Condition number and random point configurations on the sphere

PhD Thesis

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Abstract

In this dissertation, we analyze two different approaches to the problem of solving systems of polynomial equations, along with some geometric and probabilistic tools needed for these approaches.

In the first part of this thesis, we study the average conditioning of a random underdetermined polynomial system. We compare the expected values of the moments of the condition number to those of random matrices. An expression for these moments is obtained by analyzing the kernel-finding problem for random matrices. Furthermore, we compute the second moment of the Frobenius condition number.

In the second part of this dissertation, we shift our attention to the solution sets of random polynomials arising from the polynomial eigenvalue problem for random matrices. For these solution sets, we compute the expected logarithmic energy. We generalize known results for the Shub-Smale polynomials and the spherical ensemble. These two processes represent the two extremal cases of the polynomial eigenvalue problem, and we prove that the logarithmic energy lies between them. In particular, the roots of the Shub-Smale polynomials have the lowest logarithmic energy within this family.

Resumen

En esta tesis analizamos dos enfoques diferentes para el problema de resolver sistemas de ecuaciones polinomiales, así como algunas herramientas geométricas y probabilísticas necesarias para estos enfoques.

En la primera parte de este trabajo, estudiamos el condicionamiento promedio de un sistema polinomial aleatorio indeterminado. Comparamos los valores esperados de los momentos del número de condición con los correspondientes a matrices aleatorias. Esta relación se obtiene mediante el análisis del problema de encontrar el núcleo de matrices aleatorias. En particular, se calcula el segundo momento del número de condición de Frobenius.

En la segunda parte de esta tesis, centramos nuestra atención en los conjuntos de solucion de polinomios aleatorios que surgen al considerar el problema de autovalores polinomiales para matrices aleatorias. Para estos conjuntos de soluciones, calculamos la energía logarítmica esperada. Generalizamos algunos resultados conocidos para los polinomios de Shub-Smale y el spherical ensemble. Estos dos procesos representan los casos extremos del problema de autovalores polinomiales, y demostramos que la energía logarítmica se encuentra entre estos dos extremos. En particular, las raíces de los polinomios de Shub-Smale son las que presentan la menor energía logarítmica dentro de esta familia.

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Contents

Chapter 1

Introduction

The problem of solving systems of polynomial equations is a classical subject with a rich and influential history. It played a pivotal role in the development of complex numbers and group theory and served as a major motivation behind the emergence of Algebraic Geometry and Abstract Algebra.

This thesis is closely connected to the study of polynomial systems. Specifically, we explore two distinct aspects of this problem. First, we investigate the average conditioning for random underdetermined polynomial systems. Later, we shift our attention to the solution sets of random polynomials chosen according to a given probability distribution.

The next two sections provide an overview of these two approaches and highlight the main contributions of this dissertation.

1.1 Condition number

Solving systems of equations is a fundamental problem, which has been deeply studied from different points of view, such as algebraic, geometric and numerical approaches.

A classic numerical method for solving such systems, is the so-called Newton's iteration. Shub and Smale introduced Newton's operator for underdetermined systems of equations in their work [SS4] (cf Dégot [D3]). The primary objective of their efforts was to develop and analyse effective algorithms for computing approximations to complete intersection algebraic subvarieties of $\mathbb{C}^n$.

The condition number associated with a numerical problem measures the sensitivity of the considered problem to variations of the input (see Blum et al.

[BCSS], Bürgisser-Cucker [BC]). The condition number was introduced by Turing [T] and von Neuman-Goldstine [vNG], while studying the propagation of errors for linear equation solving and matrix inversion.

We study here polynomial systems $f : \mathbb{C}^n \to \mathbb{C}^r$ with $r \leq n$. When $r = n$, if $x \in \mathbb{C}^n$ is a simple zero of f and $\dot{f}$ is a first order variation of the system, the corresponding first order variation $\dot{x}$ of x is equal to $\dot{x} = K(f,x)\dot{f} = -Df(x)^{-1}\dot{f}(x)$, where $Df(x)$ is the derivative of f at x. The map $K(f,x)$ is linear and the condition number $\mu(f,x)$ of this problem is the operator map of this map

$$\mu(f,x) := \max_{\dot{f} \neq 0} \frac{\|Df(x)^{-1}\dot{f}(x)\|}{\|\dot{f}\|}.$$

Shub and Smale in their famous series "Complexity of Bézout Theorem", show that the complexity of a certain continuation algorithm for solving the polynomial system $f(x) = 0$ depends mainly on the condition number $\mu(f,x)$. Thus, the condition number appears as a crucial continuous invariant, related to f and x, which measures the complexity of solving polynomial systems. Ever since then, condition numbers have played a leading role in the study of both accuracy and complexity of numerical algorithms.

As pointed out by Demmel [D4], computing the condition number of any numerical problem is a time-consuming task that suffers from intrinsic stability problems. For this reason, understanding the behaviour of the condition number in such a way that we can rely on probabilistic arguments is a useful strategy.

Thus, in the first part of this thesis, we focus on understanding the moments of the condition number for random underdetermined polynomial systems. Namely, we compared said moments to the ones of the condition number for random matrices. In order to be more precise in our statement we need to introduce some preliminary notations.

1.1.1 Preliminaries

For every positive integer $d \in \mathbb{N}$, let $\mathcal{H}_d^n$ be the complex vector space of all homogeneous polynomials of degree d in $(n+1)$-complex variables with coefficients in $\mathbb{C}$.

We denote by a multi-index $j := (j_0, \cdots, j_n) \in \mathbb{Z}^{n+1}$, $j_i \geq 0$ for $i = 0, \cdots, n$, and consider $|j| = j_0 + \cdots + j_n$. Then, for $x = (x_0, \cdots, x_n) \in \mathbb{C}^{n+1}$, we write

$$x^j := x_0^{j_0} \cdots x_n^{j_n}.$$

We consider the Bombieri-Weyl Hermitian product in $\mathcal{H}_d^n$, defined as follows. Let $h, g \in \mathcal{H}_d^n$, be two elements, $h(x) = \sum_{|j|=d} a_j x^j$, $g(x) = \sum_{|j|=d} b_j x^j$, we define

$$\langle h, g \rangle_d = \sum_{|j|=l} a_j \bar{b}_j \binom{d}{j}^{-1},$$

where $\binom{d}{j} = \dfrac{d!}{j_0! \cdots j_n!}$ (see Shub-Smale [SS1]).

For any list of positives degrees $(d) := (d_1, \cdots, d_r)$, $r \leq n$, let

$$\mathcal{H}_{(d)}^{r,n} := \prod_{i=1}^{r} \mathcal{H}_{d_i}^n$$

be the complex vector space of homogeneous polynomial systems $h := (h_1, \cdots, h_r)$ of respective degrees d_i.

We denote by $\mathcal{D}_r$ the Bézout number associated with the list (d), i.e.

$$\mathcal{D}_r := \prod_{i=1}^{r} d_i,$$

when $n = r$ this is the number of generic roots of the system.

The previously defined Hermitian product induces a Hermitian product in $\mathcal{H}_{(d)}^{r,n}$ as follows. For any two elements $h = (h_1, \cdots, h_r)$, $g = (g_1, \cdots, g_r) \in \mathcal{H}_{(d)}^{r,n}$, we define

$$\langle h, g \rangle := \sum_{i=1}^{r} \langle h_i, g_i \rangle_{d_i}.$$

The Hermitian product $\langle \cdot, \cdot \rangle$ induces a Riemannian structure in the space $\mathcal{H}_{(d)}^{r,n}$.

The space $\mathbb{C}^{n+1}$ is equipped with the canonical Hermitian inner product $\langle \cdot, \cdot \rangle$ which induces the usual Euclidean norm $\| \cdot \|$, and we denote by $\mathbb{P}(\mathbb{C}^{n+1})$ its associated projective space. This is a smooth manifold which carries a natural Riemannian metric, namely, the real part of the Fubini-Study metric on $\mathbb{P}(\mathbb{C}^{n+1})$ given in the following way: for a non-zero $x \in \mathbb{C}^{n+1}$,

$$\langle w, w' \rangle_x := \frac{\langle w, w' \rangle}{\|x\|^2},$$

for all w, w' in the Hermitian complement $x^{\perp}$ of x. This induces the norm $\| \cdot \|_x$ in $T_x\mathbb{P}(\mathbb{C}^{n+1})$.

The space $\mathcal{H}_{(d)}^{r,n} \times \mathbb{P}(\mathbb{C}^{n+1})$ is endowed with the Riemannian product structure (see Blum et al. [BCSS]).

1.1.2 Average Conditioning

The condition number associated to a computational problem measures the sensitivity of the outputs of the considered problem, to variations of the input (see Bürgisser-Cucker [BC]). In [D2] Dedieu defined the condition number of a polynomial $f:\mathbb{C}^n\to\mathbb{C}$ at a point $x\in\mathbb{C}^n$, such that $f(x)=0$ and $Df(x)$ is surjective, as

$$\mu(f,x) := \|Df(x)^{\dagger}\|_{op},$$

where $Df(x)^{\dagger}$ is the Moore-Penrose pseudo inverse of the linear map $Df(x)$, i.e. the derivative of f at x, and $\|Df(x)^{\dagger}\|_{op}$ is the operator norm of $Df(x)^{\dagger}$.

Following this idea, and using the normalized condition number μ_{norm} introduced in Shub-Smale [SS1], Dégot [D3] suggested an extension of this condition number for the undetermined case which was adjusted into a projective quantity by Beltrán-Pardo in [BP1].

As done in [BS1], we will consider the Frobenius condition number, just by considering the Frobenius norm instead of the operator one.

Given $h\in\mathcal{H}_{(d)}^{r,n}$ and $x\in\mathbb{C}^{n+1}$ such that $h(x)=0$ and $Dh(x)$ has rank r, then the Frobenius condition number of h at x is defined by

$$\mu_F^r(h,x) := \|h\|\|Dh(x)^{\dagger}\Delta(d_i^{1/2}\|x\|^{d_i-1})\|_F,$$

where $\|\cdot\|_F$ is the Frobenius norm (i.e. $\mathrm{trace}(L^*L)^{1/2}$ where L^* is the adjoint of L). If the rank of $Dh(x)$ is strictly smaller than r, we set $\mu_F^r(h,x):=\infty$.

When $r=n$, we will write $\mu_F(h,x)$ instead.

Let $\Sigma' := \{(h,x)\in\mathcal{H}_{(d)}^{r,n}\times\mathbb{C}^{n+1} \,:\, h(x)=0;\ \mathrm{rank}(Dh(x))<r\}$ and $\Sigma\subseteq\mathcal{H}_{(d)}^{r,n}$ be the projection of Σ' onto the first coordinate, commonly referred as the discriminant variety. Observe that for all $h\in\mathcal{H}_{(d)}^{r,n}\setminus\Sigma$, thanks to the inverse image of a regular value theorem, the zero set

$$V_h := \{x\in\mathbb{P}(\mathbb{C}^{n+1}) \,:\, h(x)=0\},$$

is a complex smooth submanifold of $\mathbb{P}(\mathbb{C}^{n+1})$ of dimension $n-r$. Then it is endowed with a complex Riemannian structure that induces a finite volume form.

Now, for $h\in\mathcal{H}_{(d)}^{r,n}\setminus\Sigma$ it makes sense to consider the 2-nd moment of the Frobenius condition number of h, $\mu_{F,Av}^{r,2}(h)$, as the average of $(\mu_F^r(h,x))^2$ over its zero set V_h, i.e.

$$\mu_{F,Av}^{r,2}(h) := \frac{1}{\mathrm{vol}(V_h)}\int_{x\in V_h}\mu_F^r(h,x)^2\,dV_h. \tag{1.1}$$

1.1.3 Main Contributions

We study the average conditioning for a random underdetermined polynomial system. The expected value of the moments of the condition number are compared to the moments of the condition number of random matrices. An expression for these moments is given by studying the kernel finding problem for random matrices. Furthermore, the second moment of the Frobenius condition number is computed.

The main contribution of the thesis in this direction gives a closed formula for the expected value of $\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}$. To be accurate in this notion, we need to fix a probability measure in $\mathcal{H}_{(d)}^{r,n}$.

Consider the average with respect to the standard Gaussian distribution on $\mathcal{H}_{(d)}^{r,n}$, that is,

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\phi(h)) = \frac{1}{\pi^N}\int_{h\in\mathcal{H}_{(d)}^{r,n}} \phi(h)e^{-\|h\|^2}\,dh,$$

where N is the complex dimension of $\mathcal{H}_{(d)}^{r,n}$ and $\phi : \mathcal{H}_{(d)}^{r,n} \to \mathbb{R}$ is a measurable function.

Theorem (I). *The expected value, with respect to the standard Gaussian distribution, of the 2-nd moment of the relative Frobenius condition number* $\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}$ *is*

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}\left(\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}\right) = \frac{r}{n-r+1}.$$

As a matter of fact, we will be proving a more general result (see Chapter 4). That result can be extended to the case where we consider the operator norm. Furthermore, after some computations (see Chapter 3), we get the closed expression for the case of the 2-nd moment stated in Corollary 4.4.4. The proof of the general result strongly relies on Theorem 4.4.6, which states that the moments of the condition number for the polynomial case are essentially the moments of the condition number of a random matrix.

Remark 1.1.1. Observe that by taking $r = n$ in the previous statement, one recovers the average of the 2-nd moment of the relative Frobenius condition number for the determined case, namely

$$\mathop{\mathbb{E}}_{g\in\mathcal{H}_{(d)}^{n,n}}\left(\frac{\mu_{F,Av}^{2}(g)}{\|g\|^2}\right) = n$$

(see [ABB+1, Theorem 2]).

This statement provides the expected value of the 2-nd moment of the relative condition number for the underdetermined case in terms of the expected one in the determined case. From a geometric perspective, these two cases exhibit notable distinctions, and there is no inherent requirement for these expected values to be in any kind of relation. It would be interesting to understand which are the reasons behind this relation.

Remark 1.1.2. In [BP1, Theorem 1.4], an upper bound of the expected value of the average conditioning is computed, while using our argument we get an equality. Furthermore, using Cauchy-Schwartz inequality and Corollary 4.4.4, we get a sharper bound.

1.2 Random points configurations on the sphere

The problem of finding configurations of points in the 2-dimensional sphere with small logarithmic energy is a very challenging problem, with several applications. It is one of the problems listed by Smale for the XXI Century [S], and there have been several advances in different directions related to this problem.

The original motivation for this problem, according to [S], was the search for well-conditioned homogeneous polynomials as in [SS3]. It is proved in [SS2] that well-conditioned polynomials are highly probable. In [SS3] the problem was raised as to how to write a deterministic algorithm which produces a polynomial g such that all of its roots are well conditioned, this question was answered by Beltrán et al. in [BEMOC]. It was also realized that a polynomial whose roots (seen in the Riemann sphere) have low logarithmic energy is well conditioned. So one would like to use this relation in the other sense, from a well-conditioned polynomial construct a low logarithmic energy configuration on the sphere.

Given N points in $\mathbb{R}^3$, the logarithmic energy of the configuration is defined as

$$V(x_1,\ldots,x_N) = -\sum_{1\le i<j\le N} \ln \|x_i - x_j\|.$$

The problem of minimizing this energy in the unit sphere S^2 is considered a very hard optimization problem, also known as the Fekete problem. Not only are the configurations of points that minimize the energy not completely understood even for a small number of points (for instance, $N = 7$), but also the asymptotic value of the minimum is not known with enough precision. More precisely, let

$$V_N = \min_{x_1,\dots,x_N \in S^2} V(x_1,\dots,x_N)$$

be the minimum of the energy in the sphere. The 7th Smale problem consists of finding a configuration of points $x_1,\dots,x_N$ in the sphere, in polynomial time in N, such that its logarithmic energy $V(x_1,\dots,x_N)$ is close enough to the minimum, namely $V(x_1,\dots,x_N) - V_N \leq c \ln N$, for a universal constant c.

One of the major obstacles is that the value of V_N itself is not known with precision up to the $\ln N$ term, and therefore problem number 7 of Smale's list is still far from being solved.

Indeed, the value of V_N is [BS2]

$$V_N = \frac{\kappa}{2} N^2 - \frac{N \ln N}{4} + CN + o(N), \tag{1.2}$$

where $\kappa = \frac{1}{2} - \ln 2$ and C is an unknown constant. As far as it is known, this constant C is bounded (lower bound by [L, BL] and recently improved by [M1], upper bound by [BHS, BS2])

$$-0.0284228\ldots \leq C \leq \ln 2 + \frac{1}{4}\ln\frac{2}{3} + \frac{3}{2}\ln\frac{\sqrt{\pi}}{\Gamma(1/3)} = -0.027802\ldots,$$

and the upper bound is conjectured to be actually the value for C [BHS, BS2].

Despite the intrinsic difficulties of finding these optimal configurations of points, or even the value of the minimal energy, there have been some very exciting advances throughout the last decades. For instance, the diamond ensemble proposed by Beltrán and Etayo [BE], which achieves configurations of points with logarithmic energy very close to the conjectured minimum, and two other random processes, which we describe in more detail in what follows.

The first one, proposed by Armentano, Beltrán, and Shub in [ABS], consists of taking the roots of a random polynomial. Specifically, let

$$p_N(z) = \sum_{k=0}^{N} a_k \binom{N}{k}^{1/2} z^k \tag{1.3}$$

with a_k i.i.d. complex standard Gaussian coefficients $\mathcal{N}_{\mathbb{C}}(0,1)$, i.e., each coefficient is $\alpha + i\beta$, with α, β real independent zero-mean and $1/2$ variance Gaussian. Now compute the roots of p_N in $\mathbb{C}$, and project them to S^2 through the inverse stereographic projection. The authors prove that the expected logarithmic energy of the resulting ensemble in S^2 is

$$\frac{\kappa}{2}N^2 - \frac{N\ln N}{4} - \frac{\kappa}{2}N. \tag{1.4}$$

Observe that the expression coincides up to the first two terms with (1.2), and the constant for the linear term is $-\frac{\kappa}{2} \approx 0.096\ldots$.
More recently, in [MY] the authors prove a central limit theorem for the logarithmic energy resulting from this random process, where they show that the fluctuations are of order $\sqrt{N}$, and therefore a typical realization of this process will have energy close to the expression in (1.4).

The second approach consists of taking the eigenvalues of a random matrix. Specifically, let A and B be two random matrices with i.i.d. complex standard Gaussian entries $\mathcal{N}_{\mathbb{C}}(0,1)$. Now compute the eigenvalues of the matrix AB^{-1}, and project them to S^2 through the inverse stereographic projection. Alishahi and Zamani [AZ] proved that the expected value of the logarithmic energy for this configuration, so-called *spherical ensemble*, is

$$\frac{\kappa}{2}N^2 - \frac{N\ln N}{4} + \left(\frac{\ln 2}{2} - \frac{\gamma}{4}\right)N - \frac{1}{8} + O\left(\frac{1}{N}\right), \tag{1.5}$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant. Observe that the first two terms coincide with the known expression in (1.2), and the constant for the linear term is $\left(\frac{\ln 2}{2} - \frac{\gamma}{4}\right) \approx 0.2022\ldots$.

1.2.1 Main Contributions

We compute the expected logarithmic energy of solutions to the polynomial eigenvalue problem for random matrices. We generalize some known results for the Shub-Smale polynomials, and the spherical ensemble. These two processes are the two extremal particular cases of the polynomial eigenvalue problem, and we prove that the logarithmic energy lies between these two cases. In particular, the roots of the Shub-Smale polynomials are the ones with the lowest logarithmic energy of the family.

Let us consider the following polynomial in $\mathbb{C}$,

$$F(z) = \det\left(\sum_{i=0}^{d} G_i \binom{d}{i}^{1/2} z^i\right),$$

where each G_i is an $r \times r$ random matrix with independent entries distributed as $\mathcal{N}_{\mathbb{C}}(0,1)$. The problem of finding the zeros of this function is known as the *polynomial eigenvalue problem* (PEVP). Observe that $F(z)$ has, generically, $N = dr$ roots in $\mathbb{C}$, which can be projected to the unit sphere through the inverse stereographic projection as before. We will call this configuration of points the PEVP-ensemble.

Now, for a given number of points N, one can choose different pairs of its divisors (d,r) forming $N = dr$. Notably, for $r = 1$ and $d = N$, we obtain exactly the random polynomials as in (1.3). In the other extreme, for $r = N$ and $d = 1$, we obtain $F(z) = \det(G_0 + G_1 z)$, whose roots coincide with the eigenvalues of $-G_0 G_1^{-1}$, and therefore we recover the spherical ensemble.

Some numerical experiments suggest that the expected logarithmic energy of intermediate instances (meaning $1 < d < N$) lies between the energy of the two extremal cases and decreases linearly with d. The main result of this paper, which we state below, gives a precise computation of the expected logarithmic energy for the PEVP-ensemble. The numerical experiments, along with the analysis of this dependence on d, are presented in Section 5.3.

Theorem (II). *Let $F(z)$ be the random complex polynomial of degree N defined as*

$$F(z) = \det\left(\sum_{i=0}^{d} G_i \binom{d}{i}^{1/2} z^i\right),$$

where G_i are $r \times r$ matrices with i.i.d. entries following $\mathcal{N}_{\mathbb{C}}(0,1)$. Then, with the definitions above, we have

$$\mathbb{E}(V(x_1,\cdots,x_N)) = \frac{\kappa}{2}N^2 - \frac{N\log d}{4} - \frac{N}{4}\Big(1 + \psi(r+1) - \psi(2) - 2\ln 2\Big)$$

where $\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$ is the digamma function, i.e., the logarithmic derivative of the gamma function $\Gamma(n)$.

(See Section 5.2.2 for the proof.)

Observe that this result generalizes the computed expected logarithmic energies of the ensembles by [AZ] and [ABS]. Moreover, for $r = N, d = 1$, we get

$$\mathbb{E}(V(x_1,\cdots,x_N)) \quad = \quad \frac{\kappa}{2}N^2 - \frac{N\psi(N+1)}{4} + N\left(\frac{\ln 2}{2} - \frac{\gamma}{4}\right).$$

This is actually the exact value for the expected value of the spherical ensemble, which, to the best of our knowledge, had not been computed before. Using the usual approximation of $\psi(N+1)$, we obtain the same asymptotic expression as in (1.5). A more detailed asymptotic analysis of this expression is given in Section 5.3.

Remark 1.2.1. Observe that in the matrix $\sum_{i=0}^{d} G_i \binom{d}{i}^{1/2} z^i$, each entry is a Shub-Smale polynomial, like the ones in (1.3). The distribution of the roots of these polynomials, projected to S^2, are invariant under the orthogonal group. Now, since the determinant is a homogeneous polynomial in the entries of the matrix, the zeros of the resulting process $F(z)$ are also invariant under the same group of isometries (Proposition 2.1.1 of [K3]). This invariance in S^2 is a desirable property if we want the resulting points of a process in which every configuration is possible to be well distributed in the sphere.

