3*. If instead a boundary component is a component of $\partial M_{\bar{\epsilon}}$ then we deform g to satisfy Conditions 1 and *2* (and therefore Remark 3.

Consider now the torus T^2_2. If it is isotopic to T^2_1 there is nothing to prove. Thus we have to prove that it is impossible that T^2_2 is not isotopic to T^2_1. We first note

[6]In [14] the manifold N is assumed to be strictly convex, but this can be substituted by just mean convex.

that there is a positive lower bound for the areas of the surfaces isotopic to T_2^2 in the region $\mathcal{R}$. To see that we use a theorem due to Schoen and Yau [15] saying that if the inclusion $i : T_2^2 \rightarrow \mathcal{R}$ induces a one-to-one homomorphism in π_1 then there is a minimal surface which minimizes the area functional among all the maps from T^2 in $\mathcal{R}$ inducing the same homomorphism in π_1. Thus one can take as the positive lower bound the area of such surface. Define $\bar{A}$ the limit of $A(T^2)$ where T^2 varies in the set of all tori isotopic to T_2^2. Now, a theorem due to Meeks-Simon-Yau [16] shows that there is a finite set of (non-necessarily oriented) compact minimal surfaces $\{\mathcal{S}_1, \ldots, \mathcal{S}_J\}$ with total area $\bar{A}$ and multiplicities $n_1, \ldots, n_J$, which are the measure theoretical limit of an isotopy of $\mathcal{S}$. We claim that $J = 1$, $n_1 = 1$ and that that the limit is isotopic to T_2^2. Indeed following [16], there is a sequence of γ-reductions $\tilde{\mathcal{S}}_k <_\gamma \tilde{\mathcal{S}}_{k-1} <_\gamma \ldots <_\gamma T_2^2$ of T_2^2 such that $\tilde{\mathcal{S}}_k$ is isotopic to $n_1\mathcal{S}_1 + \ldots + n_J\mathcal{S}_J$ (see *Remark* (3.27) in [16]). Topologically, a γ-reduction of a surface $\mathcal{S}$ consists of excising a strip ($\sim S^1 \times [0,1]$) and gluing back two discs (say D_1 and D_2) in such a way that these two discs, together with the strip, bound an open ball which does not intersects $\mathcal{S}$. By Lemma 2 the inclusion $i : T_2^2 \rightarrow M$ induces a one-to-one map in π_1, therefore one of the boundaries of the strip (there are two and are isotopic in $\mathcal{S}$) has to bound a disc (say D) in $\mathcal{S}$. Now, D, together with the disc that among D_1 or D_2 has the same boundary as D, form a surface homeomorphic to the two-sphere. By Lemma 2 such sphere bounds a three-ball. Thus a γ-reduction of a surface isotopic to T_2^2 produces as an outcome a sphere and a torus, with the torus isotopic to T_2^2. Again by Lemma 2 the sphere must be isotopically contractible to a point and will not count in the limit.

On the other hand [16] the limit $\mathcal{S}_1 \sim T_2^2$ (where $\sim$ here mean equivalence up to isotopy) is stable, but stable surfaces with genus greater than zero do not exist on manifolds with a positive lower bound on the scalar curvature. The proof of Proposition 4 would then be finished if we can show that $\mathcal{S}_1$ does not enters the region near the boundary of $\mathcal{R}$ where the metric was deformed. Note first that T_2^2 is not isotopic to any of the K-torus not in $\partial M_{\bar{\epsilon}}$ that may form part of the boundary of the region $\mathcal{R}$. If this were the case then such torus has to be T_1^2 which we have assumed is not (note for that, that the curve joining T_1^2, T_2^2 and $\partial M_{\bar{\epsilon}}$ intersects T_2^2 only once, and therefore has intersection number one). Thus by *Adjustment 3* $\mathcal{S}_1$ cannot enter the region where the metric g was deformed near any K-torus in the boundary of $\mathcal{R}$. If instead $\mathcal{S}$ gets close enough to $\partial M_{\bar{\epsilon}}$ that it touches the region near $\partial M_{\bar{\epsilon}}$ where g was modified, then according to Remark 3 the area of $\mathcal{S}_1$ would be greater than the area of T_2^2 which is impossible.