Layout of this dissertation

In Chapter 2, we introduce the some geometric tools and explore its intricacies, applications, and related concepts. The goal is to present various contexts in which those tools can be applied, with a focus on those relevant to the development of this dissertation.

In Chapter 3, we present the probabilistic tools that underpin this thesis and carry out key computations essential to the main results, including the well-known Kac-Rice formula for random fields.

In Chapter 4, we study the average condition number for a random underdetermined polynomial system. The expected value of the moments of the condition number are compared to the moments of the condition number of random matrices. An expression for these moments is given by studying the kernel finding problem for random matrices. In particular, we compute the second moment of the Frobenius condition number.

In Chapter 5, we compute the expected logarithmic energy of solutions to the polynomial eigenvalue problem for random matrices, generalize some known results for the Shub-Smale polynomials, and the spherical ensemble.

Chapter 2

Applications of the coarea formula

In this chapter, we introduce the coarea formula and explore its intricacies, applications, and related concepts. The aim is to present various contexts in which the coarea formula can be applied, tailored specifically to the development of this thesis, as it is one of the main tools employed. We refer the reader to [BCSS] and [N] for further background. Let us begin with the following toy example to build some intuition.

Let $B = B_{\mathbb{R}^3}(0,1) \subseteq \mathbb{R}^3$ be the unit ball. To compute its volume, one simply integrates the constant function 1 over B and apply Fubini's theorem. This can be interpreted as follows.

Let $\varphi : B \subseteq \mathbb{R}^3 \to \mathbb{R}$ be the projection onto the first coordinate, i.e. $\varphi(p) = x_p$. Given $t \in [-1,1]$, the fibre $\varphi^{-1}(t)$ is the set

$$\varphi^{-1}(t) = \left\{ (t, \sqrt{1-t^2}u) : u \in B_{\mathbb{R}^2}(0,1) \right\},$$

where we naturally identify $\mathbb{R} \times \mathbb{R}^2$ with $\mathbb{R}^3$. Let A_t denote its area, which is

$$A_t = \int_{p \in \varphi^{-1}(t)} 1,\ d\varphi^{-1}(t)(p) = \pi(1-t^2).$$

Therefore, the volume of B is given by

$$\operatorname{vol}(B) = \int_B 1\, dB = \int_{-1}^{1} A_t\, dt = \int_{-1}^{1} \left(\int_{p \in \varphi^{-1}(t)} 1\, d\varphi^{-1}(t)(p) \right) dt = \frac{4}{3}\pi.$$

Now, let us compute the area of S^2.

Analogously, define $\varphi : S^2 \subset \mathbb{R}^3 \to \mathbb{R}$ by $\varphi(p) = x_p$, the projection onto the first coordinate. Given $t \in [-1,1]$, the fibre $\varphi^{-1}(t)$ is the set

$$\varphi^{-1}(t) = \left\{ (t, \sqrt{1-t^2}u) : u \in S^1 \right\}$$

and let L_t denote its length, which is

$$L_t = \int_{p \in \varphi^{-1}(t)} 1 \, d\varphi^{-1}(t)(p) = 2\pi\sqrt{1-t^2}.$$

Thus, naively integrating gives

$$\text{area}(S^2) = \int_{S^2} 1 \, dS^2 = \int_{-1}^{1} L_t \, dt = 4\pi \int_0^1 \sqrt{1-t^2} \, dt = \pi^2.$$

This contradicts the well-known fact, proven by Archimedes in the 3-rd century BC, that

$$\text{area}(S^2) = 4\pi.$$

What went wrong?

At any point $p \in S^2$ the gradient of the map φ satisfies

$$\|\nabla\varphi(p)\| = \sqrt{1-t^2},$$

where $t = \varphi(p)$. This can be verified by choosing a suitable orthonormal base of the tangent space T_pS^2. Recall that T_pS^2 can be identified with $p^\perp \subset \mathbb{R}^3$ and $p = (t, \sqrt{1-t^2}u)$ for some $u \in S^1$, so it is easy to check that

$$\left\{ (\sqrt{1-t^2}, -tu), (0, u^\perp) \right\} \subseteq T_pS^2$$

is an orthonormal basis. In this base the gradient is exactly

$$\nabla\varphi(p) = \left(\sqrt{1-t^2}, 0 \right),$$

so the norm is as claimed.

This shows that φ does not preserve distances, for example, take the unit vector $v_1 = (\sqrt{1-t^2}, -tu)$, then the derivative of φ in the direction of v_1

$$D\varphi(p)v_1 = \sqrt{1-t^2}$$

has norm strictly less than 1 if $t \neq 0$. Roughly speaking, the fibres $\varphi^{-1}(t)$ "tighten" as t approaches ± 1. To account for this contraction, we must divide by $\|\nabla\varphi(p)\|$. Doing so gives

$$\operatorname{vol}(S^2) = \int_{-1}^{1} \left(\int_{p\in\varphi^{-1}(t)} \frac{1}{\|\nabla\varphi(p)\|} \, d\varphi^{-1}(t)(p) \right) dt = 4\pi,$$

as expected.

We now turn to the linear case, which is the cornerstone of the coarea formula.

Let $A : U \to V$ a surjective linear map from to vector spaces of dimension $n+k$ and n, respectively. Taking the singular value decomposition (SVD), there exists orthonormal bases $\{u_1, \cdots, u_{n+k}\}$ of U and $\{v_1, \cdots, v_n\}$ of V, such that

$$A(u_i) = \begin{cases} \lambda_i v_i & \text{if } 1 \leq i \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Recall that the $\lambda_i' s$ are the singular values of A, i.e., the square roots of the eigenvalues of the positive symmetric operator $AA^T : V \to V$. The square root of its determinant,

$$NJ_A := \sqrt{\det AA^T}$$

is called the *normal Jacobian* of A. Analogously $NJ_A = |\det A|_{\ker^\perp}|$, where $A|_{\ker^\perp}$ is the restriction of A to the orthogonal complement of its kernel.

Thanks to this, can reformulate the situation as follows.

Let $X = (x_1, \cdots, x_n)$ and $Y = (x_{n+1}, \cdots, x_{n+k})$, and consider the map $A : \mathbb{R}^{n+k} \to \mathbb{R}^n$ defined by

$$A(X,Y) = D(X),$$

where D is the diagonal matrix with non-zero entries $\lambda_1, \cdots, \lambda_n$. Then, given a measurable function $\phi : U \to [0,\infty)$, applying change of variables and Fubini's theorem we get

$$\begin{aligned} \int_{\mathbb{R}^{n+k}} \phi(X,Y)\, d(X,Y) &= \int_{\mathbb{R}^{n+k}} \phi\left(\frac{x_1}{\lambda_1}, \cdots, \frac{x_n}{\lambda_n}, x_{n+1}, \cdots, x_{n+k} \right) \frac{1}{NJ_A} \, dX\, dY \\ &= \int_{\mathbb{R}^n} \left(\int_{A^{-1}(X)} \frac{\phi(X,Y)}{NJ_A} \, dY \right) dX, \end{aligned}$$

which is the first formulation of the coarea formula.

Remark 2.0.1. It is easy to check that the normal Jacobian of the map $A : U \to V$ quantifies the volume distortion between the image of a unit ball in U and the unit ball in V.

Going back to the first example, the normal Jacobian would we $\|\nabla \varphi(p)\|$.

In the next section, we will present the coarea formula in the setting most relevant to this thesis, along with some of its applications.

2.1 Coarea formula and applications to probability theory

Firstly, let us formulate the coarea formula in a more general setting and derive some known results for random variables.

Suppose $\varphi : M \to N$ is a surjective map from a Riemannian manifold M to a Riemannian manifold N, whose derivative $D\varphi(x) : T_xM \to T_{\varphi(x)}N$ is surjective for almost all $x \in M$. The horizontal space $H_x \subset T_xM$ is defined as the orthogonal complement of $\operatorname{Ker} D\varphi(x)$. The horizontal derivative of φ at x is the restriction of $D\varphi(x)$ to H_x. The normal Jacobian $NJ_\varphi(x)$ is the absolute value of the determinant of the horizontal derivative, defined almost everywhere on M. Namely,

$$NJ_\varphi(x) = \sqrt{\det(D\varphi(x) D\varphi(x)^T)}.$$

Theorem ([BCSS, p. 241] The coarea formula)**.** *Let M, N be Riemannian manifolds of respective dimensions $m \geq n$, and let $\varphi : M \to N$ be a smooth surjective map, whose derivative $D\varphi(x) : T_xM \to T_{\varphi(x)}N$ is surjective for almost all $x \in M$. Then, for any positive measurable function, $\Phi : M \to [0, +\infty)$ we have*

$$\int_{x \in M} \Phi(x)\, dM = \int_{y \in N} \int_{x \in \varphi^{-1}(y)} \frac{\Phi(x)}{NJ_\varphi(x)}\, d\varphi^{-1}(y)\, dN$$

and

$$\int_{x \in M} NJ_\varphi(x) \Phi(x)\, dM = \int_{y \in N} \int_{x \in \varphi^{-1}(y)} \Phi(x)\, d\varphi^{-1}(y)\, dN,$$

where $d\varphi^{-1}(y)$ is the induced volume measure in the manifold $\varphi^{-1}(y)$. □

Remark 2.1.1. Let $M = A + iB \in \mathbb{C}^{n \times n}$ be a complex matrix with A, B real matrices, and $M_{\mathbb{R}} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ in $\mathbb{R}^{2n \times 2n}$ its real representation. Observe that $M_{\mathbb{R}}$

is conjugate to $\begin{pmatrix} A-iB & 0 \\ 0 & A+iB \end{pmatrix}$ via the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} -iI_n & iI_n \\ I_n & I_n \end{pmatrix}$, where I_n is the identity matrix of size n. Therefore,

$$\det M_{\mathbb{R}} = |\det M|^2.$$

Taking this into account, we can extend the coarea formula to the complex context. In this case, the normal Jacobian would be

$$NJ_{\varphi}(x) = |\det(D\varphi(x)D\varphi(x)^*)|,$$

where $D\varphi(x)^*$ is the conjugate transpose of $D\varphi(x)$ (see [HKPV, Lemma 3.1.2]).

Let us state it in the univariate case for simplicity, but the general case is analogous.

Let $f : \mathbb{C} \to \mathbb{C}$ be a smooth surjective map such that $f'(z) \neq 0$ for almost all $z \in \mathbb{C}$. Then, since the fibres are countable sets, for any positive measurable function $\phi : \mathbb{C} \to [0, +\infty)$ we have

$$\int_{\mathbb{C}} \phi(z)\, d\mathbb{C} = \int_{u\in\mathbb{C}} \sum_{z\in f^{-1}(u)} \frac{\phi(z)}{|f'(z)|^2}\, d\mathbb{C} \tag{2.1}$$

and

$$\int_{\mathbb{C}} \phi(z)|f'(z)|^2\, d\mathbb{C} = \int_{u\in\mathbb{C}} \sum_{z\in f^{-1}(u)} \phi(z)\, d\mathbb{C}. \tag{2.2}$$

We refer the reader to Chapter 3 for all needed background of probability.

Lemma 2.1.2. *Let $X(\cdot) : \mathbb{C}^D \to \mathbb{C}^d$ be a random field with density $\rho_X(\cdot)$, and $\varphi : \mathbb{C}^d \to \mathbb{C}^l$ be a smooth surjective map, whose derivative $D\varphi(x) : \mathbb{C}^d \to \mathbb{C}^l$ is surjective for almost all $x \in \mathbb{C}^d$. Then, the pointwise density of the random field $Y(\cdot) : \mathbb{C}^D \to \mathbb{C}^l$ defined by $Y(\cdot) = \varphi(X(\cdot))$ is*

$$\rho_Y(y) = \int_{x\in\varphi^{-1}(y)} \rho_X(x) \frac{1}{NJ_{\varphi}(x)}\, d\varphi^{-1}(y).$$

Proof. It follows from the coarea formula, applied to the fact that

$$\mathbb{P}(Y \in B) = \mathbb{P}\left(X \in \varphi^{-1}(B)\right) = \int_{x\in\varphi^{-1}(B)} \rho_X(x)\, dx,$$

for every Borel subset $B \subset \mathbb{C}^l$. □

Remark 2.1.3. Please note, that given a random field $X(\cdot):\mathbb{C}^D \to \mathbb{C}^d$ and a point $z \in \mathbb{C}^D$, we get a random vector $X(z) \in \mathbb{C}^d$. Thus, the previous lemma, and many other lemmas for random fields, can be stated for random vectors.

First we will consider a useful example that will be revisited throughout this chapter.

Let $\mathcal{P}_N$ be the set of complex polynomials of degree less or equal to $N \in \mathbb{N}$ equipped with the Bombieri-Weyl norm, that is the norm induced by the following inner product.

For $f, g \in \mathcal{P}_N$, $f(z) = \sum_0^N a_k z^k$, $g(z) = \sum_0^N b_k z^k$ we define

$$\langle f, g\rangle_N = \sum_{k=0}^{N} a_k \bar{b}_k \binom{N}{k}^{-1}. \tag{2.3}$$

For any $z \in \mathbb{C}$ consider $\varphi_z : \mathcal{P}_N \to \mathbb{C}$ the linear map defined by $\varphi_z(f) = f(z)$, namely, the evaluation map. In the space $\mathcal{P}_N$, take the orthonormal basis of monomials $\left\{\binom{N}{k} Z^k\right\}$ for $0 \le k \le N$. A straightforward computation shows that

$$\nabla\varphi_z(f) = \left(\binom{N}{0}^{1/2} z^0, \binom{N}{1}^{1/2} z, \cdots, \binom{N}{N}^{1/2} z^N\right),$$

and thus, the normal Jacobian satisfies

$$NJ_{\varphi_z}(f) = \|\nabla\varphi_z(f)\|^2 = \left(1+|z|^2\right)^N.$$

If we consider $a_k \in \mathbb{C}$ i.i.d. standard Gaussian random variables for $0 \le k \le N$ then the elliptic polynomials, also called Kostlan-Shub-Smale polynomials, are defined by

$$f = \sum_{k=0}^{N} \binom{N}{k}^{1/2} a_k Z^k. \tag{2.4}$$

Taking into account the orthonormal basis defined before it is clear that an elliptic polynomial can be seen as the random vector of coefficients $X \in \mathbb{C}^{N+1}$. Thus, we obtain that the density of these random polynomials is $\rho_X(f) = \frac{e^{-\|f\|^2}}{\pi^{N+1}}$.

Now, given $z \in \mathbb{C}$, consider the complex random variable $f(z)$. Following Lemma 2.1.2 we have that the density of $f(z)$ for any $w \in \mathbb{C}$ satisfies,

$$\rho_{f(z)}(w) = \frac{1}{\pi^{N+1}(1+|z|^2)^N} \int_{\varphi^{-1}(w)} e^{-\|f\|^2}\, d\varphi^{-1}(w).$$

Please observe that the fibre $\varphi^{-1}(w)$ is exactly the set $\{f \in \mathcal{P}_N : f(z) = w\}$. Let $g_1 \in \mathcal{P}_N$ be the following polynomial $g_1 = \sum_{k=0}^{N} \binom{N}{k} \bar{z}^k Z^k$. It is clear that $\langle f, g_1 \rangle_N$ is equal to $f(z)$ for any $f \in \mathcal{P}_N$, in particular $\|g_1\|^2 = g_1(z) = (1+|z|^2)^N$. Then, the fibre $\varphi^{-1}(w)$ is in fact, the affine hyperplane given by $\{f \in \mathcal{P}_N : \langle f, g_1 \rangle_N = w\}$.

Projection of the polynomial f orthogonally onto the hyperplane yields, a decomposition $f = h + \frac{w}{(1+|z|^2)^N} g_1$ such that $h(z) = 0$ and $\|f\|^2 = \|h\|^2 + \frac{|w|^2}{(1+|z|^2)^N}$. Furthermore, since the integral of $e^{-\|h\|^2}$ over the hyperplane $\{h \in \mathcal{P}_N : h(z) = 0\}$ is π^N, we have,

$$\rho_{f(z)}(w) = \frac{e^{-\frac{|w|^2}{(1+|z|^2)^N}}}{\pi(1+|z|^2)^N}. \tag{2.5}$$

This means that $f(z)$ is a centred Gaussian random variable with variance equal to $(1+|z|^2)^N$.

Remark 2.1.4. Since $f(z)$ is a linear combination of Gaussian random variables, the previous statement follows, in another way, with basic properties of random variables.

Corollary 2.1.5. *Let $A \in \mathbb{C}^{r\times r}$ be a random matrix with i.i.d. centred Gaussian entries $\mathcal{N}_{\mathbb{C}}(0,\sigma^2)$. Then, the density at* 0 *of the random variable* $\det(A)$ *is equal to*

$$\rho_{\det(A)}(0) = \frac{1}{\pi\sigma^{2r}\Gamma(r)}.$$

Proof. Using the previous lemma, it follows that

$$\rho_{\det(A)}(0) = \frac{1}{\pi^{r^2}\sigma^{2r^2}} \int_{\Sigma} e^{\frac{-\|A\|^2}{\sigma^2}} \frac{1}{NJ_{\det}(A)}\, d\Sigma$$

where $\Sigma = \{A \in \mathbb{C}^{r\times r} \ : \ \det(A) = 0\}$.

By Jacobi's formula, the derivative of the determinant at A in the direction $\dot{A}$ is equal to $\mathrm{tr}(\mathrm{adj}\,(\mathrm{A})\dot{\mathrm{A}})$, where the adjugate matrix of A, $\mathrm{adj}\,(\mathrm{A})$, is the transpose of the cofactor matrix.

After a routine computation, it follows that

$$NJ_{\det}(A) = \|\mathrm{adj}\,(\mathrm{A})\|_{\mathrm{F}}^2.$$

Let $\hat{\Sigma}$ be the set of rank $r-1$ matrices of $\mathbb{C}^{r\times r}$. It is clear that $\hat{\Sigma}$ is the smooth part of the algebraic variety Σ. It follows that integrating over Σ is the same as integrating over $\hat{\Sigma}$, then

$$\rho_{\det(A)}(0) = \frac{1}{\pi^{r^2}\sigma^{2r^2}} \int_{\hat{\Sigma}} e^{\frac{-\|A\|^2}{\sigma^2}} \frac{1}{\|\mathrm{adj}\,(\mathrm{A})\|_{\mathrm{F}}^2}\, d\hat{\Sigma}. \tag{2.6}$$

Given a matrix $A \in \hat{\Sigma}$, the kernel is a one dimensional subspace of $\mathbb{C}^r$ given by the orthogonal complement of the column space of A. Furthermore, given a direction $v \in \mathbb{P}(\mathbb{C}^r)$, it is easy to check that there exist a matrix $A \in \hat{\Sigma}$ such that the column space is orthogonal to v. Therefore, there is a well-defined smooth surjective submersion

$$\varphi : \hat{\Sigma} \to \mathbb{P}(\mathbb{C}^r), \qquad \varphi(A) = v, \qquad \text{such that } \; Av = 0, \tag{2.7}$$

A straightforward computation, using the derivative of an implicit function, shows that the differential map $D\varphi(A) : T_A\hat{\Sigma} \to T_v\mathbb{P}(\mathbb{C})$ is defined by

$$D\varphi(A)\dot{A} = -A^{\dagger}\dot{A}v,$$

where $A^{\dagger}$ is the Moore-Penrose pseudo-inverse of A. The tangent space to $\hat{\Sigma}$ at A, can be parametrized as $\{\dot{X}A + A\dot{Y} \; : \dot{X}, \dot{Y} \in \mathbb{C}^{r\times r}\}$ (see Arnold et al. [AGZV]).

One can verify that the Hermitian complement to $\ker D\varphi(A)$ is the image of $\{(0, \dot{w}v^*) \; : \; \dot{w}^*v = 0\}$ by the map $(\dot{X}, \dot{Y}) \mapsto \dot{X}A + A\dot{Y}$. So, applying $D\varphi(A)$, we get the following composition

$$\{(0, \dot{w}v^*) \; : \; \dot{w}^*v = 0\} \mapsto A\dot{w}v^* \mapsto \dot{w},$$

which is the identity.

Then, the normal Jacobian of the map φ is the inverse of the normal Jacobian of the first map, which is in fact the square of the product of the non-zero singular values of A. In conclusion, the normal Jacobian of φ satisfies,

$$NJ_{\varphi}(A) = \frac{1}{\|\mathrm{adj}\,(\mathrm{A})\|_{\mathrm{F}}^2}$$

Applying the coarea formula to $\varphi : \hat{\Sigma} \to \mathbb{P}(\mathbb{C}^r)$, equation (2.6) can be rewritten as

$$\rho_{\det(A)}(0) = \frac{1}{\pi^{r^2}\sigma^{2r^2}} \int_{\mathbb{P}(\mathbb{C}^r)} \int_{\mathbb{C}^{r\times(r-1)}} e^{\frac{-\|A\|_F^2}{\sigma^2}}\, dA\, d\mathbb{P}(\mathbb{C}^r).$$

Recall that the volume of $\mathbb{P}(\mathbb{C}^r)$ is $\frac{\pi^{r-1}}{\Gamma(r)}$. Noting that the inner integral does not depend on v, it follows that

$$\rho_{\det(A)}(0) = \frac{\pi^{r-1}}{\pi^{r^2}\sigma^{2r^2}\Gamma(r)} \int_{A\in\mathbb{C}^{r\times(r-1)}} e^{\frac{-\|A\|_F^2}{\sigma^2}}\,dA.$$

Finally, applying Fubini's theorem and taking polar coordinates, we get

$$\rho_{\det(A)}(0) = \frac{1}{\pi\sigma^{2r}\Gamma(r)}.$$

□

Another great application of the coarea formula that follows Lemma 2.1.2 is for computing the conditional expectation.

Suppose $X \in \mathbb{C}^d$ is a random vector with density $p_X(\cdot)$, and we define the random vector $Y \in \mathbb{C}^l$ as $\varphi(X)$ for some measurable, differentiable map $\varphi : \mathbb{C}^d \to \mathbb{C}^l$.

Then the total expectation can be written using the coarea formula as,

$$\mathbb{E}(f(X)) = \int_{\mathbb{C}^d} f(x)p_X(x)\,dx = \int_{\mathbb{C}^l}\left(\int_{x\in\varphi^{-1}(y)} f(x)p_X(x)\frac{1}{NJ_\varphi(x)}\,d\varphi^{-1}(y)\right)dy.$$

Hence, the conditional expectation of $f(X)$ given $Y = \varphi(X) = y$ is

$$\mathbb{E}(f(X) \mid Y = y) = \frac{1}{p_Y(y)}\int_{\varphi^{-1}(y)} f(x)p_X(x)\frac{1}{NJ_\varphi(x)}\,d\varphi^{-1}(y), \tag{2.8}$$

where the density of Y is given by Lemma 2.1.2, namely

$$p_Y(y) = \int_{\varphi^{-1}(y)} p_X(x)\frac{1}{NJ_\varphi(x)}\,d\varphi^{-1}(y).$$

The following lemma has a similar flavour to the Gaussian regression (3.3), in the sense that reformulates the conditional expectation.

Lemma 2.1.6. *Let $A \in \mathbb{C}^{r\times r}$ be a random matrix with i.i.d. centred Gaussian entries $\mathcal{N}_{\mathbb{C}}(0,\sigma^2)$. If $\phi : \mathbb{C}^{r\times r} \to [0,+\infty)$ is a measurable function such that $\phi(MU) = \phi(M)$ for every unitary matrix U, then*

$$\mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times r}}\big(\phi(A) \mid \det(A) = 0\big) = \mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times r}}(\phi(A) \mid A_r = 0)$$

where A_r is the r-th column of A.

Proof. By definition, the conditional expectation of $\phi(A)$ conditional to $\{\det(A) = 0\}$ is equal to the integral of $\phi(A)$ with respect to the conditional density, i.e., by (2.8)

$$\mathbb{E}(\phi(A) \mid \det(A) = 0) = \frac{1}{\rho_{\det(A)}(0)} \int_{\Sigma} \frac{\phi(A)}{NJ_{\det}(A)} \frac{e^{\frac{-\|A\|^2}{\sigma^2}}}{\pi^{r^2}\sigma^{2r^2}} \, d\Sigma.$$

Since $\rho_{\det(A)}(0) = \frac{1}{\pi\sigma^{2r}\Gamma(r)}$, and $\phi(AU) = \phi(A)$ for every unitary matrix U, applying the coarea formula to the same map φ, defined in (2.7), as in the previous corollary, we get

$$\mathbb{E}(\phi(A) \mid \det(A) = 0) = \int_{\mathbb{C}^{r\times(r-1)}} \phi(A) \frac{e^{\frac{-\|A\|^2}{\sigma^2}}}{\pi^{r(r-1)}\sigma^{2r(r-1)}} \, dA = \mathbb{E}(\phi(A) \mid A_r = 0).$$

$\square$

2.2 Double-fibration technique, solution varieties and volumes

Another important application of the coarea formula is the well-known double-fibration technique, which proceeds as follows.

Consider a double-fibration setting, where $\mathcal{V}$ is a subvariety of the product $M \times N$ and consider both projections $\pi_1 : \mathcal{V} \to M$ and $\pi_2 : \mathcal{V} \to N$,

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] & \\ M & & N \end{tikzcd}$$

In order to integrate some real-valued function over M whose value at some point x is an average over the fibre $\mathcal{V}_x$, we lift it to $\mathcal{V}$ and then pushforward to N using the projections. The original expected value over M is then written as an integral over N which involves the quotient of normal Jacobians of the projections π_1 and π_2. More precisely,

$$\int_M \int_{\mathcal{V}_x} \phi(x,y) d\mathcal{V}_x \, dM = \int_N \left(\int_{V_y} \phi(x,y) \frac{NJ_{\pi_1}(x,y)}{NJ_{\pi_2}(x,y)} d\mathcal{V}_y \right) dN, \tag{2.9}$$

where the quotient of the normal Jacobians satisfies

$$\frac{NJ_{\pi_1}(a,x)}{NJ_{\pi_2}(a,x)} = \left|\det\left(D\pi_2(a,x)D\pi_1(a,x)^{\dagger}\right)\right|^2$$

Furthermore, if there exist a local map $G : U_a \to V_x$ from a neighbourhood of $a \in M$ to a neighbourhood of $x \in N$ then, the quotient of the normal Jacobians is given by

$$\frac{NJ_{\pi_1}(a,x)}{NJ_{\pi_2}(a,x)} = |\det DG(a)DG(a)^*|^{-1},$$

where $DG(a)^*$ is the transpose conjugate of the differential $DG(a)$ of the map G at a (see Blum et al. [BCSS, Section 13.2], Dedieu [D1], also Remark 2.0.1).

In many examples, one has that $\mathcal{V}$ is given by the zero set of a function on M or as an incidence variety. That is why, we call $\mathcal{V}$ the solution variety. Let us see some examples.

2.2.1 Counting Roots of Elliptic Polynomials

In this subsection we will consider the roots of complex polynomials. This can be extended to polynomial systems, but for simplicity and as a didactic approach we do it for the univariate case (see Chapter 4).

Let $\mathcal{P}_N$ be the set of complex polynomials of degree less or equal to $N \in \mathbb{N}$ equipped with the Bombieri-Weyl norm.

Considering the evaluation map $ev : \mathcal{P}_N \times \mathbb{C} \to \mathbb{C}$ given by $ev(f,z) = f(z)$ one can see the pairs of polynomial and roots as the double-fibration setting where $\mathcal{V} = \{(f,z) \in \mathcal{P}_N \times \mathbb{C} : f(z) = 0\}$,

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] & \\ \mathcal{P}_N \arrow[rr, dashed] & & \mathbb{C} \end{tikzcd}$$

Let us compute the quotient of normal Jacobians.

Please observe that thanks to the Implicit Function Theorem, one gets, when $f'(z) \neq 0$,

$$f'(z)\dot{z} + \dot{f}(z) = 0,$$

which means that $\dot{z} = \frac{-\dot{f}(z)}{f'(z)}$.

In turn, we get that

$$\frac{NJ_{\pi_1}(f,z)}{NJ_{\pi_2}(f,z)} = \frac{|f'(z)|^2}{(1+|z|^2)^N}. \tag{2.10}$$

To compute the average number of roots of random elliptic polynomials one would compute the following

$$\begin{aligned}
\mathbb{E}\left(\sum_{x:f(x)=0} 1\right) &= \int_{\mathbb{C}}\int_{\mathcal{V}_z} \frac{|f'(z)|^2 e^{\|f\|^2}}{\pi^{N+1}(1+|z|^2)^N}\, d\mathcal{V}_z\, dz \\
&= \int_{\mathbb{C}} \frac{N\pi^N(1+|z|^2)^{N-2}}{\pi^{N+1}(1+|z|^2)^N} \int_0^\infty t^3 e^{-t^2}\, dt\, dz \\
&= \frac{N}{\pi}\int_{\mathbb{C}} \frac{1}{(1+|z|^2)^2}\, dz \\
&= N,
\end{aligned}$$

where the second equality follows [ABS, Proposition 3] and the last arises from taking polar coordinates.

Remark 2.2.1. The previous computation is obvious using the Fundamental theorem of algebra, from where we know that $\sum_{x:f(x)=0} 1 = N$ a.s.

Please observe that, by (2.8), the inner integral in the first step can be seen as

$$\int_{\mathcal{V}_z} \frac{|f'(z)|^2 e^{\|f\|^2}}{\pi^{N+1}(1+|z|^2)^N}\, d\mathcal{V}_z = \mathbb{E}\left(|f'(z)|^2 \mid f(z)=0\right)\rho_{f(z)}(0).$$

The right-hand is easy to compute. Using Gaussian regression 3.3 [W, Proposition 1], one has that the conditional expectation is equal to $N(1+|z|^2)^{N-2}$ while following (2.5), $\rho_{f(z)} = \frac{1}{\pi(1+|z|^2)^N}$. This yield,

$$\mathbb{E}\left(|f'(z)|^2 \big| f(z)=0\right)\rho_{f(z)}(0) = \frac{N}{\pi(1+|z|^2)^2}. \tag{2.11}$$

After taking polar coordinates we get,

$$\mathbb{E}\left(\sum_{x:f(x)=0} 1\right) = \int_{\mathbb{C}} \frac{N}{\pi(1+|z|^2)^2}\, dz = N.$$

2.2.2 The linear solution variety

In this section we consider the problem of solving linear systems, in particular we prove the generalization, to the underdetermined case, of the Proposition 6.6 of [ABB+2]. This example is going to be used in Section 4.4.

Given $v \in \mathbb{C}^{n+1}$ such that $Mv = 0$, then for any $\lambda \in \mathbb{C}$ we have that $w = \lambda v$ also satisfies the equation $Mw = 0$. Therefore, the natural space for studying the solutions of this equation is the set of direction, or the set of lines, in $\mathbb{C}^{n+1}$, namely the projective space $\mathbb{P}(\mathbb{C}^{n+1})$.

Consider

$$\mathcal{V}^{\mathrm{lin}} = \{(M,v) \in \mathbb{C}^{r\times(n+1)} \times \mathbb{P}(\mathbb{C}^{n+1}) : \; Mv = 0\}.$$

The linear solution variety $\mathcal{V}^{\mathrm{lin}}$, for the underdetermined case, is a $(r+1)n$-dimensional smooth submanifold of $\mathbb{C}^{r\times(n+1)} \times \mathbb{P}(\mathbb{C}^{n+1})$, and it inherits the Riemannian structure of the ambient space.

The linear solution variety is equipped with the two canonical projections $\pi_1^{\mathrm{lin}} : \mathcal{V}^{\mathrm{lin}} \to \mathbb{C}^{r\times(n+1)}$ and $\pi_2^{\mathrm{lin}} : \mathcal{V}^{\mathrm{lin}} \to \mathbb{P}(\mathbb{C}^{n+1})$.

For $M \in \mathbb{C}^{r\times(n+1)}$, $(\pi_1^{\mathrm{lin}})^{-1}(M)$ is a copy of the projective linear subspace corresponding to the kernel of M in $\mathbb{P}(\mathbb{C}^{n+1})$, and for $v \in \mathbb{P}(\mathbb{C}^{n+1})$, $(\pi_2^{\mathrm{lin}})^{-1}(v)$ is a copy of the linear subspace of $\mathbb{C}^{r\times(n+1)}$ consisting of the matrices $A \in \mathbb{C}^{r\times(n+1)}$ such that $Av = 0$. Also, the set of critical points Σ' is the set of pairs $(M,v) \in \mathcal{V}^{\mathrm{lin}}$ such that $\operatorname{rank}(M) < r$ (see Blum et al. [BCSS, Section 13.2]).

In this case the tangent space to $\mathcal{V}^{\mathrm{lin}}$ at (M,v) is the set of pairs $(\dot{M},\dot{v})$ in $\mathbb{C}^{r\times(n+1)} \times \mathbb{C}^{n+1}$ satisfying the following linear equations

$$\dot{M}v + M\dot{v} = 0, \qquad v^*\dot{v} = 0.$$

Then, if $(M,v) \notin \Sigma'$, for any $\dot{v} \in \operatorname{Ker} M^{\perp}$, since $M^{\dagger}M = id|_{\operatorname{Ker} M^{\perp}}$, we have

$$\begin{aligned} M\dot{v} &= -\dot{M}v \\ M^{\dagger}M\dot{v} &= -M^{\dagger}\dot{M}v \\ \dot{v} &= -M^{\dagger}\dot{M}v. \end{aligned}$$

It is clear then, that we have the following decomposition in orthogonal subspaces,

$$T_{(M,v)}\mathcal{V}^{\mathrm{lin}} = \{(0,\dot{v}) : \dot{v} \in \operatorname{Ker} M\} \oplus \{(\dot{M},\dot{v}) : \dot{v} = \varphi(\dot{M})\}$$

where $\varphi(\dot{M}) = -M^\dagger \dot{M}$. A routine computation shows that, if $\|v\| = 1$, then $\varphi\varphi^*$ is equal to $M^\dagger (M^\dagger)^*$. Writing down the singular value decomposition of M, it follows that $\det(\varphi\varphi^*) = \det(MM^*)^{-1}$.

Then,

$$\frac{NJ_{\pi_1^{\mathrm{lin}}}(M,v)}{NJ_{\pi_2^{\mathrm{lin}}}(M,v)} = |\det MM^*|.$$

Proposition 2.2.2. *Let $\phi : \mathbb{C}^{r\times(n+1)} \to [0,\infty)$ be a measurable unitary invariant function in the sense that $\phi(MU^*) = \phi(M)$ for any unitary matrix $U \in \mathcal{U}(n+1)$. Then,*

$$\mathop{\mathbb{E}}_{M\in\mathbb{C}^{r\times(n+1)}}(\phi(M)) = \frac{\Gamma(n-r+1)}{\Gamma(n+1)} \mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times n}}(\phi((0|A)) \cdot |\det AA^*|),$$

where $(0|A)$ is the matrix whose first column is all 0's and the following columns are the same as A.

Proof. Let $\chi(M,v) = \phi(M)e^{-\|M\|_F^2}$. Then, applying the double-fibration technique we get that

$$\int_{M\in\mathbb{C}^{r\times(n+1)}} \int_{v\in\mathbb{P}(\mathrm{Ker} M)} \chi(M,v)\, d\mathbb{P}(\mathrm{Ker} M)\, d\mathbb{C}^{r\times(n+1)}$$

is equal to

$$\int_{v\in\mathbb{P}(\mathbb{C}^{n+1})} \int_{M:Mv=0} \chi(M,v) \cdot |\det MM^*|\, dM\, d\mathbb{P}(\mathbb{C}^{n+1}).$$

Now, since χ does not depend on v, and $\mathbb{P}(\mathrm{Ker} M)$ is a copy of $\mathbb{P}(\mathbb{C}^{n-r+1})$, we have that the first integral is equal to

$$\frac{\pi^{n-r}}{\Gamma(n-r+1)} \int_{M\in\mathbb{C}^{r\times(n+1)}} \phi(M)e^{-\|M\|_F^2}\, dM.$$

Also, by parametrizing $\{M : Mv = 0\}$ by $\{(0|A)U_v^* : A \in \mathbb{C}^{r\times n}\}$, where U_v is any matrix in $\mathcal{U}(n+1)$ such that $U_v e_1 = v$, and since $\chi(M,v) \cdot |\det MM^*|$ is equal to $\chi(MU^*, Uv) \cdot |\det MU^*UM^*|$ for all $U \in \mathcal{U}(n+1)$ by hypothesis, we have that the second integral is equal to

$$\frac{\pi^n}{\Gamma(n+1)} \int_{A\in\mathbb{C}^{r\times n}} \phi(0|A) \cdot |\det AA^*| e^{-\|A\|_F^2}\, dM.$$

In conclusion, dividing by π^{rn}, we get

$$\frac{1}{\Gamma(n-r+1)} \mathop{\mathbb{E}}_{M\in\mathbb{C}^{r\times(n+1)}}(\phi(M)) = \frac{1}{\Gamma(n+1)} \mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times n}}(\phi((0|A))\cdot|\det AA^*|)$$

□

2.2.3 The solution variety for the finding kernel problem

It will be useful for computations done in Section 4.4 to consider a scheme similar to that of the previous subsection for the case of finding kernels of rectangular matrices.

We denote by $G(k,l)$ the Grassmannian of k-planes in $\mathbb{C}^l$, i.e. the set of k-dimensional linear subspace of $\mathbb{C}^l$. It can be seen that $G(k,l)$ is a projective variety of dimension $k(l-k)$ and degree $\Gamma(k(k-l)+1)\prod_{i=1}^{k}\frac{\Gamma(i)}{\Gamma(k+i)}$, so its volume with regard to the usual Riemannian metric, satisfy

$$\mathrm{vol}(G(k,l)) = \pi^{k(l-k)}\prod_{i=1}^{k}\frac{\Gamma(i)}{\Gamma(k+i)} \tag{2.12}$$

(see Harris [H], Mumford [M2]).

Now, consider

$$\mathcal{V}^{\ker} = \{(M,V)\in\mathbb{C}^{r\times n}\times G(n-r,n):\ MV=0\}$$

where $G(n-r,n)$ is the Grassmannian of $(n-r)$-planes in $\mathbb{C}^n$.

The linear kernel variety $\mathcal{V}^{\ker}$ is an rn-dimensional smooth submanifold of $\mathbb{C}^{r\times n}\times G(n-r,n)$, and it inherits the Riemannian structure of the ambient space.

The linear kernel variety is equipped with the two canonical projections $\pi_1^{\ker}$ and $\pi_2^{\ker}$.

In this case the tangent space to $\mathcal{V}^{\ker}$ at (M,V) is the set of pairs $(\dot{M},\dot{V})$ in $\mathbb{C}^{r\times n}\times\mathbb{C}^{n\times(n-r)}$ satisfying the following linear equations

$$\dot{M}V + M\dot{V} = 0, \qquad V^*\dot{V} = 0.$$

Then, if $(M,V)\notin\Sigma'$, for any $\dot{V}$ such that $V^*\dot{V}$, we have

$$\begin{aligned} M\dot{V} &= -\dot{M}V \\ M^\dagger M\dot{V} &= -M^\dagger\dot{M}V \\ \dot{V} &= -M^\dagger\dot{M}V. \end{aligned}$$

It is clear then, that the tangent space at (M,V) is the set of pairs $(\dot{M},\dot{V})$ such that $\dot{V} = \Phi(\dot{M})$, where $\Phi(\dot{M}) = -M^{\dagger}\dot{M}V$. A routine computation shows that $\det(\Phi\Phi^*) = |\det MM^*|^{n-r}$.

It follows that,

$$\frac{NJ_{\pi_1^{\ker}}(M,V)}{NJ_{\pi_1^{\ker}}(M,V)} = |\det MM^*|^{n-r}.$$

Applying the double-fibration technique and (2.12) we get the following.

Proposition 2.2.3. *Let $\phi : \mathcal{V}^{\ker} \to [0,\infty)$ be a measurable unitarily invariant function in the sense that $\phi(M,V) = \phi(MU^*,UV)$ for any unitary matrix $U \in \mathcal{U}(n)$. Then,*

$$\mathop{\mathbb{E}}_{M\in\mathbb{C}^{r\times n}}(\phi(M,\operatorname{Ker} M)) = \prod_{i=1}^{n-r}\frac{\Gamma(i)}{\Gamma(r+i)}\mathop{\mathbb{E}}_{B\in\mathbb{C}^{r\times r}}\left(\phi((0|B),V_{n-r})\cdot|\det B|^{2(n-r)}\right)$$

where V_{n-r} is the subspace generated by the first $n-r$ vectors in the canonical base.

Remark 2.2.4. Observe that the previous proposition is in fact a generalization in another sense than Proposition 2.2.2 of the same result. It is clear from the fact that if $r = n-1$, the previous proposition is exactly Proposition 6.6 of [ABB+2].

Remark 2.2.5. Observe that the previous proposition can be proved by applying Proposition 2.2.2 successively $n-r$ times.

2.2.4 Solutions to the Polynomial Eigenvalue Problem

In this subsection, we lay the foundations for the geometric framework required in Chapter 5.

Let $d \in \mathbb{N}$ and $A_k \in \mathbb{C}^{r\times r}$ be random matrices with i.i.d. standard Gaussian entries for $0 \le k \le d$.

We consider the solutions of the Polynomial Eigenvalue Problem for the matrices $\{A_k\}$, i.e. the zero set of the determinant of the polynomial matrix

$$F(z) = \det\left(\sum_{k=0}^{d}\binom{d}{k}^{1/2} z^k A_k\right). \tag{2.13}$$

Denote by $\mathcal{M}_r(\mathcal{P}_d)$ the set of $r\times r$ matrices of polynomials of degree less than d. One can consider $F(z) = \det(A(z))$ where $A \in \mathcal{M}_r(\mathcal{P}_d)$ has i.i.d. random elliptic polynomial entries.

Consider the solution variety $\mathcal{V} = \{(A,z) \in \mathcal{M}_r(\mathcal{P}_d) \times \mathbb{C} : F(z) = 0\}$ and its two projections

$$\begin{tikzcd}
 & \mathcal{V} \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] & \\
\mathcal{M}_r(\mathcal{P}_d) \arrow[rr, dashed] & & \mathbb{C}
\end{tikzcd}.$$

Please observe that the implicit function theorem defines a local map, $G : U_{(A,z)} \to \mathbb{C}$, around a well suited pair (A,z). Also, the same result gives an expression of its differential, namely, if one differentiate the condition $\det(A(z)) = 0$ we get

$$\mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\mathrm{A}'(\mathrm{z}))\dot{\mathrm{z}} + \mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\dot{\mathrm{A}}(\mathrm{z})) = 0,$$

implying

$$DG(A)\dot{A} = -\frac{\mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\dot{\mathrm{A}}(\mathrm{z}))}{\mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\mathrm{A}'(\mathrm{z}))}.$$

Considering the orthogonal basis $\left\{ \binom{d}{k}^{1/2} Z^k E_{ij} \right\}$ of monomial matrices, where the matrices E_{ij} are the elemental matrices with just a one in the ij entry, the Jacobian matrix is

$$DG(A) = -\frac{1}{F'(z)} \left(\binom{d}{k}^{1/2} z^k \mathrm{adj}\,(\mathrm{A(z)})_{\mathrm{ij}} \right),$$

where $F'(z) = \mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\mathrm{A}'(\mathrm{z}))$ is the derivative of $F(z) = \det(A(z))$ as a function from $\mathbb{C}$ to $\mathbb{C}$.

Then, the quotient of normal Jacobians satisfies,

$$\frac{NJ_{\pi_2}(A,z)}{NJ_{\pi_1}(A,z)} = |\det(DG(A)DG(A)^*)|^{-1} = \frac{|F'(z)|^2}{(1+|z|^2)^d \|\mathrm{adj}\,(\mathrm{A(z)})\|_{\mathrm{F}}^2}. \tag{2.14}$$

Let $\phi : \mathcal{V} \to [0,\infty)$ be a measurable function, applying the double-fibration technique, we get

$$\int_{\mathcal{M}} \sum_{i=1}^{N} \phi(A,z_i)\, d\mathcal{M}_r(\mathcal{P}_d) = \int_{\mathbb{C}} \int_{\mathcal{V}_z} \phi(A,z) \frac{|F'(z)|^2}{(1+|z|^2)^d \|\mathrm{adj}\,(\mathrm{A(z)})\|_{\mathrm{F}}^2}\, d\mathcal{V}_z\, dz. \tag{2.15}$$

If one takes a probability distribution on $\mathcal{M}(\mathcal{P}_d)$ with density ρ, applying Lemma 2.1.2 and (2.8) to equation (2.15), one gets

$$\mathbb{E}\left(\sum_{i=1}^{N}\phi(A,z_i)\right)=\int_C \mathbb{E}\left(\phi(A,z)\mid \det(A(z))=0\right)\rho_{\det(A(z))}(0)\,dz. \tag{2.16}$$

Remark 2.2.6. This equation is the so-called weighted Kac-Rice formula for the PEVP (see Section 3.3.2).

2.3 Point processes

In this section we revisit some of the previous examples and present new ones, but in the context of Point processes. These examples are going to be used in Chapter 5. We refer the reader to [HKPV] for all the needed technicalities in this marvellous subject. The idea behind this section is to illustrate how the double-fibration technique can be used in this setting.