□

2.4 Stable minimal tubes and the final argument in the proof of Theorem 1.

We need one more result. What we shall show is that if one of the boundary curves of a stable minimal tube (joining two small curves, almost geodesic loops) is very collapsed an lies in a region which is essentially one-collapsed (Proposition 7) or two collapsed (Proposition 8), then (at the right scale) the tube is close to a flat tube. As mentioned in the introduction, this result is necessary in order to use size relations on stable minimal surfaces to prevent three-dimensional collapse.

In order to simplify notation let $D(r) = B((0,0),r)$ in $\mathbb{R}^2$. On $(\mathbb{R}^2 \setminus D(1)) \times S^1$ consider the coordinates (r,θ,ϕ) where (r,θ) are polar coordinates in $\mathbb{R}^2$ and ϕ is the arc length in the factor S^1. Let $l(\theta_0, r_0)$ be the ray $l(\theta_0, r_o) = \{(t,\theta,\phi)/1 \leq r \leq r_0, \theta = \theta_0, \phi = 0\}$. On $(\mathbb{R}^2 \backslash D(1)) \times S^1$ consider the metric $g = r^2 d\theta^2 + dr^2 + f(r)^2 d\theta_1^2$ where $f : [1,\infty) \to [1,2]$ is such that $f(r) = 1$ when $r \geq 2$, $f(1) = 2$, $f'(1) = -4$, f is decreasing and smooth. With this function f, the boundary $(\partial D(1)) \times S^1$ has mean curvature equal to one in the outward direction. Consider now a minimal surface $\mathcal{S}$ given by the embedding $\varphi : [1,\infty) \times S^1 \to (\mathbb{R}^2 \backslash D(1)) \times S^1$ with $\varphi(\{1\} \times S^1)$ a curve in $(\partial D(1)) \times S^1$ and isotopic to the factor S^1. Suppose that $\mathcal{S}$ minimizes area among isotopies of compact support, i.e.: for every domain Ω of compact closure and homotopy $\bar{\varphi} : [0,1] \times [1,\infty) \times S^1 \to (\mathbb{R}^2 \setminus D(1)) \times S^1$ leaving $\Omega^c \cap \mathcal{S}$ invariant we have $\Omega \cap \varphi(t, \mathcal{S}) \geq \Omega \cap \mathcal{S}$. Under this setup we will prove below that outside $(D(2) \setminus D(1)) \times S^1$, $\mathcal{S}$ is the metric product of the ray $l(\theta)$ and S^1.

Proposition 7 *The surface $\mathcal{S}$ considered in the paragraph above is the metric product of a ray $l(\theta)$ and S^1.*

Proof:

We note first the following direct comparison. Let $\mathcal{S}$ be an embedded surface isotopic to $l(r,\theta) \times S^1$ in $(D(r) \setminus D(1)) \times S^1$, then $Area(\mathcal{S}) \leq Area(l(r,\theta) \times S^1)$ (just project $\mathcal{S}$ along θ into $l(r,\theta) \times S^1$). Another comparison we will use is this. Let C be an embedded curve in $(\partial D(r)) \times S^1$ isotopic to S^1 and consider the circle $\{(r,0)\} \times S^1$. These two curves enclose a region in $(\partial D(r)) \times S^1$ whose area is less than $2\pi r$. An important consequence of these two comparisons is the following. *Let $\varphi : [1,\infty) \times S^1$ be the embedding defining $\mathcal{S}$. Consider the composition $r \circ \varphi$ and a regular value $r_0 > 2$. Then $(r \circ \varphi)^{-1}(r_0)$ consists of a set of closed curves in $(1,\infty) \times S^1$ only one of which (say C) is isotopic to the factor S^1 and is closest to $\{1\} \times S^1$. Consider the region Γ enclosed by it and $\{1\} \times S^1$, then $Area(\varphi(\Gamma)) \leq Area(l(r_0) \times S^1) + 2\pi r_0 \leq 2(r_0 - 1) + r\pi r_0$.* It follows that the area of the projection of $\varphi([1,r_0] \times S^1)$ into the (x,y)-plane has the same upper bound.