A point process is a random variable taking values in the space of discrete subsets of a metric space. Many physical phenomena can be modelled by random discrete sets. For example, the arrival times of people in a queue, the arrangement of stars in a galaxy, energy levels of heavy nuclei of atoms etc. The single most important such process, known as the Poisson process has been widely studied and applied. The Poisson process is characterized by independence of the process when restricted to disjoint subsets of the underlying space. This assumption of independence is acceptable in some examples, but naturally, not all. If one looks at the distribution of like-charge particles confined by an external field, knowing that a particular location holds a particle makes it unlikely for there to be any other close to it.

Given a point process $\mathcal{X}$, the joint intensities of the point process with respect to the measure μ in the underlying metric space Λ are functions (if any exist) $\rho_k:\Lambda^k\to[0,\infty)$ for $k\geq 1$, such that for any non-negative measurable function $f:\Lambda^k\to[0,\infty)$,

$$\mathbb{E}\left(\sum f(x_1,\cdots,x_k)\right)=\int_{\Lambda^k} f(x_1,\cdots,x_k)\rho_k(x_1,\cdots,x_k)\,d\mu(x_1)\cdots d\mu(x_k), \tag{2.17}$$

where the sum is taken over all distinct k-tuples of point in the process $\mathcal{X}$.

So the question that remains is, how can we find these intensity functions?

A particular case of point processes, are the so-called Determinantal Point Processes (DPP's) where one has a Hermitian, positive semidefinite kernel $K : \Lambda \times \Lambda \to \mathbb{R}$ such that,

$$\rho_k(x_1, \cdots, x_k) = \det\left([K(x_i, x_j)]_{i,j=1}^k\right).$$

In general is very difficult to prove that a point process is determinantal, and there are cases where it can be proved that it is not determinantal, so what can one do in such cases?

This is where the double-fibration technique comes to play, let us see it in the following section.

Elliptic Polynomials

Let $a_k \in \mathbb{C}$ be i.i.d standard Gaussian random variables. Then, the elliptic polynomials, also called Kostlan-Shub-Smale polynomials, are defined by

$$f(z) = \sum_{k=0}^{N} \binom{N}{k}^{1/2} a_k z^k.$$

We consider the point process on S^2 given by the inverse stereographic projection of the zero set of elliptic polynomials.

Now, let us compute its first and second intensity.

Consider the solution variety $\mathcal{V} = \{(f, z) \in \mathcal{P}_N \times \mathbb{C} : f(z) = 0\}$ and its two projections

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] & \\ \mathcal{P}_N \arrow[rr, dashed] & & \mathbb{C} \end{tikzcd}.$$

Following (2.10), the quotient of normal Jacobians is $\frac{NJ_{\pi_1}(f,z)}{NJ_{\pi_2}(f,z)} = \frac{|f'(z)|^2}{\left(1+|z|^2\right)^N}$.

So, for any measurable function $\phi : \mathbb{C} \to [0, +\infty)$ applying the double-fibration technique (2.9) one has that

$$\int_{\mathcal{P}_N} \sum_{\{z:f(z)=0\}} \phi(z) \frac{e^{-\|f\|^2}}{\pi^{N+1}} \, d\mathcal{P}_N = \int_{\mathbb{C}} \phi(z) \int_{\mathcal{V}_z} \frac{|f'(z)|^2 e^{-\|f\|^2}}{\pi^{N+1} \left(1+|z|^2\right)^N} \, d\mathcal{V}_z,$$

meaning,

$$\rho_1(z) = \int_{\mathcal{V}_z} \frac{|f'(z)|^2 e^{-\|f\|^2}}{\pi^{N+1}\left(1+|z|^2\right)^N} d\mathcal{V}_z = \frac{N}{\pi\left(1+|z|^2\right)^2}. \tag{2.18}$$

Please observe that reorganizing the last equality and following (2.11)

$$\rho_1(z) = \mathbb{E}(|f'(z)|^2 | f(z)=0)\rho_{f(z)}(0) = \frac{N}{\pi\left(1+|z|^2\right)^2}.$$

For the second intensity, one can work in an analogous manner, but considering the solution variety $\mathcal{V} = \{(f,(z,w)) \in \mathcal{P}_N \times \mathbb{C}^2 : f(z) = 0 = f(w)\}$.

The quotient of normal Jacobians in this case satisfies

$$\frac{NJ_{\pi_2}(f,(z,w))}{NJ_{\pi_1}(f,(z,w))} = \frac{|f'(z)|^2|f'(w)|^2}{\left(1+|z|^2\right)^N\left(1+|w|^2\right)^N\left(1-|\rho(z,w)|^{2N}\right)}$$

where $\rho(z,w) := \frac{1+z\bar{w}}{\left(1+|z|^2\right)^{1/2}\left(1+|w|^2\right)^{1/2}}$ which its module is in fact, the cosine of the angle between the two roots z,w embedded in $\mathbb{R}^3$.

So, the second intensity function for the zero set of elliptic polynomials can be computed by

$$\rho_2(z,w) = \int_{\mathcal{V}_{(z,w)}} \frac{|f'(z)|^2|f'(w)|^2 e^{-\|f\|^2}}{\pi^{N+1}\left(1+|z|^2\right)^N\left(1+|w|^2\right)^N\left(1-|\rho(z,w)|^{2N}\right)} d\mathcal{V}_{(z,w)}$$

Or in an equivalent way, using (2.8),

$$\rho_2(z,w) = \mathbb{E}(|f'(z)|^2|f'(w)|^2 \mid f(z)=f(w)=0)\rho_{(f(z),f(w))}(0,0)$$

Once again, doing a Gaussian regression between the random vectors $Y = \begin{pmatrix} f'(z) \\ f'(w) \end{pmatrix}$ and $X = \begin{pmatrix} f(z) \\ f(w) \end{pmatrix}$, one gets

$$\mathbb{E}(|f'(z)|^2|f'(w)|^2 \mid f(z)=f(w)=0) = \mathbb{E}(|\eta_1|^2|\eta_2|^2),$$

for some η_1,η_2 Gaussian random vectors independent of $f(z)$ and $f(w)$. The last expectation is in turn equal to the permanent $per((\eta\eta^*))$ using Wick's formula (see (3.4)). Recall that the permanent of a matrix is analogous to the determinant, but without taking into account the signature of the permutation.

Then, it follows

$$\rho_2(z,w) = \frac{per(C - BA^{-1}B^*)}{\pi^2(1+|z|^2)^N(1+|w|^2)^N(1-|\rho(z,w)^{2N}|)}, \tag{2.19}$$

where $A = \mathbb{E}(XX^*)$, $B = \mathbb{E}(XY^*)$ and $C = \mathbb{E}(YY^*)$. If one would like to compute the k-th intensity one would do so in an analogous manner (see [HKPV][Corollary 3.4.2]).

Generalized Eigenvalue Problem (Spherical ensemble)

Let $A, B \in \mathbb{C}^{N\times N}$ be two random matrices with i.i.d. standard Gaussian entries.

We consider the point process on S^2 given by the inverse stereographic projection of the solutions of the Generalized Eigenvalue Problem for the matrices A, B, i.e. the zero set of the polynomial $\det(A + zB)$.

It was proved by Krishnapur in [K3] that this point process on the complex plane is a DPP with kernel

$$K(z,w) = \frac{N}{\pi}\frac{(1+z\bar{w})^{N-1}}{(1+|z|^2)^{\frac{N+1}{2}}(1+|w|^2)^{\frac{N+1}{2}}},$$

with respect to the Lebesgue measure in $\mathbb{C}$.

This gives rise to the following first and second order intensities,

$$\rho_1(z) = K(z,z) = \frac{N}{\pi}\frac{1}{(1+|z|^2)^2}, \tag{2.20}$$

and

$$\rho_2(z,w) = \det\begin{pmatrix} K(z,z) & K(z,w) \\ K(w,z) & K(w,w) \end{pmatrix} = \frac{N^2}{\pi^2}\frac{\left(1-|\rho(z,w)|^{2(N-1)}\right)}{(1+|z|^2)^2(1+|w|^2)^2}. \tag{2.21}$$

Polynomial Eigenvalue Problem (PEVP ensemble)

Let $d \in \mathbb{N}$ and $A_k \in \mathbb{C}^{r\times r}$ be random matrices with i.i.d. standard Gaussian entries for $0 \le k \le d$.

We consider the point process on S^2 given by the inverse stereographic projection of the solutions of the Polynomial Eigenvalue Problem for the matrices $\{A_k\}$, i.e. the zero set of the polynomial

$$F(z) = \det\left(\sum_{k=0}^{d} \binom{d}{k}^{1/2} z^k A_k\right).$$

It is easy to see that when $d = 1$ one recovers the Spherical Ensemble and when $r = 1$ one recovers the zero set of Elliptic Polynomials.

Denote by $\mathcal{M}_r(\mathcal{P}_d)$ the set of $r \times r$ matrices of polynomials of degree less than d. One can consider $F(z) = \det(A(z))$ where $A \in \mathcal{M}_r(\mathcal{P}_d)$ has i.i.d. random elliptic polynomial entries.

Consider the first solution variety $\mathcal{V} = \{(A,z) \in \mathcal{M}_r(\mathcal{P}_d) \times \mathbb{C} : F(z) = 0\}$ and its two projections

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] & \\ \mathcal{M}_r(\mathcal{P}_d) \arrow[rr, dashed] & & \mathbb{C} \end{tikzcd}.$$

As seen in (2.14), one has that the quotient of normal Jacobians is $\frac{NJ_{\pi_2}(A,z)}{NJ_{\pi_1}(A,z)} = \frac{|F'(z)|^2}{(1+|z|^2)^d \|\mathrm{adj}\,(\mathrm{A(z)})\|_{\mathrm{F}}^2}$.

Then, proceeding in the same way as before,

$$\begin{aligned}
\rho_1(z) &= \int_{\mathcal{V}_z} \frac{|F'(z)|^2 e^{-\|A\|^2}}{\pi^{r^2(d+1)}(1+|z|^2)^d \|\mathrm{adj}\,(\mathrm{A(z)})\|_{\mathrm{F}}^2}\, d\mathcal{V}_z \\
&= \mathbb{E}\left(\left|\mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\mathrm{A}'(\mathrm{z}))\right|^2 \middle| \det(\mathrm{A(z)}) = 0\right) \rho_{\det(A(z))}(0)
\end{aligned}$$

and after some manipulations we get

$$\rho_1(z) = \frac{N}{\pi(1+|z|^2)^2}. \tag{2.22}$$

Remark 2.3.1. Please observe that in the three cases, the first intensities (2.18), (2.20) and (2.22) are the same. This can be seen because all these point processes are invariant under the same group of transformations, meaning that one can not distinguish them simply by their point distributions.

Chapter 3

Computations for complex random Gaussian Fields

In this chapter, we introduce the probabilistic tools that underpin the thesis and carry out key computations essential for the developments of the main results. We refer the reader to [AW], [AT] and [HKPV] for further background.

3.1 Introduction

A complex Gaussian random field is a collection of complex random vectors indexed by points in a domain, often a complex manifold or subset of $\mathbb{C}^n$, such that every finite set of these vectors has a complex multivariate Gaussian distribution. Namely, it is a generalization of a real Gaussian field where in this case the field values are complex-valued random variables. These fields arise naturally in quantum physics, signal processing, and complex geometry (e.g., random holomorphic sections or waves).

The motivation behind working with random fields lie in the Kac-Rice formula (3.9), which is a fundamental tool in the proof of the main result of Chapter 5.

3.2 Complex Gaussian random vectors

Throughout this dissertation, we shall encounter complex Gaussian random variables. As conventions vary, we begin by establishing our terminology. By $\mathcal{N}(\mu, \sigma^2)$, we mean the distribution of the real-valued random variable with probability density $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Here $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are the mean and variance respectively.

A standard complex Gaussian is a complex-valued random variable with probability density $\frac{1}{\pi}e^{-|z|^2}$ with respect to the Lebesgue measure on the complex plane, and we denote it by $\mathcal{N}_{\mathbb{C}}(0,1)$. Equivalently, one may define it as $X + iY$, where X and Y are i.i.d. $\mathcal{N}\left(0, \frac{1}{2}\right)$ random variables.

Let $X_1, \cdots, X_n$ be i.i.d. standard complex Gaussians. Then we say that the random vector

$$X := \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \in \mathbb{C}^n$$

is a standard complex Gaussian vector. Then if $A \in \mathbb{C}^{m\times n}$ is a matrix, the random vector

$$Y = AX + \mu \in \mathbb{C}^m \tag{3.1}$$

is said to be a complex Gaussian random vector with mean μ and covariance $\Sigma = AA^*$, where A^* is the conjugate transpose of A. We denote it by $\mathcal{N}_{\mathbb{C}^m}(\mu, \Sigma)$. When μ is equal to 0, we say that is a centred Gaussian random vector.

Note that if X has $\mathcal{N}_{\mathbb{C}^m}(\mu, \Sigma)$ distribution, then for every $j, k \leq m$, we have

$$\mathbb{E}\left((X_k - \mu_k)\left(X_j - \mu_j\right)\right) = 0, \quad \text{and} \quad \mathbb{E}\left((X_k - \mu_k)\overline{\left(X_j - \mu_j\right)}\right) = \Sigma_{kj}. \tag{3.2}$$

Remark 3.2.1. If a complex Gaussian random vector satisfies the first equation of (3.2), is sometimes called a circularly symmetric random vector. Since we will not consider the case where this is not satisfied, we include it as a part of the definition of complex Gaussian random vector.

Let us now examine some important properties of Gaussian random vectors.

3.2.1 Regression Formula

One of the powerful features of multivariate Gaussian distributions is the explicit form of conditional expectations and covariances.

Let $(X, Y) \in \mathbb{C}^{n+m}$ be a jointly Gaussian random vector with mean and covariance given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_{\mathbb{C}^{n+m}}\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}\right).$$

If Σ_{YY} is not singular, then consider the random Gaussian vector

$$Z = X - \Sigma_{XY}\Sigma_{YY}^{-1}Y.$$

Note that the variance of Z satisfies $\Sigma_{ZZ} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}^{*}$, and that Z is independent of Y. Essentially, Z is obtained by subtracting from X its orthogonal projection on Y (seen as elements in L^1).

Hence, for any bounded function f, we have

$$\mathbb{E}\left(f(X) \mid Y = y\right) = \mathbb{E}\left(f\left(Z + \Sigma_{XY}\Sigma_{YY}^{-1}y\right)\right), \tag{3.3}$$

for almost every y.

This equation is called Gaussian regression (see [AW, Proposition 1.2].

Another common way to present this equation, is to say that the distribution of the random vector X given Y i.e. the conditional distribution of $(X \mid Y = y)$ is

$$\mathcal{N}_{\mathbb{C}^n}\left(\mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(y - \mu_Y), \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}\right),$$

see [HKPV, Exercise 2.1.3]).

3.2.2 Wick Formula

Wick's formula, also known as Isserlis' theorem, expresses higher-order moments of centred Gaussian variables in terms of their second moments.

Lemma ([HKPV, Lemma 2.1.7] Wick's formula)**.** *Let $(X,Y) \in \mathbb{C}^{n+m}$ be a jointly centred Gaussian random vector with mean and covariance given by*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}_{\mathbb{C}^{n+m}}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}.\right)$$

Then,

$$\mathbb{E}\left(X_1 \cdots X_n \overline{Y}_1 \cdots \overline{Y}_n\right) = \operatorname{per}\left(\Sigma_{XY}\right).$$

In particular,

$$\mathbb{E}\left(|X_1 \cdots X_n|^2\right) = \operatorname{per}\left(\Sigma_{XX}\right). \tag{3.4}$$

3.2.3 Computations for random vectors

Let us turn to some computation of the expected value of the norm of random Gaussian vectors. These computations are needed for sections 3.2.4 and 4.4.

Lemma 3.2.2. *Let v be a standard Gaussian random vector in $\mathbb{C}^n$ and $\alpha \in \mathbb{R}$ with $\alpha > -2n$. Then,*

$$\mathop{\mathbb{E}}_{v\in\mathbb{C}^n}(\|v\|^{\alpha}) = \frac{\Gamma(n+\alpha/2)}{\Gamma(n)}.$$

Proof. By a simple calculation, taking polar coordinates, we have

$$\begin{aligned}\mathop{\mathbb{E}}_{v\in\mathbb{C}^n}(\|v\|^{\alpha}) &= \frac{1}{\pi^n}\int_{v\in\mathbb{C}^n}\|v\|^{\alpha}e^{-\|v\|^2}d\mathbb{C}\\ &= \frac{\mathrm{vol}(S^{2n-1})}{\pi^n}\int_0^{\infty}\rho^{2n+\alpha-1}e^{-\rho^2}d\rho = \frac{\Gamma(n+\alpha/2)}{\Gamma(n)},\end{aligned}$$

where S^{2n-1} is the unit sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$. □

Lemma 3.2.3. *Let v be a standard Gaussian random vector in $\mathbb{C}^n$ and $\alpha,\beta \in \mathbb{R}$ with $2\alpha+\beta > 1-2n$. Then,*

$$\mathop{\mathbb{E}}_{v\in\mathbb{C}^n}\left(\|v\|^{2\alpha}\|\Pi_{e_n^{\perp}}v\|^{\beta}\right) = \frac{\Gamma(n+\alpha+\beta/2)}{n\Gamma(n-1)},$$

where $\Pi_{e_n^{\perp}}v$ is the projection of v on the Hermitian complement of e_n.

Proof. By definition of the norm in $\mathbb{C}^n$ and the binomial expansion, we have that

$$\|v\|^{2\alpha} = \left(\|\Pi_{e_n^{\perp}}v\|^2 + |v_n|^2\right)^{\alpha} = \sum_{i=0}^{\alpha}\binom{\alpha}{i}(\|\Pi_{e_n^{\perp}}v\|^2)^{\alpha-i}(|v_n|^2)^i.$$

Since the entries of v are independent, it follows

$$\begin{aligned}\mathop{\mathbb{E}}_{v\in\mathbb{C}^n}\left(\|v\|^{2\alpha}\|\Pi_{e_n^{\perp}}v\|^{\beta}\right) &= \mathbb{E}\left(\sum_{i=0}^{\alpha}\binom{\alpha}{i}\left(\|\Pi_{e_n^{\perp}}v\|^2\right)^{\alpha+\beta/2-i}\left(|v_n|^2\right)^i\right)\\ &= \sum_{i=1}^{\alpha}\binom{\alpha}{i}\mathop{\mathbb{E}}_{w\in\mathbb{C}^{n-1}}\left(\|w\|^{2\alpha+\beta-2i}\right)\mathop{\mathbb{E}}_{z\in\mathbb{C}}\left(|z|^{2i}\right).\end{aligned}$$

Then, applying the previous lemma

$$
\begin{aligned}
\mathop{\mathbb{E}}_{v\in\mathbb{C}^n}\left(\|v\|^{2\alpha}\|\Pi_{e_n^\perp}v\|^{\beta}\right) &= \sum_{i=0}^{\alpha}\frac{\Gamma(\alpha+1)\Gamma(n+\alpha+\beta/2-i-1)}{\Gamma(\alpha-i+1)\Gamma(n-1)}\\
&= \frac{\Gamma(\alpha+1)}{\Gamma(n-1)}\frac{\Gamma(n+\alpha+\beta/2)}{n\Gamma(\alpha+1)}\\
&= \frac{\Gamma(n+\alpha+\beta/2)}{n\Gamma(n-1)}.
\end{aligned}
$$

□

Lemma 3.2.4. *Let v be a Gaussian random vector in $\mathbb{C}^n$ with diagonal variance matrix $\Delta(\sigma_1,\cdots,\sigma_n)$ and $u\in\mathbb{C}^n$ an independent standard Gaussian random vector. Then,*

$$
\mathop{\mathbb{E}}_{u\in\mathbb{C}^n}\left(\mathbb{E}\left(|\Pi_u v|^2\right)\right) = \frac{\sum_{i=1}^n \sigma_i^2}{n}.
$$

Proof. Since,

$$
|\Pi_u v|^2 = \frac{|\langle v,u\rangle|^2}{\|u\|^2} = \left|\left\langle v,\frac{u}{\|u\|}\right\rangle\right|^2 = \sum_{i,j=1}^n v_i\bar{v}_j\frac{u_i\bar{u}_j}{\|u\|^2}.
$$

Taking expectation, since the entries of v are independent we have

$$
\mathbb{E}\left(|\Pi_u v|^2\right) = \sum_{i=1}^n \sigma_i^2\frac{|u_i|^2}{\|u\|^2}.
$$

Taking expectation again, since the entries of u are i.i.d., the result follows. □

3.2.4 Random matrices computations

Random matrices can be viewed as random vector in $\mathbb{C}^{r\times r}$. However, since our interest lies in computing expectations that depend on the determinant of random matrices, we believe it is more appropriate to present these results in a separate section.

Our main approach to working with the determinant follows the ideas of Azaïs and Wschebor [AW], where $|\det A|$ is interpreted as the volume of the parallelotope in $\mathbb{C}^r$ generated by the columns $A_1,\cdots,A_r$ of A.

Observe that $\mathrm{vol}(A_1,\cdots,A_k) = \mathrm{vol}(A_1,\cdots,A_{k-1})\|\Pi_{V_{k-1}^\perp}A_k\|$, where V_{k-1} is the subspace generated by $A_1,\cdots,A_{k-1}$, and $\|\Pi_{V_{k-1}^\perp}A_k\|$ is the Euclidean norm in $\mathbb{C}^r$ of the orthogonal projection of the vector A_k onto the Hermitian complement subspace V_{k-1}.

The following is a generalization of the Proposition 7.1 of Armentano et al. [ABB+2].

Lemma 3.2.5. *Let A be a standard Gaussian random matrix in $\mathbb{C}^{r\times r}$ and $k>0$. Then,*

$$\mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times r}}\left(\|A^{-1}\|_F^2\cdot|\det A|^{2k}\right) = \frac{r}{k}\prod_{i=1}^{r}\frac{\Gamma(k+i)}{\Gamma(i)}.$$

Proof. Observe that from a direct application of Cramer's rule we have

$$\|A^{-1}\|_F^2 = \frac{1}{|\det A|^2}\sum_{i,j=1}^{r}|\det(A^{ij})|^2.$$

It follows,

$$\begin{aligned}\mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times r}}\left(\|A^{-1}\|_F^2\cdot|\det A|^{2k}\right) &= \mathbb{E}\left(\sum_{i,j=1}^{r}|\det(A^{ij})|^2\cdot|\det A|^{2(k-1)}\right)\\ &= r^2\,\mathbb{E}\left(|\det(A^{nn})|^2\cdot|\det A|^{2(k-1)}\right).\end{aligned}$$

Using the ideas of Azaïs-Wschebor [AW], considering $|\det A|$ as the volume of the parallelepiped generated by the columns of A we get,

$$\mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times r}}\left(\|A^{-1}\|_F^2\cdot|\det A|^{2k}\right) = r^2\mathop{\mathbb{E}}_{z\in\mathbb{C}}\left(|z|^{2(k-1)}\right)\cdot\prod_{i=2}^{r}\mathop{\mathbb{E}}_{v\in\mathbb{C}^i}\left(\|v\|^{2(k-1)}\|\Pi_{e_i^\perp}v\|^2\right).$$

By Lemma 3.2.2 and Lemma 3.2.3 we get

$$\begin{aligned}\mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times r}}\left(\|A^{-1}\|_F^2\cdot|\det A|^{2k}\right) &= r^2\frac{\Gamma(k)}{\Gamma(1)}\prod_{i=2}^{r}\frac{\Gamma(i+k)}{i\Gamma(i-1)}\\ &= r\frac{\Gamma(k)}{\Gamma(k+1)}\prod_{j=1}^{r}\frac{\Gamma(k+j)}{\Gamma(j)} = \frac{r}{k}\prod_{j=1}^{r}\frac{\Gamma(k+j)}{\Gamma(j)}.\end{aligned}$$

□

Lemma 3.2.6. *Let A be an $r \times r$ random complex matrix with standard Gaussian i.i.d. entries. Then*

$$\mathbb{E}(\log|\det A|) = \frac{r(\psi(r+1)-1)}{2},$$

where $\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$ is the digamma function.