Indeed one can show that there is no curve in the region enclosed by C and $\{1\} \times S^1$. To see this assume that there is another curve C_1 bounding a disc $\tilde{\Delta}$. If C_1 encloses some other curves $\{C_2, \ldots, C_J\}$ then consider the region Δ enclosed by $\{C_1, C_2, \ldots, C_J\}$, if not consider the disc $\tilde{\Delta}(= \Delta)$. The region Δ gets mapped into a piece of $\mathcal{S}$ in $(\mathbb{R}^2 \setminus D(r_0)) \times S^1$. We claim that such piece has area greater than the region $\bar{\Delta}$ enclosed by $\varphi(C_1)$ in $(\partial D(r_0)) \times S^1$. The claim follows from two facts: (i) the projection from $\varphi(\Delta)$ into $(\partial D(r_0)) \times S^1$ along radial geodesics (i.e. perpendicular to $(\partial D(r_0)) \times S^1$) is area decreasing and that the projection of $\varphi(\Delta)$ covers the full $\bar{\Delta}$, and (ii) $\bar{\Delta}$ is isotopic to $\varphi(\tilde{\Delta})$ through an isotopy leaving invariant $\varphi(([1,\infty)\times S^1)\setminus\tilde{\Delta})$. Thus there are no curves between C and $\{1\}\times S^1$. Denote the region enclosed by them as Γ_{r_0}.

For any $n>2$ consider a regular value r_0 of $r\circ\varphi$ in $[n,n+1)$ and consider the cover of $(D(r_0)\setminus D(1))\times S^1$ where the S^1 factor has been unwrapped n-times. Denote it by $U(n,r_0)$. Consider the lift $\mathcal{S}_n$ of $\mathcal{S}$ and the lift $\varphi(\Gamma_{r_0})_n$ of $\varphi(\Gamma_{r_0})$. From the observation above it follows that the area of the connected component of $\mathcal{S}_n\cap U(n,r_0)$ is less or equal than cn^2 for some numeric c (i.e. quadratic growth). From standard estimates for the decay of the second fundamental form of $\mathcal{S}$ and the quadratic area growth shown before [11], it follows that the sequence of surfaces $\{\varphi(\Gamma_{r_0})\}$ converges (as n diverges) on the scaled spaces $\{U(n,r_0),(\frac{1}{n^2})g\}$ to a surface $\mathcal{S}_\infty$ which is scale invariant and whose projection into the (x,y)-plane has at most linear (in r) area growth[7]. It follows that the projection must be a set of lines thourgh the origin with measure zero (otherwise, as the projection is scale invariant, if it were not a set of lines with measure zero, it would have quadratic area growth). We do not get from here that the projection of $\mathcal{S}$ is a single line as we want. To conclude that we observe that the projection of $\varphi(\Gamma_{r_0})\cap((\partial D(r_0))\times S^1)$ into the (x,y)-plane, as connected, is concentrating on a narrowing set of angles (θ_n^1,θ_n^2) (where θ is the arc length of the projection of $(\partial U(n,r_0))$ into the (x,y)-plane and in the metric $(1/n^2)g$). Following the same comparison as in the beginning, the surface $\varphi(\Gamma_{r_0})$ must therefore be enclosed in the region between $l(\theta_n^1)\times S^1$ and $l(\theta_n^2)\times S^1$. As $|\theta_n^1-\theta_n^2|\to 0$, it follows that $\mathcal{S}$ must be as we claimed.