Proof. Using the linearity of the expectation and the properties of the logarithm, iterating this process, we conclude that

$$\mathbb{E}(\log(\mathrm{vol}(A_1,\cdots,A_r))) = \sum_{i=1}^{r} \mathop{\mathbb{E}}_{v_i \in \mathbb{C}^i}(\log(\|v_i\|)), \tag{3.5}$$

where v_i is a standard Gaussian complex vector in $\mathbb{C}^i$, and abusing notation, $\|v_i\|$ stands for the Euclidean norm of v_i in its corresponding space.

Using polar coordinates,

$$\mathop{\mathbb{E}}_{v_i \in \mathbb{C}^i}(\log(\|v_i\|)) = \frac{2}{\Gamma(i)} \int_0^{\infty} \rho^{2i-1} e^{-\rho^2} \log\rho \, d\rho = \frac{\psi(i)}{2},$$

where the last equality follows from taking $t = \rho^2$ and differentiating the integral expression of $\Gamma(i)$ with respect to i, or see directly Gradshteyn and Ryzhik [GR, 4.352.4].

Since $\psi(n) = \left(\sum_{i=1}^{n-1} \frac{1}{i}\right) - \gamma$, where γ is the Euler-Mascheroni constant, and $n\psi(n+1) = n\psi(n)+1$, it follows

$$\sum_{i=1}^{r} \psi(i) = r(\psi(r+1)-1).$$

Then, we conclude

$$\mathbb{E}(\log(|\det A|)) = \frac{r(\psi(r+1)-1)}{2}.$$

□

Lemma 3.2.7. *Let A be an $r \times r$ random complex matrix with standard Gaussian i.i.d. entries.*

Then, $\mathop{\mathbb{E}}_{A \in \mathbb{C}^{r\times r}}\left(|\det A|^2 \log|\det A|\right)$ is equal to

$$\frac{r!}{2}\left((r+1)\psi(r+2) - r - \psi(2)\right).$$

Proof. Let us define,

$$I_r(k) := \mathop{\mathbb{E}}_{A_i\in\mathbb{C}^r}\left(\mathrm{vol}(A_1,\cdots,A_k)^2\log\left(\mathrm{vol}(A_1,\cdots,A_k)\right)\right)$$

$$V_r(k) := \mathop{\mathbb{E}}_{A_i\in\mathbb{C}^r}\left(\mathrm{vol}(A_1,\cdots,A_k)^2\right)$$

where the A_i's are independent standard Gaussian random vectors in $\mathbb{C}^r$.

Observe that

$$\mathrm{vol}(A_1,\cdots,A_k) = \mathrm{vol}(A_1,\cdots,A_{k-1})\|\Pi_{V_{k-1}^{\perp}}A_k\|, \tag{3.6}$$

where V_{k-1} is the subspace generated by $A_1,\cdots,A_{k-1}$. Now, using the independence of the random vector, and proceeding iteratively, it is easy to check that

$$\begin{aligned}
I_r(k) =& I_r(k-1)\mathop{\mathbb{E}}_{v\in\mathbb{C}^{r-k+1}}\left(\|v\|^2\right) + V_r(k-1)\mathop{\mathbb{E}}_{v\in\mathbb{C}^{r-k+1}}\left(\|v\|^2\log\|v\|\right)\\
&\vdots\\
=& I_r(1)\prod_{i=r-k+1}^{r}\mathop{\mathbb{E}}_{v\in\mathbb{C}^i}\left(\|v\|^2\right) + \frac{\Gamma(r+1)}{\Gamma(r-k+1)}\sum_{i=r-k+1}^{r-1}\frac{1}{i}\mathop{\mathbb{E}}_{v\in\mathbb{C}^i}\left(\|v\|^2\log\|v\|\right)\\
=& \frac{\Gamma(r+1)}{\Gamma(r-k+1)}\sum_{i=r-k+1}^{r}\frac{1}{i}\mathop{\mathbb{E}}_{v\in\mathbb{C}^i}\left(\|v\|^2\log\|v\|\right).
\end{aligned}$$

Using the iterative formula for $I_r(r)$, we get that

$$\mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times r}}\left(|\det A|^2\log|\det A|\right) = \frac{r!}{2}\left((r+1)\psi(r+2)-r-\psi(2)\right).$$

$\square$

Remark 3.2.8. If one has a similar setting as the previous lemma, but the last column has a diagonal covariance matrix whose entries are $d_1,\cdots,d_r$ and $\sum_{i=1}^{r} d_i = N$, the previous lemma can be extended to this case using Lemma 3.2.4 and following the same reasoning,

$$\mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times r}}\left(|\det A|^2\log|\det A|\right) = \frac{N\Gamma(r)}{2}\left((r+1)\psi(r+2)-r-\psi(2)-\log\left(\frac{N}{r}\right)\right).$$

Corollary 3.2.9. *Let A be an $r \times r$ random complex matrix with standard Gaussian i.i.d. entries and M be a fixed $r \times r$ matrix.*

Then, $\underset{A \in \mathbb{C}^{r\times r}}{\mathbb{E}} \left(|\det AM|^2 \log |\det AM| \right)$ *is equal to*

$$|\det M|^2 \frac{r!}{2} \left((r+1)\psi(r+2) - r - \psi(2) + \log |\det M|^2 \right).$$

Proof. Using the multiplicative nature of the determinant, the properties of the logarithm, and the linearity of the expectation, one gets that the expectation $\underset{A \in \mathbb{C}^{r\times r}}{\mathbb{E}} \left(|\det AM|^2 \log |\det AM| \right)$ is equal to

$$|\det M|^2 \left(\underset{A \in \mathbb{C}^{r\times r}}{\mathbb{E}} \left(|\det A|^2 \log |\det A| \right) + \log |\det M| \underset{A \in \mathbb{C}^{r\times r}}{\mathbb{E}} |\det A|^2 \right).$$

Now observe that using the same recursive argument as in (3.6), one gets that $\underset{A \in \mathbb{C}^{r\times r}}{\mathbb{E}} |\det A|^2 = \prod_{i=1}^{r} \underset{v_i \in \mathbb{C}^i}{\mathbb{E}} \|v_i\|^2 = r!$, where again v_i is a standard Gaussian vector in $\mathbb{C}^i$. The result follows. $\square$

Remark 3.2.10. Please observe, that if one has another fixed matrix B multiplying on the other side, one gets an analogous result.

3.3 Complex Gaussian Random Fields

A complex random field will denote a random function

$$f : M \subset \mathbb{C}^D \to \mathbb{C}^d, \quad D \geq 1, D \geq d.$$

Namely, a complex random field is a collection $\{f(z) : z \in M\}$ of complex-valued random vectors indexed by a set M (typically a domain of $\mathbb{C}^D$). In the case where $D = d = 1$, it is called a complex random process. This notion can be extended to manifolds.

We say that a complex random field

$$f : M \subset \mathbb{C}^D \to \mathbb{C}^d$$

is a complex Gaussian random field if for any finite set $\{z_1, \dots, z_k\} \subset M$, the random vector

$$(f(z_1), \dots, f(z_k)) \in \mathbb{C}^{kd}$$

is complex Gaussian random vector.

These fields are characterized by their mean function

$$\mu(z) = \mathbb{E}[f(z)]$$

and the covariance kernel

$$K(z,w) = \mathbb{E}[f(z)f(w)^*].$$

Remark 3.3.1. As in Remark 3.2.1, complex Gaussian field is called Hermitian (or proper) if

$$\Delta(z,w) = \mathbb{E}[f(z)f(w)^T] = 0 \quad \text{for all} \quad z,w \in M.$$

From our definition this condition is always satisfied, and since we are not going to consider the case where this is not satisfied, we do not do this distinction.

3.3.1 Examples

Let us see some examples of complex Gaussian random field that will appear throughout this dissertation.

Elliptic Polynomials

Let f be an elliptic polynomial defined in (2.4). This give rise to a random map

$$f : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto f(z)$$

which is in fact a complex random process.

It is clear, that for any finite set $\{z_1, \cdots, z_k\}$ we have that

$$\begin{pmatrix} f(z_1) \\ \vdots \\ f(z_k) \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{N}z_1 & \binom{N}{2}^{1/2} z_1^2 & \cdots & \binom{N}{N}^{1/2} z_1^N \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \sqrt{N}z_N & \binom{N}{2}^{1/2} z_N^2 & \cdots & \binom{N}{N}^{1/2} z_N^N \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_N \end{pmatrix}.$$

Since $\begin{pmatrix} a_0 \\ \vdots \\ a_N \end{pmatrix}$ is a standard complex Gaussian random vector, by definition (3.1) we have that

$$\begin{pmatrix} f(z_1) \\ \vdots \\ f(z_k) \end{pmatrix} \sim \mathcal{N}_{\mathbb{C}^k}(0, \Sigma),$$

where $\Sigma_{ij} = \mathbb{E}\left(f(z_i)\overline{f(z_j)}\right)$.

It follows that this is a complex Gaussian random process.

Polynomial Systems

Let $\mathcal{P}_{(d)}^{r,n}$ be the space of complex polynomial systems, defined in the same way as $\mathcal{H}_{(d)}^{r,n}$, but for the case of polynomials of degree at most d in n-complex variables (see Chapter 4).

Consider a polynomial system $f = (f_1, \cdots, f_r) \in \mathcal{P}_{(d)}^{r,n}$ such that every f_i is an elliptic polynomial. In an analogous way as the previous case, this gives rise to a random map

$$f : \mathbb{C}^n \to \mathbb{C}^r, \qquad z \mapsto f(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_r(z) \end{pmatrix}$$

which is in fact a random field.

It is clear, that for any finite set $\{z_1, \cdots, z_k\}$ we have that

$$\begin{pmatrix} f(z_1) \\ \vdots \\ f(z_k) \end{pmatrix} \sim \mathcal{N}_{\mathbb{C}^{rk}}(0, \Sigma),$$

where Σ is given by the entries $\mathbb{E}\left(f_i(z_j)\overline{f_l(z_k)}\right)$.

It follows that this is a complex Gaussian random field.

Polynomial Eigenvalue Problem

Consider the random polynomial defined in (2.13). Then the map $F : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \det\left(\sum_{k=0}^{d} \binom{d}{k}^{1/2} z^k A_k\right),$$

where the matrices $A_i \in \mathbb{C}^{r\times r}$ are standard Gaussian random matrices (see Section 2.2.4), yields a random process. Note that if $r = 1$ we recover the elliptic polynomials which is a Gaussian process, but if $r > 1$ this process is not Gaussian.

Gaussian Analytic Functions

A random analytic function on a region $\Lambda \subset \mathbb{C}$ is called a Gaussian analytic function if for any finite subset $\{z_1, \cdots, z_k\}$ of Λ the random vector $(f(z_1), \cdots, f(z_k))$ is centred complex Gaussian random vector.

Namely, a random Gaussian analytic function is a random Gaussian field $f : \Lambda \to \mathbb{C}$ who is analytic.

3.3.2 Kac-Rice Formula

The Kac-Rice formula provides an expression for the expected measure of a level set by a random field under certain regularity and non-degeneracy conditions. Here we are going to present the case for complex random fields.

Let $f : \mathbb{C} \to \mathbb{C}$ a smooth random process and for every $u \in \mathbb{C}$ and every Borel set $B \subset \mathbb{C}$ consider

$$N_u(f,B) := \#\{z \in B : f(z) = u\}.$$

By the coarea formula (2.2), for any measurable function $\phi : \mathbb{C} \to [0,+\infty)$ we get

$$\int_{\mathbb{C}} \phi(u) N_u(f,B)\, du = \int_B \phi(f(z)) |f'(z)|^2\, dz. \tag{3.7}$$

Taking expectation on both sides, applying the Fubini Theorem, and conditioning by $f(z) = u$, we get

$$\int_{\mathbb{C}} \phi(u) \mathbb{E}(N_u(f,B))\, du = \int_{\mathbb{C}} \phi(u)\, du \int_B \mathbb{E}\left(|f'(z)|^2 \mid f(z) = u\right) \rho_{f(z)}(u)\, dz.$$

Since ϕ is arbitrary, we have that for almost every $u \in \mathbb{C}$ we get

$$\mathbb{E}(N_u(f,B)) = \int_B \mathbb{E}\left(|f'(z)|^2 \mid f(z) = u\right) \rho_{f(z)}(u)\, dz. \tag{3.8}$$

Remark 3.3.2. Note that the previous strategy still works if one considers a weight in the level set, namely consider a map $\varphi : \mathbb{C} \to [0,+\infty)$ (it could be pretty general), then

$$\mathbb{E}\left(\sum_{z \in B:\, f(z)=u} \varphi(z)\right) = \int_B \mathbb{E}\left(\varphi(z)|f'(z)|^2 \mid f(z) = u\right) \rho_{f(z)}(u)\, dz, \tag{3.9}$$

for almost all levels $u \in \mathbb{C}$.

Some practitioners regard the result above as sufficient for practical purposes. However, this is rarely adequate. In many cases, as the one of interest in this dissertation, the level of interest is $u = 0$, and a result that holds for almost every level u does not necessarily apply at the level $u = 0$. Therefore, it is essential to establish formula (3.9) for all levels. Doing so requires substantial effort, particularly in the case of non-Gaussian models.

By Hammersley's formula (see [HKPV, Theorem 3.3.1]), for any Gaussian analytic function f with strictly positive definite covariance kernel and such that $\det\left(K(z_i, z_j)\right)$ does not vanish anywhere, the counting measure on the level set at any level $u \in \mathbb{C}$ is given by

$$\mathbb{E}\left(|f'(z)|^2 \mid f(z) = u\right) p_{f(z)}(u).$$

Therefore, equation (3.9) holds for every level.

This formula can be extended to the setting of random fields. We will state a general result for the case of real-valued Gaussian random fields. A complex random field can be viewed as a real one by separating it into its real and imaginary parts, so the results apply in that context as well.

Theorem ([AAL, Theorem 2.2] Kac-Rice formula)**.** *Let $f : M \to \mathbb{R}^d$ be a Gaussian random field, $M \subset \mathbb{R}^D$ an open subset ($D \geq d$), satisfying the following:*

- *The sample paths of $f(\cdot)$ are a.s. C^1;*
- *for each $t \in M$, $f(t)$ has a positive definite variance-covariance matrix.*

Then, for every Borel subset $B \subset M$ and every level $u \in \mathbb{R}$, one has

$$\mathbb{E}(\sigma_{D-d}(\mathcal{L}_u(B))) = \int_B \mathbb{E}\left(\sqrt{\det\left(Df(t)Df(t)^T\right)} \mid f(t) = u\right) p_{f(t)}(u)\, dz, \quad (3.10)$$

where $\sigma_{D-d}(\mathcal{L}_u(B))$ is the corresponding dimensional measure of the level set $\mathcal{L}_u(B) = \{z \in B : f(z) = u\}$.

Remark 3.3.3. The Kac-Rice formula (3.10) has been extended to different settings, for instance for the non-Gaussian case (see [AAL]). Also for the case where one puts weights on the level set (see [AAL, Theorem 6.1]). To each point $z \in M$, we associate a weight $g(z)$ that depends on the location $z \in M$. Then, under certain conditions, we get

$$\mathbb{E}(G_u(f, B)) = \int_B \mathbb{E}\left(g(z)|\Delta(z)|^2 \mid f(z) = u\right) p_{f(z)}(u)\, dz, \quad (3.11)$$

where

$$G_u(f,B) = \int_{z\in B:\, f(z)=u} g(z)\, d\sigma_{D-d}(z).$$

Remark 3.3.4. When $D = d$, the level set $\mathcal{L}_u(B)$, under the condition of this theorem, consists of isolated points. Then, $\sigma_{D-d}(\mathcal{L}_u(B)) = N_u(f,B)$ is just the number of solutions in B. Then, the formula (3.10) takes its most known form

$$\mathbb{E}(N_u(f,B)) = \int_B \mathbb{E}\left(|\det Df(z)| \mid f(z) = u\right) p_{f(z)}(u)\, dz. \tag{3.12}$$

Remark 3.3.5. When $D = d$, since the level sets consists of isolated points, one can consider the Point Process given by the solution of $f(z) = u$. Note that in such case the first intensity function is given by the integrand of (3.12). Furthermore, in general the k-th intensity function is given by the integrand of the generalization of (3.12) to higher moments (see (2.19) and [HKPV, Corollary 3.4.2]).

Chapter 4

Polynomial Systems

In this chapter we study complex systems of polynomial equations. We are going to compute the condition number and its average. The results obtained in this work led to [C], which has been submitted for publication. The first part of this chapter is dedicated to placing the correct setting for this subject and also presenting the known results for the determined systems. On the last section, the new work for the underdetermined polynomial systems is displayed.

4.1 Introduction

Solving systems of equations is a fundamental problem, which has been deeply studied from different points of view, such as algebraic, geometric and numerical approaches.

A classic numerical method of solving such systems, is the called Newton's iteration. In this chapter, we establish the average values of a key quantity influencing the computational performance of Newton's operator in underdetermined scenarios. Shub and Smale introduced Newton's operator for underdetermined systems of equations in their work [SS4] (cf Dégot [D3]). The primary objective of their efforts was to develop and analyse effective algorithms for computing approximations to complete intersection algebraic subvarieties of $\mathbb{C}^n$.

This key quantity is the condition number, which measures the sensitivity of the set of solutions of the considered system, to variations of the equations (see Blum et al. [BCSS], Bürgisser-Cucker [BC]).

The condition number was introduced by Turing [T] and von Neuman-Goldstine [vNG], while studying the propagation of errors for linear equation solving and matrix inversion. Ever since then, condition numbers have played a leading role in the study of both accuracy and complexity of numerical algorithms.

As pointed out by Demmel [D4], computing the condition number of any numerical problem is a time-consuming task that suffers from intrinsic stability problems. For this reason, understanding the behaviour of the condition number in such a way that we can rely on probabilistic arguments is a useful strategy.

In order to be more precise in our statement we need to introduce some preliminary notations.

4.2 Preliminaries

For this section we refer the reader to [BCSS, Chapters 10 and 12].

4.2.1 Space of homogeneous polynomials and projective space

For every positive integer $d \in \mathbb{N}$, let $\mathcal{H}_d^n$ be the complex vector space of all homogeneous polynomials of degree d in $(n+1)$-complex variables with coefficients in $\mathbb{C}$, whose dimension is $\binom{n+d}{n}$.

We denote by a multi-index $j := (j_0, \cdots, j_n) \in \mathbb{Z}^{n+1}$, $j_i \geq 0$ for $i = 0, \cdots, n$, and consider $|j| = j_0 + \cdots + j_n$. Then, for $x = (x_0, \cdots, x_n) \in \mathbb{C}^{n+1}$, we write

$$x^j := x_0^{j_0} \cdots x_n^{j_n}.$$

It follows that any element $h \in \mathcal{H}_d^n$ can be written as

$$h(x) = \sum_{|j|=d} a_j x^j,$$

where $a_j = a_{j_0, \cdots, j_n} \in \mathbb{C}$.

We consider the Bombieri-Weyl Hermitian product in $\mathcal{H}_d^n$, defined as follows. Let $h, g \in \mathcal{H}_d^n$, be two elements, $h(x) = \sum\limits_{|j|=d} a_j x^j$, $g(x) = \sum\limits_{|j|=d} b_j x^j$, we define

$$\langle h, g \rangle_d = \sum_{|j|=d} a_j \overline{b}_j \binom{d}{j}^{-1},$$

where $\binom{d}{j} = \dfrac{d!}{j_0! \cdots j_n!}$ (see Shub-Smale [SS1]).

For any list of positives degrees $(d) := (d_1, \cdots, d_r)$, $r \leq n$, let

$$\mathcal{H}_{(d)}^{r,n} := \prod_{i=1}^{r} \mathcal{H}_{d_i}^{n}$$

be the complex vector space of homogeneous polynomial systems $h := (h_1, \cdots, h_r)$ of respective degrees d_i. It is easy to check that $\mathcal{H}_{(d)}^{r,n}$ is a complex vector space of dimension $N := \sum_{i=1}^{r} \binom{n+d_i}{n}$.

Remark 4.2.1. In the case where $r = n$, the space $\mathcal{H}_{(d)}^{n,n}$ is usually denoted by $\mathcal{H}_{(d)}$ as for example in [SS1], [BCSS], [ABB$^+$1], [BP2].

We denote by $\mathcal{D}_r$ the Bézout number associated with the list (d), i.e.

$$\mathcal{D}_r := \prod_{i=1}^{r} d_i.$$

The previously defined Hermitian product induces a Hermitian product in $\mathcal{H}_{(d)}^{r,n}$ as follows. For any two elements $h = (h_1, \cdots, h_r)$, $g = (g_1, \cdots, g_r) \in \mathcal{H}_{(d)}^{r,n}$, we define

$$\langle h, g \rangle := \sum_{i=1}^{r} \langle h_i, g_i \rangle_{d_i}. \tag{4.1}$$

The Hermitian product $\langle \cdot, \cdot \rangle$ induces a Riemannian structure in the space $\mathcal{H}_{(d)}^{r,n}$.

The space $\mathbb{C}^{n+1}$ is equipped with the canonical Hermitian inner product $\langle \cdot, \cdot \rangle$ which induces the usual Euclidean norm $\| \cdot \|$, and we denote by $\mathbb{P}(\mathbb{C}^{n+1})$ its associated projective space. This is a smooth manifold which carries a natural Riemannian metric, namely, the real part of the Fubini-Study metric on $\mathbb{P}(\mathbb{C}^{n+1})$ given in the following way: for a non-zero $x \in \mathbb{C}^{n+1}$,

$$\langle w, w' \rangle_x := \frac{\langle w, w' \rangle}{\|x\|^2},$$

for all w, w' in the Hermitian complement $x^\perp$ of x. This induces the norm $\| \cdot \|_x$ in $T_x\mathbb{P}(\mathbb{C}^{n+1})$.