□

The same argument also shows that

Proposition 8 *An embedded minimal surface $\varphi:[0,\infty)\times S^1\to T^2\times[0,\infty)$, with $\varphi(\{0\}\times S^1)$ an embedded and non-contractible curve in $T^2\times\{0\}$, minimizing area among isotopic variations of compact support, must be the product of a geodesic loop (in the same isotopy class as the curve) in T^2 and $[0,\infty)$.*

We proceed now to close the argument by contradiction of the proof of Theorem 1.

Proof (of Theorem 1):

Suppose that the claim of Theorem 1 is false and there is a pointed sequence $\{(M_i,g_i,o_i)\}$ with $\nu^\delta\to 0$, $\nu^\delta_{g_i}(o_i)\geq\nu_0$, $\|Ric_{g_i}\|_{L^p_{g_i}}\leq\Lambda$ and $R_{g_i}\geq R_0>0$. Extract a limit space $\{(\bar{M},\bar{g},\bar{o})\}$. We are going to consider on it the ends that follow from Corollary 2.

If an end is of the type as in *item 1* or *item3* of Corollary 2 then proceed as follows. Pick a sequence $\{T_i^2\}_{i=1}^{i=\infty}$ of K-tori in the end whose sizes go to zero, i.e. the length of its shortest geodesic loop goes to zero. Consider the region enclosed by T_1^2 and T_n^2 that we know is diffemorphic to $T^2\times[-1,1]$ and denote it by $[T_1^2,T_n^2]$.

Let $\{p_n\}$ be a sequence of points in T_n^2 and let $\{(F_n,g_{F_n})\}$ be the sequence of associated spaces. We know there is no S^1-soul in F_n at a distance $d/2$ from p_n

[7] The limit surface may degenerate at the origin but this is not a problem.

(in the metric g_{F_n}). But the same must be for any $d' > d$ and for n sufficiently large for otherwise there would be a pseudo-cusp (of size $d'/2$) closing the end and this is impossible. Thus (F_n, g_{F_n}) converges to a product $T^2 \times \mathbb{R}$. Deform now the metric $\bar{g}$ only inside a 1-neighborhood in the metric $\bar{g}^*$ of T_1^2 and T_n^2, to a smooth metric $\bar{g}_D$ in such a way that

1. satisfy *Conditions 1* and *2*,
2. the metric $\tilde{\bar{g}}_{D,p_n}$ where $p_n \in \ell_n$ converges strongly in $H^{2,p}$ to a product metric in $T^2 \times [0, \infty)$ (for some flat metric in the first factor),
3. the metric $\tilde{\bar{g}}_{D,p_n}$ where $p_n \in \ell_n$ converges strongly in C^2 inside $B_{\bar{g}^*}(T_n^2, 1/2)$ (of course with the same flat metric in the first factor as in the previous *item*).

Let ℓ_n be the shortest geodesic loop of T_n^2. Consider the class of all embedded tubes $\phi([0,1] \times S^1) \rightarrow [T_1^2, T_n^2]$ such that $\phi(\{0\} \times S^1)$ is isotopic to ℓ_n in T_n^2 and $\phi(\{1\} \times S^1)$ is a closed curve in T_1^2. Let $\mathcal{S}_n$ be an embedded surface that has minimal area in the class. The surfaces $\mathcal{S}_n$, inside $(([T_n^2, T_1^2], \tilde{\bar{g}}_{D,p_n}))$ thus converge to a stable surface in the flat product $T^2 \times [0, \infty)$ which by Proposition 8 must be a flat tube, i.e. $\lim \mathcal{S}_n = S^1 \times [0, \infty)$. Consider the surfaces $\bar{\mathcal{S}}_n$ which, for given n, are simply the domains of $\mathcal{S}_n$ made of points where the metric $\bar{g}$ was not deformed (and therefore at those points we have $R \geq R_0 > 0$). Note that these surfaces are stable and by Remark 4 their area is uniformly bounded below. The *non-collapse at a finite distance property* of stable surfaces (given at the introduction) applied to the surfaces $\bar{\mathcal{S}}_n$ now shows that such a sequence of surfaces does not exists, thus ruling out the ends of type as in *items 1* and *3* of Corollary 2.