4.2.2 Solution, critical and discriminant varieties and condition number

Given $h \in \mathcal{H}_{(d)}^{r,n}$, if $h(x) = 0$ then, $h(\lambda x) = 0$ for any $\lambda \in \mathbb{C}$. Then, the solutions to the polynomial system $h(x) = 0$, can be thought in the projective space $\mathbb{P}(\mathbb{C}^{n+1})$.

We may define the solution variety as the pairs $(h, x) \in \mathcal{H}_{(d)}^{r,n} \times \mathbb{P}(\mathbb{C}^{n+1})$ of polynomial systems and solutions, namely,

$$\mathcal{V} = \left\{ (h,x) \in \mathcal{H}_{(d)}^{r,n} \times \mathbb{P}(\mathbb{C}^{n+1}) / h(x) = 0 \right\},$$

then we have the following (see [BCSS, Proposition 10.1]).

Proposition 4.2.2. *The solution variety $\mathcal{V}$ is a smooth connected subvariety of $\mathcal{H}_{(d)}^{r,n} \times \mathbb{P}(\mathbb{C}^{n+1})$ of codimention r. The tangent space to $\mathcal{V}$ at (h,x) is the vector space of $(\dot{h}, \dot{x}) \in T_h\mathcal{H}_{(d)}^{r,n} \times T_x\mathbb{P}(\mathbb{C}^{n+1})$ such that $\dot{h}(x) + Dh(x)\dot{x} = 0$, where $Dh(x)v$ is the differential of the map $h : \mathbb{C}^{n+1} \to \mathbb{C}^r$ at $x \in \mathbb{C}^{n+1}$ applied to $v \in \mathbb{C}^{n+1}$.*

If we consider the two canonical projections $\pi_1 : \mathcal{V} \to \mathcal{H}_{(d)}^{r,n}$, $\pi_2 : \mathcal{V} \to \mathbb{P}(\mathbb{C}^{n+1})$, we have the following diagram:

$$\begin{tikzcd}
 & \mathcal{V} \arrow[dl, bend right, "\pi_1"'] \arrow[dr, bend left, "\pi_2"] & \\
\mathcal{H}_{(d)}^{r,n} & & \mathbb{P}(\mathbb{C}^{n+1})
\end{tikzcd}$$

We define the critical variety $\Sigma' \subseteq \mathcal{V}$ as the set of critical points of the projection π_1, i.e. the set of point (h,x) such that $D\pi_1(h,x)$ is not surjective. It can be proved that

$$\Sigma' = \{(h,x) \in \mathcal{V} \ : \ \operatorname{rank}(Dh(x)) < r\}$$

(see [SS1]).

The discriminant variety $\Sigma \subseteq \mathcal{H}_{(d)}^{r,n}$ is defined as the image of Σ' by π_1, namely, the set of systems h such that $\operatorname{rank}(Dh(x)) < r$ for some of its solutions. It can be proved that it is a zero measure set (see [SS1]).

Recall that the condition number associated to a computational problem measures the sensitivity of the outputs of the considered problem, to variations of the input (see Bürgisser-Cucker [BC] or Blum et al. [BCSS]).

For the case where $r = n$, Shub and Smale in [SS1], defined the normalized condition number at a pair $(h,x) \in \mathcal{V} \setminus \Sigma'$ as,

$$\mu_{norm}(h,x) = \|h\| \left\| (Df(x)|_{x^\perp})^{-1} \Delta \left(\|x\|^{d_i - 1} d_i^{1/2} \right) \right\|_{op} \tag{4.2}$$

and ∞ in other case.

In [D2] Dedieu defined the condition number of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$ at a point $x \in \mathbb{C}^n$, such that $f(x) = 0$ and $Df(x)$ is surjective, as

$$\mu(f,x) := \|Df(x)^\dagger\|_{op},$$

where $Df(x)^\dagger$ is the Moore-Penrose pseudo inverse of the linear map $Df(x)$, i.e. the derivative of f at x, and $\|Df(x)^\dagger\|_{op}$ is the operator norm of $Df(x)^\dagger$.

Following this idea, and using the normalized condition number μ_{norm} (4.2), Dégot [D3] suggested an extension of this condition number for the undetermined case which was adjusted into a projective quantity by Beltrán-Pardo in [BP1].

So putting all together, given $h \in \mathcal{H}_{(d)}^{r,n}$ and $x \in \mathbb{C}^{n+1}$ such that $h(x) = 0$ and $Dh(x)$ has rank r, the normalized condition number of h at x is defined by

$$\mu^r(h,x) := \|h\| \left\| Dh(x)^\dagger \Delta(d_i^{1/2} \|x\|^{d_i - 1}) \right\|_{op}, \tag{4.3}$$

where $\|\cdot\|_{op}$ is the operator norm. If the rank of $Dh(x)$ is strictly smaller than r, we set $\mu^r(h,x) := \infty$.

As done in [BS1] and [ABB$^+$1], we will also consider the Frobenius condition number, just by considering the Frobenius norm instead of the operator one, that is,

$$\mu_F^r(h,x) := \|h\| \left\| Dh(x)^\dagger \Delta(d_i^{1/2} \|x\|^{d_i - 1}) \right\|_F. \tag{4.4}$$

Recall that the Frobenius norm of a matrix L is $\mathrm{trace}(L^*L)^{1/2}$ where L^* is the adjoint of L, and its denoted by $\|L\|_F$. Note that $\mu^r(h,x) \le \mu_F^r(h,x) \le \sqrt{n}\mu^r(h,x)$.

When $r = n$, we will write $\mu_{norm}(h,x)$ and $\mu_F(h,x)$ instead.

4.2.3 Unitary invariance

The unitary group $U(n+1)$ acts on $\mathbb{C}^{n+1}$ as the group of linear automorphisms that preserve the Hermitian product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{n+1}$. More precisely,

$$\langle \sigma v, \sigma w \rangle = \langle v, w \rangle, \text{ for all } v,\ w \in \mathbb{C}^{n+1},\ \sigma \in U(n+1).$$

This action, induces an action of $U(n+1)$ on $\mathcal{H}_{(d)}^{r,n}$, given by $\sigma h(x) := h(\sigma^{-1}x)$, for all $x \in \mathbb{C}^{n+1}$, and also induces a natural action of $U(n+1)$ on $\mathbb{P}(\mathbb{C}^{n+1})$. Then we have the following result, which is the reason why we consider the Bombieri-Weyl Hermitian structure on $\mathcal{H}_{(d)}^{r,n}$.

Theorem 4.2.3. *The Hermitian structure on $\mathcal{H}_{(d)}^{r,n}$ defined in (4.1) is invariant under the action of $U(n+1)$. That is,*

$$\langle \sigma h, \sigma g \rangle = \langle h, g \rangle, \quad \textit{for all } h, g \in \mathcal{H}_{(d)}^{r,n},\ \sigma \in U(n+1).$$

Remark 4.2.4. There are several proofs of this fact, for example see [K1]. We are going to do a different proof because of its simplicity.

Proof. Please observe that it is sufficient to prove that the result holds for the case $r = 1$.

Given $x \in \mathbb{C}^{n+1}$ define the polynomial $K_d(\cdot, x) \in \mathcal{H}_d^n$ as the polynomial given by $\langle \cdot, x \rangle^d$, that is

$$K_d(y,x) = \sum_{|\alpha|=d} \binom{d}{\alpha} \bar{x}^{\alpha} y^{\alpha}.$$

It follows the definition of $K_d(\cdot,x)$ that $\langle h, K_d(\cdot,x)\rangle = h(x)$ for all $h \in \mathcal{H}_d^n$, and by a straightforward computation, one has that $\sigma K_d(\cdot,x) = \mathrm{K}_d(\cdot,\sigma x)$. Putting both things together one gets, for any $x, y \in \mathbb{C}^{n+1}$,

$$\langle \sigma K_d(\cdot,x), \sigma K_d(\cdot,y)\rangle = K_d(\sigma y, \sigma x) = K_d(y,x).$$

Also, observe that if $h \in \mathcal{H}_d^n$ is such that

$$\langle h, K_d(\cdot,x)\rangle = 0 \quad \text{for all } x \in \mathbb{C}^{n+1},$$

since $\langle h, K_d(\cdot,x)\rangle = h(x)$, then it satisfies $h(x) = 0$ for all $x \in \mathbb{C}^{n+1}$. This means, that the only polynomial orthogonal to every $K_d(\cdot,x)$ is the zero polynomial. In conclusion, the subspace generated by $\{K_d(\cdot,x) : x \in \mathbb{C}^{n+1}\}$ is exactly $\mathcal{H}_d^n$.

Then, since every $h \in \mathcal{H}_d^n$ is a linear combination of some of the $K_d(\cdot,x)$'s, by linearity the invariance follows. □

Taking the product action, the unitary group acts on $\mathcal{H}_{(d)}^{r,n} \times \mathbb{P}(\mathbb{C}^{n+1})$ by

$$\sigma(h,x) := (\sigma h, \sigma x),$$

furthermore, this action is a good one, in the sense of the following proposition.

Proposition 4.2.5. *Considering the action of $U(n+1)$ on $\mathcal{H}_{(d)}^{r,n} \times \mathbb{P}(\mathbb{C}^{n+1})$. Then, the solution variety $\mathcal{V}$ and the critical variety Σ' are invariant under this action. Furthermore, $\mu_F^r : \mathcal{V} \setminus \Sigma' \to \mathbb{R}$ is unitarily invariant; that is, for all $\sigma \in U(n+1)$, $\mu_F^r(\sigma(h,x)) = \mu_F^r(h,x)$.*

4.3 Average conditioning for determined systems

In this section we are going to work with the case $r = n$, so taking into account Remark 4.2.1 we will denote the space $\mathcal{H}_{(d)}^{n,n}$ simply as $\mathcal{H}_{(d)}$.

The different results of this section can be found in [ABB+1], [BP2], we will specify when needed. Some of the proofs are going to be omitted, only the relevant ones for the development of the following section will remain.

Given $h \in \mathcal{H}_{(d)} \setminus \Sigma$, the fibre $V_h = \{x \in \mathbb{P}(\mathbb{C}^{n+1}) : h(x) = 0\}$ has exactly $\mathcal{D}$ elements. Then, given a real number $\alpha > 0$, the α-th moment of the condition number of h can be defined as the average over its fibre V_h of the α-th power of the condition number, namely,

$$\mu_{Av}^{\alpha}(h) := \frac{1}{\mathcal{D}} \sum_{x:h(x)=0} \mu(h,x)^{\alpha}, \quad \mu_{F,Av}^{\alpha}(h) := \frac{1}{\mathcal{D}} \sum_{x:h(x)=0} \mu_F(h,x)^{\alpha}. \tag{4.5}$$

Then Armentano et al. proved in [ABB+1, Theorem 2] the following for the 2-nd moment of the Frobenius condition number:

Theorem 4.3.1 (Average Frobenius condition number)**.** *For every $\hat{h} \in \mathcal{H}_{(d)}$ and $\theta > 0$,*

$$\mathop{\mathbb{E}}_{h \sim \mathcal{N}(\hat{h}, \sigma^2 Id)} \left(\frac{\mu_{F,Av}^2(h)}{\|h\|^2} \right) \leq \frac{n}{\sigma^2},$$

and equality holds in the centred case.

Remark 4.3.2. The equality (in the centred case) of Theorem 4.3.1 implies from [ABB+1, Lemma 2] with $p = -2$ that

$$\mathop{\mathbb{E}}_{h \in \mathcal{H}_{(d)}} \mu_{F,Av}^2(h) = (N-1)n,$$

where N is the dimension of $\mathcal{H}_{(d)}$.

Remark 4.3.3. The extension of the centred case of this theorem is the motivation of this chapter (see Theorem 4.4.1).

For the operator norm case, Beltrán and Pardo proved in [BP2, Theorem 23] the following for the α-th moment of the normalized condition number.

Theorem 4.3.4 (Average condition number)**.** *Let $0 < \alpha < 4$ be a real number. Then,*

$$\mathop{\mathbb{E}}_{h \in \mathcal{H}_{(d)}} \left(\frac{\mu_{Av}^{\alpha}(h)}{\|h\|^{\alpha}} \right) = \frac{\Gamma(N)}{\Gamma(N-\alpha/2)} \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\Gamma(n-k+1-\alpha/2)}{n^{n-k+1-\alpha/2}\Gamma(n-k)}.$$

In particular,

$$\mathop{\mathbb{E}}_{h \in \mathcal{H}_{(d)}} \left(\frac{\mu_{Av}^{2}(h)}{\|h\|^{2}} \right) = (N-1)\left(n\left(1+\frac{1}{n}\right)^{n+1} - 2n - 1 \right),$$

where N is the dimension of $\mathcal{H}_{(d)}$.

4.4 Underdetermined systems

In this section we are going to work with the case $r < n$. First we are going to present the condition number for this case, afterwards we are going to prove new results for the average conditioning of said condition number.

4.4.1 Average conditioning

Let $\Sigma' := \{(h,x) \in \mathcal{H}_{(d)}^{r,n} \times \mathbb{C}^{n+1} \; : \; h(x) = 0; \; \mathrm{rank}(Dh(x)) < r\}$ and $\Sigma \subseteq \mathcal{H}_{(d)}^{r,n}$ be the projection of Σ' onto the first coordinate, commonly referred as the discriminant variety. Observe that for all $h \in \mathcal{H}_{(d)}^{r,n} \setminus \Sigma$, thanks to the inverse image of a regular value theorem, the zero set

$$V_h := \{x \in \mathbb{P}(\mathbb{C}^{n+1}) \; : \; h(x) = 0\},$$

is a complex smooth submanifold of $\mathbb{P}(\mathbb{C}^{n+1})$ of dimension $n-r$. Then it is endowed with a complex Riemannian structure that induces a finite volume form.

Now, for $h \in \mathcal{H}_{(d)}^{r,n} \setminus \Sigma$ it makes sense to consider the α-th moment of the Frobenius condition number of h, $\mu_{F,Av}^{r,\alpha}(h)$, as the average of $(\mu_F^r(h,x))^{\alpha}$ over its zero set V_h, i.e.

$$\mu_{F,Av}^{r,\alpha}(h) := \frac{1}{\mathrm{vol}(V_h)} \int_{x \in V_h} \mu_F^r(h,x)^{\alpha} \, dV_h. \tag{4.6}$$

4.4.2 Main Result

The main result of this chapter gives a closed formula for the expected value of $\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}$. To be accurate in this notion, we need to fix a probability measure in $\mathcal{H}_{(d)}^{r,n}$.

Recall the average with respect to the standard Gaussian distribution on $\mathcal{H}_{(d)}^{r,n}$, that is,

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\phi(h)) = \frac{1}{\pi^N}\int_{h\in\mathcal{H}_{(d)}^{r,n}} \phi(h)e^{-\|h\|^2}\,dh, \tag{4.7}$$

where N is the complex dimension of $\mathcal{H}_{(d)}^{r,n}$ and $\phi : \mathcal{H}_{(d)}^{r,n} \to \mathbb{R}$ is a measurable function.

The main result of this chapter is the following.

Theorem 4.4.1 (Main Theorem)**.** *The expected value, with respect to the standard Gaussian distribution, of the 2-nd moment of the relative Frobenius condition number $\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}$ is equal to*

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}\left(\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}\right) = \frac{r}{n-r+1}.$$

As a matter of fact, we will be proving a more general result (see Section 4.4.3). That result can be extended to the case where we consider the operator norm. Furthermore, after some computations (see Lemma 3.2.5), we get the closed expression for the case of the 2-nd moment stated in Theorem 4.4.4. The proof of the general result strongly relies on Theorem 4.4.6, which states that the moments of the condition number for the polynomial case are essentially the moments of the condition number of a random matrix.

Remark 4.4.2. Observe that by taking $r = n$ in the previous statement, one recovers the average of the 2-nd moment of the relative Frobenius condition number for the determined case, namely

$$\mathop{\mathbb{E}}_{g\in\mathcal{H}_{(d)}^{n,n}}\left(\frac{\mu_{F,Av}^{2}(g)}{\|g\|^2}\right) = n$$

(see Theorem 4.3.1).

From Theorem 4.4.1 and the previous remark we get the following proposition.

Proposition 4.4.3. *The expected value, with respect to the standard Gaussian distribution, of* 2*-nd moment of the relative Frobenius condition number satisfies*

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}\left(\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}\right)=\frac{1}{n-r+1}\mathop{\mathbb{E}}_{g\in\mathcal{H}_{(d)}^{r,r}}\left(\frac{\mu_{F,Av}^{2}(g)}{\|g\|^2}\right).$$

This statement provides the expected value of the 2-nd moment of the relative condition number for the underdetermined case in terms of the expected one in the determined case. From a geometric perspective, these two cases exhibit notable distinctions, and there is no inherent requirement for these expected values to be in any kind of relation. It would be interesting to understand which are the reasons behind this relation.

Corollary 4.4.4. *The expected value, with respect to the standard Gaussian distribution, of the* 2*-nd moment of the Frobenius condition number* $\mu_{F,Av}^{r,2}(h)$ *satisfies:*

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}\left(\mu_{F,Av}^{r,2}(h)\right)=\frac{(N-1)r}{n-r+1},$$

where N *is the dimension of* $H_{(d)}^{r,n}$.

Proof. To compute the expected value of $\mu_{F,Av}^{r,2}$, we just need to apply Lemma 2 of [ABB+1] to $\frac{\mu_{F,Av}^{r,2}}{\|h\|^2}$ for $p=-2$, and we get

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}\left(\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}\right)=\frac{\Gamma(N-1)}{\Gamma(N)}\mathop{\mathbb{E}}_{h\in\mathbb{S}(\mathcal{H}_{(d)}^{r,n})}\left(\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}\right),$$

where $\mathbb{S}(\mathcal{H}_{(d)}^{r,n})$ is the set of $h\in\mathcal{H}_{(d)}^{r,n}$ such that $\|h\|=1$. Since $\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}=\mu_{F,Av}^{r,2}(h)$ if $\|h\|=1$, and the latter is scale invariant, we have that

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\mu_{F,Av}^{r,2}(h))=\frac{\Gamma(N)}{\Gamma(N-1)}\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}\left(\frac{\mu_{F,Av}^{r,2}(h)}{\|h\|^2}\right).$$

□

Remark 4.4.5. In Theorem 1.4 of Beltrán-Pardo [BP1], an upper bound of the expected value of the average conditioning is computed, while using our argument we get an equality. Furthermore, using Cauchy-Schwartz inequality and Corollary 4.4.4, we get a sharper bound.

4.4.3 Proof of Main Theorem

For the proof of Theorem 4.4.1, we need some background and previous results which are explained in the following subsections.

Recall that for a homogeneous polynomial system $h \in \mathcal{H}_{(d)}^{r,n}$, we denote by $V_h \subseteq \mathbb{P}(\mathbb{C}^{n+1})$ the zero set, namely, $V_h = \{x \in \mathbb{P}(\mathbb{C}^{n+1}) \ : \ h(x) = 0\}$, which can be identified with $\pi_1^{-1}(h)$. Observe that for all systems $h \in \mathcal{H}_{(d)}^{r,n} \setminus \Sigma$, the set V_h is a projective variety of dimension $n-r$ (see Harris [H]).

For every $x \in \mathbb{P}(\mathbb{C}^{n+1})$ we denote by V_x the linear subspace of $\mathcal{H}_{(d)}^{r,n}$ given as

$$V_x := \{h \in \mathcal{H}_{(d)}^{r,n} \ : \ h(x) = 0\},$$

which can be identified with $\pi_2^{-1}(x)$.

Consider the linear map $L_0 : V_{e_0} \to \mathbb{C}^{r\times n}$ given by

$$L_0(h) = \Delta(d_i^{-1/2}) Dh(e_0)|_{e_0^{\perp}}, \tag{4.8}$$

where $\Delta(a_i)$ is the diagonal matrix whose diagonal entries are exactly the a_i's. Observe that for any $g \in \mathrm{Ker}(L_0)^{\perp}$ we have that $\|g\| = \|L_0(g)\|_F$. This implies that $NJ_{L_0}(g) = 1$. So, if we apply the smooth coarea formula to $L_0 : V_{e_0} \to \mathbb{C}^{r\times n}$, we have that for any measurable mapping $\phi : V_{e_0} \to [0, +\infty)$

$$\int_{h\in V_{e_0}} \phi(h) dV_{e_0} = \int_{A\in\mathbb{C}^{r\times n}} \int_{h\in L_0^{-1}(A)} \phi(h)\, dL_0^{-1}(A)\, dA. \tag{4.9}$$

Let us prove the following theorem, which is a first step towards the general version of Theorem 4.4.1. Consider the relative Frobenius condition number $\hat{\mu}_F^r$, then the α-th moment of the relative condition number is defined as

$$\hat{\mu}_F^{r,\alpha}(h) := \frac{1}{\mathrm{vol}(V_h)} \int_{x\in V_h} \frac{\mu_F^r(h,x)^\alpha}{\|h\|^\alpha} dV_h. \tag{4.10}$$

Theorem 4.4.6. *The expected value, with respect to the standard Gaussian distribution, of the α-th moment of the relative Frobenius condition number $\hat{\mu}_F^{r,\alpha}$ satisfies:*

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}} \left(\hat{\mu}_F^{r,\alpha}(h)\right) = \mathop{\mathbb{E}}_{M\in\mathbb{C}^{r\times(n+1)}} \left(\|M^\dagger\|_F^\alpha\right).$$

Proof. Since $h \in \mathcal{H}_{(d)}^{r,n} \setminus \Sigma$, V_h is a projective variety of degree $\mathcal{D}_r$ and dimension $n-r$, then we have,

$$\mathrm{vol}(V_h) = \mathcal{D}_r \,\mathrm{vol}(\mathbb{P}(\mathbb{C}^{n-r+1})) = \mathcal{D}_r \frac{\pi^{n-r}}{\Gamma(n-r+1)}$$

(see Mumford [M2, Theorem 5.22]). Then, by definition (4.10), we have the following

$$\hat{\mu}_F^{r,\alpha}(h) = \frac{\Gamma(n-r+1)}{\pi^{n-r}\mathcal{D}_r} \int_{x\in V_h} \frac{\mu_F^r(h,x)^\alpha}{\|h\|^\alpha}\, dV_h.$$

Taking expectation with regard to the Gaussian distribution, we get

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}} \left(\hat{\mu}_F^{r,\alpha}(h)\right) = \frac{\Gamma(n-r+1)}{\pi^{N+n-r}\mathcal{D}_r} \int_{h\in\mathcal{H}_{(d)}^{r,n}} \int_{x\in V_h} \frac{\mu_F^r(h,x)^\alpha}{\|h\|^\alpha} e^{-\|h\|^2}\, dV_h\, dh.$$

Then, by the definition of μ_F^r (4.4), applying the double-fibration technique (2.9) and the unitary invariance, taking $x = e_0$, we get that

$$\int_{h\in\mathcal{H}_{(d)}^{r,n}} \int_{x\in V_h} \frac{\mu_F^r(h,x)^\alpha}{\|h\|^\alpha} e^{-\|h\|^2}\, dV_h\, dh$$

is equal to

$$\int_{x\in\mathbb{P}(\mathbb{C}^{n+1})} \int_{h\in V_x} \|Dh(x)^\dagger \Delta(\|x\|^{d_i} d_i^{1/2})\|_F^\alpha \cdot |\det(Dh(x)Dh(x)^*)| e^{-\|h\|^2}\, dV_x\, d\mathbb{P}(\mathbb{C}^{n+1})$$
$$= \mathrm{vol}(\mathbb{P}(\mathbb{C}^{n+1})) \int_{h\in V_{e_0}} \|Dh(e_0)^\dagger \Delta(d_i^{1/2})\|_F^\alpha \cdot |\det(Dh(e_0)Dh(e_0)^*)| e^{-\|h\|^2}\, dV_{e_0}$$
$$= \frac{\pi^n \mathcal{D}_r}{\Gamma(n+1)} \int_{h\in V_{e_0}} \|L_0(h)^\dagger\|_F^\alpha \cdot |\det(L_0(h)L_0(h)^*)| e^{-\|h\|^2}\, dV_{e_0},$$

where $L_0(h)$ is given by (4.8)

If we apply (4.9), the latter is equal to

$$\frac{\pi^n \mathcal{D}_r}{\Gamma(n+1)} \int_{A\in\mathbb{C}^{r\times n}} \|A^\dagger\|_F^\alpha \cdot |\det(AA^*)| e^{-\|A\|^2}\, dA \int_{h\in L_0^{-1}(A)} e^{-\|h\|^2}\, dL_0^{-1}(A).$$

Since $L_0^{-1}(A)$ is a linear subspace of dimension $N-r-rn$, the right hand side integral is equal to π^{N-r-rn}.

Then,

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}\left(\hat{\mu}_F^{r,\alpha}(h)\right) = \frac{\Gamma(n-r+1)}{\Gamma(n+1)} \mathop{\mathbb{E}}_{A\in\mathbb{C}^{r\times n}}(\|A^\dagger\|_F^\alpha \cdot |\det(AA^*)|)$$
$$= \mathop{\mathbb{E}}_{M\in\mathbb{C}^{r\times(n+1)}}\left(\|M^\dagger\|_F^\alpha\right).$$

The last equality follows from Proposition 2.2.2 applied to the unitarily invariant map $\phi(M) = \|M^\dagger\|_F^\alpha$. □

Remark 4.4.7. This result, can be seen as an extension in the Frobenius context of computations done by Beltrán-Pardo in [BP1] and [BP2], namely that $(f,x) \mapsto (\Delta(d_i^{-1/2})Df(x),x)$ gives a partial isometry from $\mathcal{V}$ to the linear variety $\mathcal{V}^{\mathrm{lin}}$, and a similar relation can be established.

Remark 4.4.8. Note that $\|M^\dagger\|_F = \left(\sum \sigma_i(M)^2\right)^{1/2}$, where $\sigma_i(M)$ are the singular values of the matrix M. According to Edelman [E, Formula 3.12], we get that if $\alpha < 2(n-r+2)$ the α-th moment of $\|M^\dagger\|_F$ is finite.

Theorem 4.4.9. *Let $0<\alpha<2(n-r+2)$, then the expected value, with respect to the standard Gaussian distribution, of the α-th moment of the relative Frobenius condition number $\hat{\mu}_F^{r,\alpha}$ is finite and satisfies:*

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\hat{\mu}_F^{r,\alpha}(h)) = \mathcal{C}_{r,n} \mathop{\mathbb{E}}_{B\in\mathbb{C}^{r\times r}}\left(\|B^{-1}\|_F^\alpha \cdot |\det B|^{2(n-r+1)}\right),$$

where $\mathcal{C}_{r,n} := \prod_{i=1}^{n-r+1} \frac{\Gamma(i)}{\Gamma(r+i)}$.

Proof. Applying Theorem 4.4.6 and Proposition 2.2.3, we have

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\hat{\mu}_F^{r,\alpha}(h)) = \prod_{i=1}^{n-r+1} \frac{\Gamma(i)}{\Gamma(r+i)} \mathop{\mathbb{E}}_{B\in\mathbb{C}^{r\times r}}(\|B^{-1}\|_F^\alpha \cdot |\det B|^{2(n-r+1)}).$$

Taking $\mathcal{C}_{r,n} = \prod_{i=1}^{n-r+1} \frac{\Gamma(i)}{\Gamma(r+i)}$, we get

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\hat{\mu}_F^{r,\alpha}(h)) = \mathcal{C}_{r,n}^{\alpha} \mathop{\mathbb{E}}_{B\in\mathbb{C}^{r\times r}}\left(\|B^{-1}\|_F^\alpha \cdot |\det B|^{2(n-r+1)}\right).$$

□

Proof of Theorem 4.4.1:

Consider the case where $\alpha = 2$.

Applying Lemma 3.2.5 we have that

$$\mathop{\mathbb{E}}_{B\in\mathbb{C}^{r\times r}}(\|B^{-1}\|_F^2 \cdot |\det B|^{2(n-r+1)}) = \frac{r}{n-r+1}\prod_{i=1}^{r}\frac{\Gamma(n-r+1+i)}{\Gamma(i)}.$$

It follows

$$\begin{aligned}\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\hat{\mu}_F^{r,2}(h)) &= \left(\prod_{i=1}^{n-r+1}\frac{\Gamma(i)}{\Gamma(r+i)}\right)\left(\frac{r}{n-r+1}\prod_{i=1}^{r}\frac{\Gamma(n-r+1+i)}{\Gamma(i)}\right)\\ &= \frac{r}{n-r+1}\end{aligned}$$

□

Now, given $h \in \mathcal{H}_{(d)}^{r,n} \setminus \Sigma$, recall that the α-th moment of the (absolute) normalized condition number is

$$\mu_{F,Av}^{r,\alpha}(h) = \frac{\Gamma(n-r+1)}{\pi^{n-r}\mathcal{D}_r}\int_{x\in V_h}\mu_F^r(h,x)^{\alpha}\,dx,$$

then reasoning in the same way as in Corollary 4.4.4 we get the following, which is a more general version of it.

Corollary 4.4.10. *The expected value, with respect to the standard Gaussian distribution, of the α-nd moment of the Frobenius condition number $\mu_{F,Av}^{r,\alpha}(h)$ satisfies:*

$$\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\mu_{F,Av}^{r,\alpha}(h)) = \frac{\Gamma(N)}{\Gamma(N-\alpha/2)}\mathop{\mathbb{E}}_{h\in\mathcal{H}_{(d)}^{r,n}}(\hat{\mu}_F^{r,\alpha}(h)).$$

where N is the dimension of $H_{(d)}^{r,n}$.

□

Remark 4.4.11 (Results for the operator norm). Observe that if one considers the operator norm instead of the Frobenius one in (4.4), one gets the operator analogue of Theorem 4.4.6 and Theorem 4.4.9. This two combined extend Theorem 4.3.4. Please observe, that the computations done for the exact formula in the case $\alpha = 2$ are messier than for the Frobenius case. Therefore, we do not include it here.

Chapter 5

Logarithmic energy and the polynomial eigenvalue problem

In this chapter we compute the expected logarithmic energy of solutions to the polynomial eigenvalue problem for random matrices. We generalize some known results for the Shub-Smale polynomials, and the spherical ensemble. These two processes are the two extremal particular cases of the polynomial eigenvalue problem, and we prove that the logarithmic energy lies between these two cases. In particular, the roots of the Shub-Smale polynomials are the ones with the lowest logarithmic energy of the family. The results obtained in this work led to [ACF], which has been accepted for publication in Constructive Approximation.

5.1 Introduction and Main Result

The problem of finding configurations of points in the 2-dimensional sphere with small logarithmic energy is a very challenging problem, with several applications. It is one of the problems listed by Smale for the XXI Century [S], and there have been several advances in different directions related to this problem.

Given N points in $\mathbb{R}^3$, the logarithmic energy of the configuration is defined as

$$V(x_1,\dots,x_N) = -\sum_{1\le i<j\le N} \ln \|x_i - x_j\|.$$

The problem of minimizing this energy in the unit sphere S^2 is considered a very hard optimization problem, also known as the Fekete problem. Not only are the configurations of points that minimize the energy not completely understood even for a small number of points (for instance, $N = 7$), but also the asymptotic value of the minimum is not known with enough precision. More precisely, let

$$V_N = \min_{x_1,\dots,x_N \in S^2} V(x_1,\dots,x_N)$$

be the minimum of the energy in the sphere. The 7th Smale problem consists of finding a configuration of points $x_1,\dots,x_N$ in the sphere, in polynomial time in N, such that its logarithmic energy $V(x_1,\dots,x_N)$ is close enough to the minimum, namely $V(x_1,\dots,x_N) - V_N \le c \ln N$, for a universal constant c.

One of the major obstacles is that the value of V_N itself is not known with precision up to the $\ln N$ term, and therefore problem number 7 of Smale's list is still far from being solved.

Indeed, the value of V_N is [BS2]

$$V_N = \frac{\kappa}{2} N^2 - \frac{N \ln N}{4} + CN + o(N), \tag{5.1}$$

where $\kappa = \frac{1}{2} - \ln 2$ and C is an unknown constant. As far as it is known, this constant C is bounded (lower bound by [L, BL] and recently improved by [M1], upper bound by [BHS, BS2])

$$-0.0284228\ldots \le C \le \ln 2 + \frac{1}{4} \ln \frac{2}{3} + \frac{3}{2} \ln \frac{\sqrt{\pi}}{\Gamma(1/3)} = -0.027802\ldots,$$

and the upper bound is conjectured to be actually the value for C [BHS, BS2].

Despite the intrinsic difficulties of finding these optimal configurations of points, or even the value of the minimal energy, there have been some very exciting advances throughout the last decades. For instance, the diamond ensemble proposed by Beltrán and Etayo [BE], which achieves configurations of points with logarithmic energy very close to the conjectured minimum, and two other random processes, which we describe in more detail in what follows.

The first one, proposed by Armentano, Beltrán, and Shub in [ABS], consists of taking the roots of a random polynomial. Specifically, let

$$p_N(z) = \sum_{k=0}^{N} a_k \binom{N}{k}^{1/2} z^k \tag{5.2}$$

with a_k i.i.d. complex standard Gaussian coefficients $\mathcal{N}_{\mathbb{C}}(0,1)$, i.e., each coefficient is $\alpha + i\beta$, with α, β real independent zero-mean and $1/2$ variance Gaussian.

Now compute the roots of p_N in $\mathbb{C}$, and project them to S^2 through the inverse stereographic projection. The authors prove that the expected logarithmic energy of the resulting ensemble in S^2 is

$$\frac{\kappa}{2}N^2 - \frac{N\ln N}{4} - \frac{\kappa}{2}N. \tag{5.3}$$

Observe that the expression coincides up to the first two terms with (5.1), and the constant for the linear term is $-\frac{\kappa}{2} \approx 0.096\ldots$.
More recently, in [MY] the authors prove a central limit theorem for the logarithmic energy resulting from this random process, where they show that the fluctuations are of order $\sqrt{N}$, and therefore a typical realization of this process will have energy close to the expression in (5.3).

The second approach consists of taking the eigenvalues of a random matrix. Specifically, let A and B be two random matrices with i.i.d. complex standard Gaussian entries $\mathcal{N}_{\mathbb{C}}(0,1)$. Now compute the eigenvalues of the matrix $B^{-1}A$, and project them to S^2 through the inverse stereographic projection. Alishahi and Zamani [AZ] proved that the expected value of the logarithmic energy for this configuration, so-called *spherical ensemble*, is

$$\frac{\kappa}{2}N^2 - \frac{N\ln N}{4} + \left(\frac{\ln 2}{2} - \frac{\gamma}{4}\right)N - \frac{1}{8} + O\left(\frac{1}{N}\right), \tag{5.4}$$

where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant. Observe that the first two terms coincide with the known expression in (5.1), and the constant for the linear term is $\left(\frac{\ln 2}{2} - \frac{\gamma}{4}\right) \approx 0.2022\ldots$.

In this chapter, we study a strategy for producing configurations of points, for which the last two examples (zeros of random polynomials and eigenvalues of random matrices) are particular cases. Namely, let us consider the following polynomial in $\mathbb{C}$,

$$F(z) = \det\left(\sum_{i=0}^{d} G_i \binom{d}{i}^{1/2} z^i\right),$$

where each G_i is an $r\times r$ random matrix with independent entries distributed as $\mathcal{N}_{\mathbb{C}}(0,1)$. The problem of finding the zeros of this function is known as the *polynomial eigenvalue problem* (PEVP). Observe that $F(z)$ has, generically, $N = dr$ roots in $\mathbb{C}$, which can be projected to the unit sphere through the inverse stereographic projection as before. We will call this configuration of points the PEVP-ensemble.

Now, for a given number of points N, one can choose different pairs of its divisors (d,r) forming $N = dr$. Notably, for $r = 1$ and $d = N$, we obtain exactly the random polynomials as in (5.2). In the other extreme, for $r = N$ and $d = 1$, we obtain $F(z) = \det(G_0 + G_1 z)$, whose roots coincide with the eigenvalues of $-G_1^{-1}G_0$, and therefore we recover the spherical ensemble.

Some numerical experiments suggest that the expected logarithmic energy of intermediate instances (meaning $1 < d < N$) lies between the energy of the two extremal cases and decreases linearly with d. The main result of this chapter, which we state below, gives a precise computation of the expected logarithmic energy for the PEVP-ensemble. The numerical experiments, along with the analysis of this dependence on d, are presented in Section 5.3.

Theorem 5.1.1 (Main Theorem)**.** *Let $F(z)$ be the random complex polynomial of degree N defined as*

$$F(z) = \det\left(\sum_{i=0}^{d} G_i \binom{d}{i}^{1/2} z^i\right),$$

where G_i are $r \times r$ matrices with i.i.d. entries following $\mathcal{N}_{\mathbb{C}}(0,1)$. Then, the expected value of the logarithmic energy of the PEVP-ensemble is equal to

$$\mathbb{E}(V(x_1,\cdots,x_N)) = \frac{\kappa}{2}N^2 - \frac{N\log d}{4} - \frac{N}{4}\Big(1+\psi(r+1)-\psi(2)-2\ln 2\Big)$$

where $\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$ is the digamma function, i.e., the logarithmic derivative of the gamma function $\Gamma(n)$.

(See Section 5.2.2 for the proof and [AS, Section 6.3] for more information on the digamma function.)

Observe that this result generalizes the computed expected logarithmic energies of the ensembles by [AZ] and [ABS]. Moreover, for $r = N, d = 1$, we get

$$\mathbb{E}(V(x_1,\cdots,x_N)) \;=\; \frac{\kappa}{2}N^2 - \frac{N\psi(N+1)}{4} + N\left(\frac{\ln 2}{2} - \frac{\gamma}{4}\right).$$

This is actually the exact value for the expected value of the spherical ensemble, which, to the best of our knowledge, had not been computed before. Using the usual approximation of $\psi(N+1)$, we obtain the same asymptotic expression as in (5.4). A more detailed asymptotic analysis of this expression is given in Section 5.3.

Remark 5.1.2. Observe that in the matrix $\sum_{i=0}^{d} G_i \binom{d}{i}^{1/2} z^i$, each entry is a Shub-Smale polynomial, like the ones in (5.2). The distribution of the roots of these polynomials, projected to S^2, are invariant under the orthogonal group. Now, since the determinant is a homogeneous polynomial in the entries of the matrix, the zeros of the resulting process $F(z)$ are also invariant under the same group of isometries (Proposition 2.1.1 of [K3], see also [K2]). This invariance in S^2 is a desirable property if we want the resulting points of a process in which every configuration is possible to be well distributed in the sphere.

The rest of the chapter is organized as follows. The proof of the main theorem reduces to decomposing the logarithmic energy as a sum of three terms, and computing the expectation of each of them individually. Each of these computations present uneven levels of difficulties. Indeed, the results of the first and last terms follow from relatively simple random matrix computations, while the second one requires more tools, like the Kac-Rice formula.

In Section 5.2.1, we instantiate some of the results from Chapter 3 to obtain the expectation of the two most straightforward terms and use the weighted Kac-Rice formula, along with other computations for random matrices, to compute the remaining expectation. We aggregate these computations in Section 5.2.2 to complete the proof of the main theorem. We conclude in Section 5.3 with some experimental results and a discussion on the dependence on d of the expected logarithmic energy. Furthermore, in Section 5.4, we extend the work done in this chapter to a general case where there is no need to assume that all polynomial entries have the same degree.

5.2 Technical considerations and proof of Main Theorem

In what follows, we compute the expected logarithmic energy of the PEVP-ensemble, but instead of doing so in the unit sphere, we will derive the expression through the Riemann sphere (i.e., the sphere of radius $1/2$ centred at $(0,0,1/2)$ as in [SS3]). Throughout the chapter, we will denote as $z_1,\dots,z_N$ the roots of $F(z)$ in $\mathbb{C}$, and as $\hat{z}_1,\dots,\hat{z}_N$ their inverse stereographic projections onto the Riemann sphere. We can then transform this configuration to the unit sphere S^2 via

$$(a,b,c) \to (2a, 2b, 2c-1).$$

Given a configuration of points $\hat{z}_1,\dots,\hat{z}_N$ in the Riemann sphere, it is easy to see that the logarithmic energies of this configuration and the corresponding configuration in S^2 are related as

$$V(x_1,\cdots,x_N) = V(\hat{z}_1,\cdots,\hat{z}_N) - \frac{N(N-1)}{2}\log 2. \tag{5.5}$$

The goal is then to compute $\mathbb{E}(V(\hat{z}_1,\cdots,\hat{z}_N))$. Following Armentano et al. [ABS], since $F(z)$ is a polynomial, the logarithmic energy can be decomposed as follows:

$$V(\hat{z}_1,\cdots,\hat{z}_N) = (N-1)\sum_{i=1}^{N}\log\sqrt{1+|z_i|^2} - \frac{1}{2}\sum_{i=1}^{N}\log|F'(z_i)| + \frac{N}{2}\log|a_N|,$$

where a_N is the leading coefficient of $F(z)$. It is easy to see that this leading coefficient is $\det(G_d)$.

Then, taking expectation we get that $\mathbb{E}(V(\hat{z}_1,\cdots,\hat{z}_N))$ is equal to

$$(N-1)\mathbb{E}\left(\sum_{i=1}^{N}\log\sqrt{1+|z_i|^2}\right) - \frac{1}{2}\mathbb{E}\left(\sum_{i=1}^{N}\log|F'(z_i)|\right) + \frac{N}{2}\mathbb{E}(\log|\det(G_d)|). \tag{5.6}$$

By computing the expectation of the three terms in the previous expression, we obtain the main result of this chapter.

5.2.1 Three little terms

This section is dedicated to compute the three terms of (5.6).

Let us focus now on the first and last terms, which follow from a straightforward computation.

Proposition 5.2.1. *In the conditions of Theorem 5.1.1, we have*

$$\mathbb{E}\left(\sum_{i=1}^{N}\log\sqrt{1+|z_i|^2}\right) = \frac{N}{2}.$$

Proof. Observe that the function $F(z)$ is in fact a *polygaf* (see for example Krishnapur [K3]). Then, following Krishnapur [K3], we have that the first intensity function is

$$\rho_1(z) = \frac{1}{4\pi}\Delta_z\log(K(z,z)), \qquad z\in\mathbb{C},$$

where $\frac{1}{4}\Delta_z u = \frac{\partial^2 u}{\partial z \partial \bar{z}}$ is the complex Laplacian of u, and $K(z,w) = (1+z\overline{w})^d$ is the covariance kernel of each polynomial in the entries of A. Then,

$$\rho_1(z) = \frac{N}{\pi} \frac{1}{(1+|z|^2)^2}, \qquad z \in \mathbb{C}.$$

It follows,

$$\mathbb{E}\left(\sum_{i=1}^{N} \log \sqrt{1+|z_i|^2}\right) = \frac{N}{\pi} \int_{\mathbb{C}} \frac{\log \sqrt{1+|z|^2}}{(1+|z|^2)^2} \, d\mathbb{C} = \frac{N}{2},$$

where the last equality follows from taking polar coordinates. □

Remark 5.2.2. The previous result can be also proved by stating that the roots are invariant under the action of the unitary matrices (cf. Remark 5.1.2).

Proposition 5.2.3. *In the conditions of Theorem 5.1.1, we have*

$$\mathbb{E}(\log|\det(G_d)|) = \frac{r(\psi(r+1)-1)}{2}.$$

Proof. The result follows from recalling that G_d is an $r \times r$ random matrix with i.i.d. standard Gaussian entries and applying Lemma 3.2.6. □

Now we will focus on the second term of (5.6).

Let us define $A(z) := \sum_{i=0}^{d} G_i \binom{d}{i}^{1/2} z^i$, where each G_i is an $r \times r$ random matrix with i.i.d. standard Gaussian entries $\mathcal{N}_{\mathbb{C}}(0,1)$. The random complex polynomial $F(z)$, can be seen as a stochastic process from $\mathbb{C}$ to $\mathbb{C}$ by taking $F(z) = \det(A(z))$.

Applying the weighted Kac-Rice formula to this particular process (2.16) (see also [AW, Theorem 6.10] or [AAL, Theorem 6.1]),

$$\mathbb{E}\left(\sum_{i=1}^{N} \log|F'(z_i)|\right) = \int_{z\in\mathbb{C}} \mathbb{E}\left(|F'(z)|^2 \log|F'(z)| \mid F(z)=0\right) \rho_{F(z)}(0)\, dz, \quad (5.7)$$

where $\rho_{F(z)}(0)$ is the density at 0 of the random variable $F(z)$, and $\mathbb{E}(\cdot \mid F(z)=0)$ is the conditional expectation, conditioned on the event $\{F(z)=0\}$.

As a direct application of Corollary 2.1.5 we get the following.

Proposition 5.2.4. *In the conditions of Theorem 5.1.1, the density $\rho_{F(z)}(0)$ satisfies*

$$\rho_{F(z)}(0) = \frac{1}{\pi(1+|z|^2)^N\Gamma(r)}.$$

Proof. It follows from the fact that $A(z) \in \mathbb{C}^{r\times r}$ is a random matrix with i.i.d. centred Gaussian entries with variance $(1+|z|^2)^d$, and $F(z) = \det(A(z))$. □

Proposition 5.2.5. *In the conditions of Theorem 5.1.1, the conditional expectation $\mathbb{E}\left(|F'(z)|^2 \log|F'(z)| \mid F(z)=0\right)$ is equal to*

$$\frac{N(1+|z|^2)^{N-2}\Gamma(r)}{2}\left((r+1)\psi(r+2) - r - \psi(2) + \log\left(d(1+|z|^2)^{N-2}\right)\right).$$

Proof. Taking $\varphi(A(z)) = |F'(z)|^2 \log|F'(z)|$, which is in fact invariant, and applying Lemma 2.1.6, we have the following equality of conditional expectations,

$$\mathbb{E}\left(|F'(z)|^2 \log|F'(z)| \mid F(z)=0\right) = \mathbb{E}\left(|F'(z)|^2 \log|F'(z)| \mid A_r(z)=0\right) \tag{5.8}$$

where $A_r(z)$ is the r-th column of $A(z)$.