The analysis of the other types of ends follows the same principle, we discuss them next. Suppose instead an end is of type as in *item 2* of Corollary 2. We claim that there is a sequence $\{p_i\}$ such that $l(p_i) \rightarrow 0$ and a sequence of associated spaces $\{(F_{p_i}, g_{p_i}, p_i)\}$ converging either to a metric product of the form $T^2 \times \mathbb{R}$ or $S^1 \times \mathbb{R}^2$ (with some flat metric on the factor T^2 which necessarily does not makes it a K-torus). To see this, note that if not then for any $d' > d$ there exists $\tilde{\epsilon}(d')$ such that for every q in the set $\{p$ inside the given end$/l(p) \leq \tilde{\epsilon}(d')\}$ there is an associated space (F_q, g_q) having a soul at a distance less or equal than $d'/2$ and $\alpha > 1/(2d')$. Now given a sequence $\{p_i\}$ with $l(p_i) \rightarrow 0$ one can do the same construction of pseudo-cusps as in Proposition 3 but inside the region $M_{\tilde{\epsilon}}$ (and stopping when reaching $\partial M_{\tilde{\epsilon}}$) to conclude that each point p_i lies in a d'-pseudo-cusp (surrounded by a torus touching $\partial M_{\tilde{\epsilon}}$) and thus contradicting the fact that $\nu^\delta(p_i) \rightarrow 0$. We will work from now on with the sequence $\{p_i\}$.

The analysis of the ends as in *item 2* is slightly different than the analysis as in the *items 1* and *3*. Instead, in this case, we have to work directly with the spaces M_i. The reason is the following. In cases as in *items 1* and *3* we were able to define the region $[T_1^2, T_n^2]$, which (although with boundary) is complete and allows us to define the stable surfaces $\mathcal{S}_n$. For ends as in *item 2* we cannot define suitable sub-domains (with boundary) that would allow us to find stable surfaces to which we could apply the size relations. This inconvenience is solved if we incorporate the spaces M_i. The procedure is as follows. Find a subsequence $\{(M_{j_i}, g_{j_i})\}$ in such a way that the pointed spaces $\{(M_{j_i}, \tilde{g}_{j_i,p_i}, p_i)\}$ converge to the metric product $T^2 \times \mathbb{R}$ or $S^1 \times \mathbb{R}^2$ (strictly speaking the sequence $\{p_i\}$ is not the same as the one considered before).

We have thus two situations, let us consider first the case when the limit of $\{(M_{j_i}, \tilde{g}_{j_i,p_i}, p_i)\}$ is $T^2 \times \mathbb{R}$. Consider a sequence of tori T^2_i containing the points p_i and converging to a leaf T^2 in the limit space $T^2 \times \mathbb{R}$. Let $\mathcal{R}$ be the end to which the sequence p_{i_k} belongs. Let $\mathcal{B}$ be one boundary component of $\mathcal{R}$. Let $\mathcal{R}_i$ be the elementary component to which p_i belongs. If there is a sub-sequence i_k such that the tori $T^2_{i_k}$ separate $\mathcal{R}_{i_k}$ into two connected components then we restrict to that subsequence. Denote them by $\mathcal{R}_{1,i_k}$ and $\mathcal{R}_{1,i_k}$. Suppose that $\mathcal{R}_{1,i_k}$ are the components having a sub-domains converging to $\mathcal{R}$. Let $\mathcal{B}_{i_k}$ be a boundary component converging to $\mathcal{B}$. Now we are going to deform the metrics g_{i_k} to smooth metrics in the 1-neighborhood in the metric $g^*_{i_k}$ around the boundary components of $\mathcal{R}_{1,i_k}$. Deform g_{i_k} to a metric g_{D,i_k} such that

1. at the torus $T^2_{i_k}$
   (a) satisfy *Conditions 1* and *2*,
   (b) the metrics $\tilde{g}_{D,i_k}$ converge strongly in $H^{2,p}$ to the flat metric in $T^2 \times [0,\infty)$,
   (c) the metrics $\tilde{g}_{D,i_k}$ converge strongly in the C^2 topology, inside $B_{g^*_{i_k}}(T^2_{i_k}, 1)$, to the (of course same) flat metric in $T^2 \times \mathbb{R}$.
2. if the boundary component is a K-tori, deform it according to the *Adjustment 3*. If the boundary component is instead a component of $M_{i_k,\bar{\epsilon}'}$ deform it according to *Conditions 1* and *2*.

Let ℓ_{i_k} be a sequence of length minimizing loops in $T^2_{i_k}$ based at p_{i_k}. Consider the class of embedded surfaces $\phi_{i_k} : S^1 \times [0,1] \rightarrow \mathcal{R}_{1,i_k}$ such that $\phi_{i_k}(S^1 \times \{0\})$ lies in $T^2_{i_k}$ and is isotopic to ℓ_{i_k}, and $\phi_{i_k}(S^1 \times \{1\})$ is a smooth curve in $\mathcal{B}_{i_k}$. Let $\mathcal{S}_{i_k}$ be an embedded surface of minimal area. We consider again the surfaces $\bar{\mathcal{S}}_{i_k}$ which for each i_k is simply the sub-domain in $\mathcal{S}_{i_k}$ which consists of those points on which the metric g_{i_k} was not modified. As the surfaces $\mathcal{S}_{i_k}$ cannot get close to the K-tori in the boundary of $\mathcal{R}_{1,i_k}$ (except $\mathcal{B}_{i_k}$ if it is a K-torus), such points are either those in $B_{g^*_{i_k}}(T^2_{i_k}, 1)$ or in $B_{g^*_{i_k}}(\partial M_{i_k,\bar{\epsilon}'}, 1)$. Thus from Remark 4 we know that the areas of $\bar{\mathcal{S}}_{i_k}$ are uniformly bounded below and by Proposition 8 they get closer and closet to flat tubes at $T^2_{i_k}$ (and at the right scale). The non-collapse at a finite distance property of stable surfaces rule out the existence of these kind of surfaces.

If instead the tori T^2_i do not separate $\mathcal{R}_i$ then we remove T^2_i thus adding two new boundaries to $\mathcal{R}_i$ diffeomorphic to it. We proceed in the same way as in the previous case. The only thing to note is that the surfaces $\bar{\mathcal{S}}_i$ cannot touch the region near the second copy of T_i^2 where the metric g_i was modified, for the same reason that they do not get close to boundary components which are K-tori.

We discuss now the possibility when the limit space obtained at p_i is of the form $\mathbb{R}^2 \times S^1$. For each p_i pick a S^i fiber through it. Remove the balls $B_{g_i^*}(S_i^1, 1)$ and in the region $B_{g_i^*}(S_i^1, 3) \setminus B_{g_i^*}(S_i^1, 1)$ deform the metric g_i^* to have the metric $r^2 d\theta^2 + dr^2 + f(r)^2 d\theta_1^2$ as was considered in Proposition 7. Let ℓ_i be the shortest geodesic loop at $\partial B_{g_i^*}(S_i^1, 1)$ which is isotopic (in M_i) to S_i^1. Again we consider an area minimizing tube $\mathcal{S}_i$ joining a curve isotopic to ℓ_i with a curve in $\mathcal{B}_i$. As $i \to \infty$ we have that by Proposition 3 the surfaces $\mathcal{S}_i$ approach a metric product of a ray times S^1 in $(\mathbb{R}^2 \setminus D^2) \times S^1$ (with the metric $r^2 d\theta^2 + dr^2 + f(r)^2 d\theta_1^2$). If, as before, we restrict to the stable surfaces $\bar{\mathcal{S}}_i$ we reach again an impossibility as these kind of surfaces are ruled out by the non-collapse at a finite distance property of stable surfaces.

□

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