Since $F(z) = \det(A(z))$, we have that $F'(z) = \mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\mathrm{A}'(\mathrm{z}))$, where $A'(z)$ is the derivative of $A(z)$. Using the definition of the adjugate matrix, it follows from a straightforward computation that

$$\mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\mathrm{A}'(\mathrm{z})) = \sum_{\mathrm{i}=1}^{\mathrm{r}} \det\left(\mathrm{A(z)} \overset{\mathrm{i}}{\leftarrow} \mathrm{A}'(\mathrm{z})\right)$$

where the matrix $\left(A(z) \overset{i}{\leftarrow} A'(z)\right)$ is obtained from $A(z)$ replacing its i-th column with the i-th column of $A'(z)$. If we expand this matrix in terms of its columns, we get the expression

$$\left(A(z) \overset{i}{\leftarrow} A'(z)\right) = \left(A_1(z)\middle|\cdots\middle|A_{i-1}(z)\middle|A_i'(z)\middle|A_{i+1}(z)\middle|\cdots\middle|A_r(z)\right). \tag{5.9}$$

In the case where $A_r(z) = 0$, we get

$$F'(z) = \det\left(A(z) \overset{i}{\leftarrow} A'(z)\right) = \det\left(A_1(z)\middle|\cdots\middle|A_{r-1}(z)\middle|A_r'(z)\right).$$

In conclusion, the right-hand side of (5.8) is

$$\mathbb{E}\left(\left|\det\left(A(z) \overset{i}{\leftarrow} A'(z)\right)\right|^2 \log\left|\det\left(A(z) \overset{i}{\leftarrow} A'(z)\right)\right| \mid A_r(z)=0\right).$$

Let us do a Gaussian regression, as in Proposition 1 of Wschebor [W]. Write $A_r'(z) = \eta(z) + \beta A_r(z)$ such that $\eta(z)$ is a Gaussian random vector independent of $A_r(z)$.

It is easy to check that $\mathbb{E}(\eta(z)\eta(z)^*) = d(1+|z|^2)^{d-2}\mathrm{Id}_r = \sigma_\eta^2 \mathrm{Id}_r$, and we have that the last conditional expectation is equal to,

$$\mathbb{E}\left(\left|\det\left(A_1(z)\right|\cdots\left|A_{r-1}(z)\right|\eta(z)\right)\right|^2 \log\left|\det\left(A_1(z)\right|\cdots\left|A_{r-1}(z)\right|\eta(z)\right)\right|\right).$$

Lastly, the last expectation is equal to

$$\mathbb{E}\left(|\det AM|^2 \log|\det AM|\right),$$

where A is an $r\times r$ matrix with i.i.d. complex standard Gaussian entries, and M is the diagonal matrix whose first $(r-1)$ entries are $(1+|z|^2)$ the last one is σ_η.

The result follows from applying Corollary 3.2.9. $\square$

Now that we have computed successfully the density $\rho_{F(z)}(0)$ and the conditional expectation $\mathbb{E}\left(|F'(z)|^2\log|F'(z)| \mid F(z)=0\right)$, let us compute the remaining term of (5.6).

Proposition 5.2.6. *In the conditions of Theorem 5.1.1,*

$$\mathbb{E}\left(\sum_{i=1}^{N}\log|F'(z_i)|\right) = \frac{N}{2}\Big(N + \log d + (r+1)\psi(r+2) - r - \psi(2) - 2\Big).$$

Proof. From (5.7), we get that the left-hand side is equal to

$$\int_{z\in\mathbb{C}} \mathbb{E}\left(|F'(z)|^2\log|F'(z)| \mid F(z)=0\right)\rho_{F(z)}(0)\,dz.$$

Furthermore, applying Proposition 5.2.5 and Proposition 5.2.4 we have that it is equal to

$$\frac{N}{2\pi}\left(\int_{z\in\mathbb{C}}\frac{(r+1)\psi(r+2)-r-\psi(2)+\log d}{(1+|z|^2)^2}\,dz + (N-2)\int_{z\in\mathbb{C}}\frac{\log(1+|z|^2)}{(1+|z|^2)^2}\,dz\right).$$

Then, the result follows from changing to polar coordinates. $\square$

5.2.2 Proof of Main Theorem

Now, we will combine everything done so far, to prove Theorem 5.1.1.

Let $F(z)$ be the complex polynomial of degree $N = rd$, defined as

$$F(z) = \det\left(\sum_{i=0}^{d} G_i \binom{d}{i}^{1/2} z^i\right),$$

where G_i are $r \times r$ complex random matrices with i.i.d. standard Gaussian entries $\mathcal{N}_{\mathbb{C}}(0,1)$.

We want to compute $\mathbb{E}(V(\hat{z}_1, \cdots, \hat{z}_N))$, the expected value of the logarithmic energy of the projection of the roots of $F(z)$ onto the Riemann sphere.

Recall (5.6), we have that $\mathbb{E}(V(\hat{z}_1, \cdots, \hat{z}_N))$ is equal to

$$(N-1)\mathbb{E}\left(\sum_{i=1}^{N} \log\sqrt{1+|z_i|^2}\right) - \frac{1}{2}\mathbb{E}\left(\sum_{i=1}^{N} \log|F'(z_i)|\right) + \frac{N}{2}\mathbb{E}(\log|\det(G_d)|),$$

which, thanks to the computations of Section 5.2.1, is in turn equal to

$$\frac{(N-1)N}{2} - \frac{N}{4}\Big(N + \log d + (r+1)\psi(r+2) - r - \psi(2) - 2\Big) + \frac{Nr(\psi(r+1)-1)}{4}.$$

In conclusion,

$$\mathbb{E}(V(\hat{z}_1, \cdots, \hat{z}_N)) = \frac{N^2}{4} - \frac{N\log d}{4} - \frac{N}{4}\Big(1 + \psi(r+1) - \psi(2)\Big).$$

The result in S^2 follows by subtracting $\frac{N(N-1)}{2}\ln 2$, see (5.5).

□

5.3 Discussion and experimental examples

As observed above, the PEVP ensemble contains the spherical ensemble [AZ] and the random polynomial roots [ABS] as particular examples. Remember that the constant for the linear term in the energy was $\left(\frac{\ln 2}{2} - \frac{\gamma}{4}\right) \approx 0.2022\ldots$ for the spherical ensemble, and $-\frac{\kappa}{2} \approx 0.096\ldots$ for the random polynomial roots. This means that the random polynomial roots are better distributed than the spherical ensemble in terms of the logarithmic energy.

In this section, we study the dependence on d of the logarithmic energy of the PEVP ensemble in order to see if for an intermediate (d,r), the logarithmic energy lies between the two extremes described above. From Theorem 5.1.1, we have that the expectation of the logarithmic energy is

$$\mathbb{E}(V(x_1,\cdots,x_N)) = \frac{\kappa}{2}N^2 - \frac{N\log d}{4} - \frac{N}{4}\Big(1+\psi(r+1)-\psi(2)-2\ln 2\Big).$$

Now, it is well known that (see for example [AS, Section 6.3])

$$\psi(r+1) = \frac{\Gamma'(r+1)}{\Gamma(r+1)} = -\gamma + \sum_{j=1}^{r}\frac{1}{j}, \tag{5.10}$$

and using the Euler-Maclaurin formula as in [AZ] we have

$$\sum_{j=1}^{r}\frac{1}{j} = \ln r + \gamma + \frac{1}{2r} - \frac{1}{12r^2} + O\left(\frac{1}{r^4}\right). \tag{5.11}$$

Combining (5.10) and (5.11) we have

$$\psi(r+1) = \ln r + \frac{1}{2r} - \frac{1}{12r^2} + O\left(\frac{1}{r^4}\right). \tag{5.12}$$

Using (5.12) and the equality $\gamma = 1-\psi(2)$ in the logarithmic energy expression, we obtain

$$\mathbb{E}(V(x_1,\cdots,x_N)) \quad = \quad \frac{\kappa}{2}N^2 - \frac{N\ln N}{4} + N\left(\frac{\ln 2}{2} - \frac{\gamma}{4}\right) - \frac{d}{8} + \frac{d^2}{48N} + NO\left(\frac{1}{r^4}\right).$$

Let us make two observations. Firstly, for $d=1$ we recover the expression for the logarithmic energy of the spherical ensemble. Secondly, for a given N, the logarithmic energy of the PEVP ensemble decreases linearly with d. Indeed, the term $\frac{d^2}{48N}$ is comparable with $\frac{d}{8}$ only for $d \geq 6N$, which is obviously not the case if d is a factor of N.

In what follows, we present some experimental results which illustrate this dependence with d, and the comparison of the computed expected value of the logarithmic energy with the empirical values.

We take two values for the number of points, $N=60$ and $N=120$, which have several divisors, namely 12 and 16, respectively. For each N and for each pair (d,r) such that $N=dr$, we sort random matrices and solve numerically the corresponding

PEVP. The obtained eigenvalues are projected to S^2, and we compute the resulting logarithmic energy. This process is repeated 100.000 times for each pair (d,n).

In Figure 5.1, we present a violin plot (a kernel density estimation) for each d, on top of the expected value computed in Theorem 5.1.1. Notice that for $d = N$, the logarithmic energy corresponds to the roots of random polynomials [ABS].

In Figure 5.2, we show, with the same experimental data, the linear dependence of the logarithmic energy on d.

In Figure 5.3, we show the difference between the expected value computed in Theorem 5.1.1 and the density estimated in the 100.000 problem samples for each d. Observe that the distributions are centred at zero, and the variance seems to decrease with d.

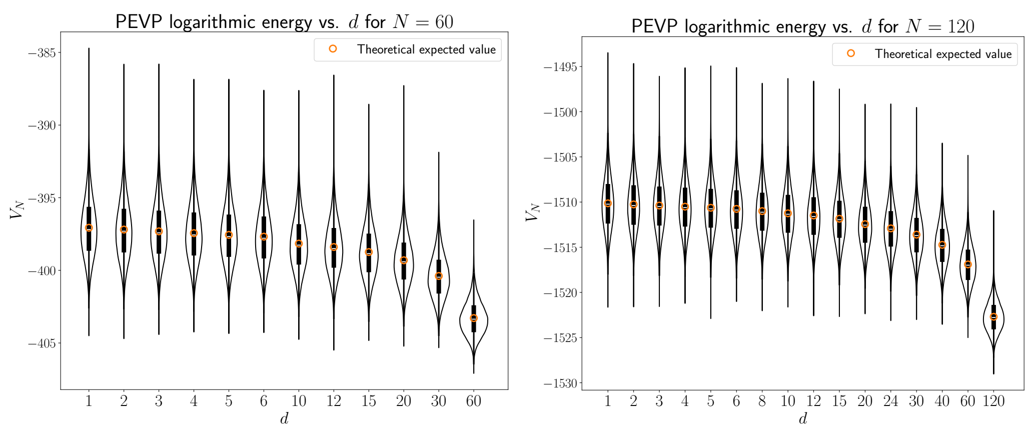

Figure 5.1: Empirical logarithmic energies for PEVP ensembles. The violin plots are computed using 100.000 repetitions for each pair (r,d).

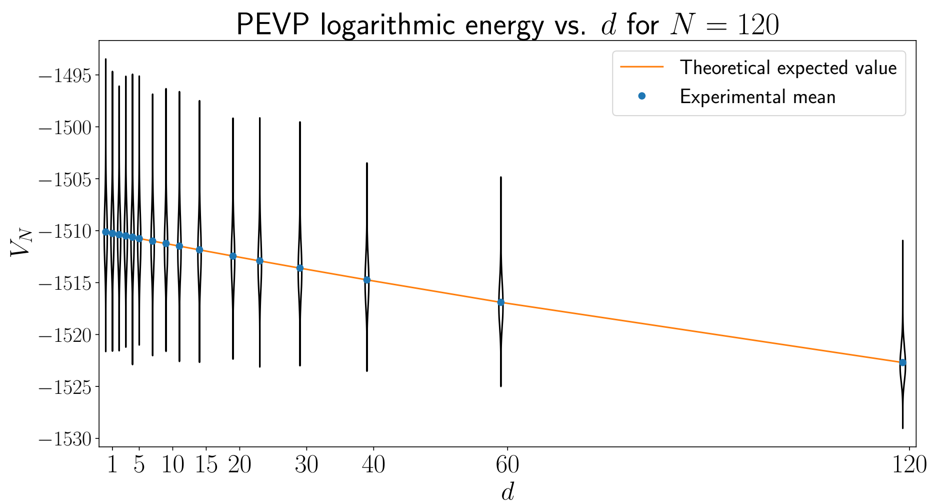

Figure 5.2: Dependence of the logarithmic energy on d. The empirical results are the same as in Fig. 5.1, but with d now correctly scaled on the x-axis, and the expected value from Theorem 5.1.1 included.

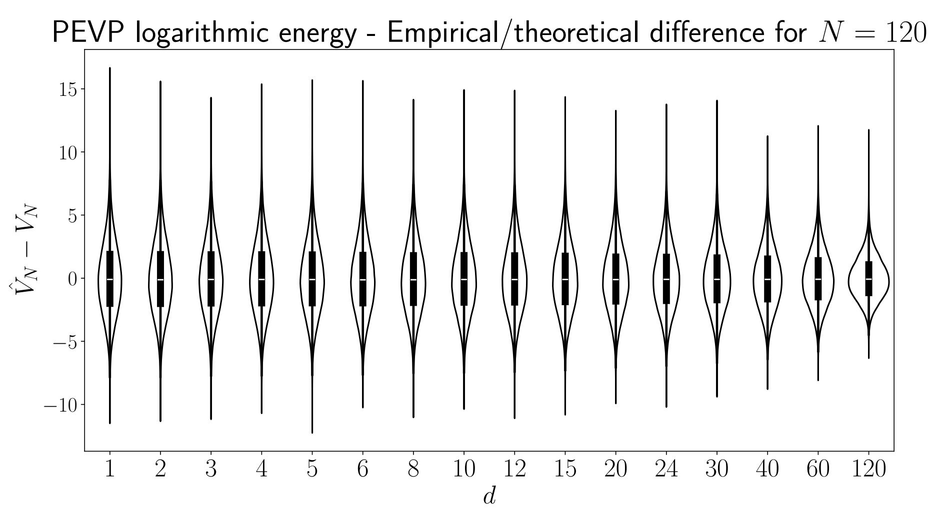

Figure 5.3: Difference between the empirical results and the expected value of Theorem 5.1.1. Notice how the difference is centred at zero for all d.

5.4 Extension for any N

In the first part of this chapter, we focused on studying the PEVP of degree d for matrices of size $r \times r$. Meaning, the point process producing $N = rd$ points. In this section we study the case for any N, extending the results previously obtained.

Let $N, r \in \mathbb{N}$ and $d = (d_1, \cdots, d_r) \in \mathbb{N}^r$ such that $\sum_{i=1}^{r} d_i = N$ and $D = \max_i d_i$. Let us consider the random polynomial

$$F(z) = \det\left(\sum_{i=0}^{D} \Delta\left(\binom{d_j}{i}^{1/2}\right) G_i z^i\right),$$

where each G_i is an $r \times r$ random matrix with independent entries distributed as $\mathcal{N}_{\mathbb{C}}(0,1)$ and $\Delta\left(\binom{d_j}{i}^{1/2}\right)$ is the diagonal matrix whose non-zero entries are $\binom{d_1}{i}^{1/2}, \cdots, \binom{d_r}{i}^{1/2}$ and recall that if $k > m$ then $\binom{m}{k} = 0$. Observe that $F(z)$ has, generically, $N = d_1 + \cdots + d_r$ roots in $\mathbb{C}$, which can be projected to the unit sphere through the inverse stereographic projection as before.

Please observe, that if we denote by $A(z)$ the random polynomial matrix

$$A(z) = \sum_{i=0}^{D} \Delta\left(\binom{d_j}{i}^{1/2}\right) G_i z^i,$$

the only difference with the work done in the previous sections is the fact that between each row of $A(z)$ we have different degree polynomials, i.e. $\deg(A(z)_{ij})$ is d_i for all j. So, in the case where every d_i is the same, we recover the previous setting.

Take $N = 7$, $r = 3$ and $d = (3,2,2)$, in such case, the random matrices that we are considering are of the form

$$G_0 + \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} G_1 z + \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} G_2 z^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} G_3 z^3, \quad (5.13)$$

where $G_i \in \mathbb{C}^{r \times r}$ have i.i.d. standard Gaussian entries.

In this sense, we get a new formulation for Theorem 5.1.1.

Theorem 5.4.1 (New Main Theorem)**.** *Let $F(z)$ be the random complex polynomial of degree N defined as*

$$F(z) = \det\left(\sum_{i=0}^{D} \Delta\left(\binom{d_j}{i}^{1/2}\right) G_i z^i\right),$$

where G_i are $r \times r$ matrices with i.i.d. entries following $\mathcal{N}_{\mathbb{C}}(0,1)$. Then, we have

$$E(V(x_1,\cdots,x_N)) = \frac{\kappa}{2}N^2 - \frac{N\log(N/r)}{4} - \frac{N}{4}\Big(1+\psi(r+1)-\psi(2)-2\ln 2\Big)$$

where $\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$ is the digamma function, i.e., the logarithmic derivative of the gamma function $\Gamma(n)$.

An interesting fact that follows the previous result is that, for fixed $N, r \in \mathbb{N}$, the expected logarithmic energy does not depend on the way one decomposes N as a sum of r positive numbers.

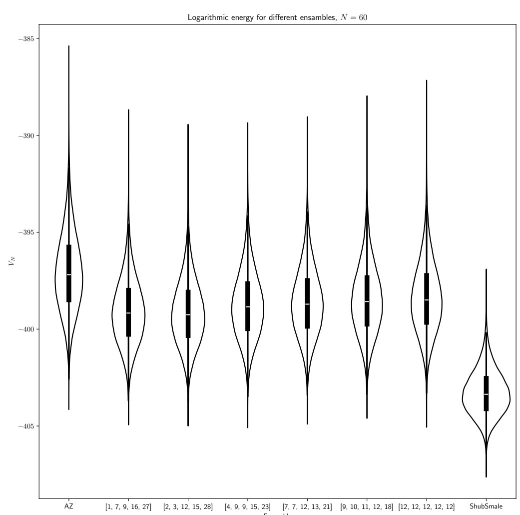

Figure 5.4: Empirical logarithmic energies for PEVP for $N = 60$ and $r = 5$ and different degree configurations. Notice how the empirical values coincide for the different ensembles with $r = 5$.

The proof of the reformulated Main Theorem is done by tweaking the proof of Theorem 5.1.1 so let us go straight to it.

Recall the expression (5.6), in that case, the leading coefficient of the polynomial $F(z)$ was $\det(G_d)$, where G_d was the matrix of leading coefficients. Now, observe that the leading coefficient for this generalization is once again the determinant of the matrix of leading coefficients G_L. In this case, the matrix G_L is the matrix whose i-th row is the i-th row of the matrix G_{d_i}. In the example (5.13), the matrix of leading coefficient would be

$$G_L = \begin{pmatrix} g_{11}^{(3)} & g_{12}^{(3)} & g_{13}^{(3)} \\ g_{21}^{(2)} & g_{22}^{(2)} & g_{23}^{(2)} \\ g_{31}^{(2)} & g_{32}^{(2)} & g_{33}^{(2)} \end{pmatrix},$$

where $g_{ij}^{(k)}$ is the ij-th entry of the matrix G_k. Then, we have an analogous expression of (5.6), namely,

$$(N-1)\mathbb{E}\left(\sum_{i=1}^{N} \log\sqrt{1+|z_i|^2}\right) - \frac{1}{2}\mathbb{E}\left(\sum_{i=1}^{N} \log|F'(z_i)|\right) + \frac{N}{2}\mathbb{E}(\log|\det(G_L)|).$$

Please note that G_L is an $r \times r$ random matrix with i.i.d. standard Gaussian entries, which means that the third term is equal to the previous case (see Proposition 5.2.3). The first term follows from Remark 5.2.2, since the set of roots is invariant under the actions of unitary matrices, which is in fact equal to the previous case.

For the middle term, we will use the same strategy is the same, but we will have to check where it differs with the previous computations.

Using Kac-Rice formula once again (2.16), we get the equality,

$$\mathbb{E}\left(\sum_{i=1}^{N} \log|F'(z_i)|\right) = \int_{z\in\mathbb{C}} \mathbb{E}\left(|F'(z)|^2 \log|F'(z)| \mid F(z)=0\right) P_{F(z)}(0)\,dz.$$

Observe that the density $\rho_{F(z)}(0)$ is the same as before, so we only need to compute the conditional expectation.

Proposition 5.4.2. *In the conditions of Theorem 5.4.1, the conditional expectation* $\mathbb{E}\left(|F'(z)|^2 \log|F'(z)| \mid F(z)=0\right)$ *is equal to*

$$\frac{N(1+|z|^2)^{N-2}\Gamma(r)}{2}\left((r+1)\psi(r+2) - r - \psi(2) + \log\left(\frac{N}{r}(1+|z|^2)^{N-2}\right)\right).$$

Proof. The proof is in essence the same as the proof of Proposition 5.2.5, but with some adjustments.

Taking $\varphi(A(z)) = |F'(z)|^2 \log |F'(z)|$, which is in fact invariant, and applying Lemma 2.1.6, we have the following equality of conditional expectations,

$$\mathbb{E}\left(|F'(z)|^2 \log |F'(z)| \mid F(z) = 0\right) = \mathbb{E}\left(|F'(z)|^2 \log |F'(z)| \mid A_r(z) = 0\right) \quad (5.14)$$

where $A_r(z)$ is the r-th column of $A(z)$.

Since $F(z) = \det(A(z))$, we have that $F'(z) = \mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\mathrm{A}'(\mathrm{z}))$, where $A'(z)$ is the derivative of $A(z)$. Using the definition of the adjugate matrix, it follows from a straightforward computation that

$$\mathrm{tr}(\mathrm{adj}\,(\mathrm{A(z)})\mathrm{A}'(\mathrm{z})) = \sum_{\mathrm{i}=1}^{\mathrm{r}} \det\left(\mathrm{A(z)} \overset{\mathrm{i}}{\leftarrow} \mathrm{A}'(\mathrm{z})\right)$$

where the matrix $\left(A(z) \overset{i}{\leftarrow} A'(z)\right)$ is defined in (5.9).

In the case where $A_r(z) = 0$, we get

$$F'(z) = \det((A(z)|A'(z))_r) = \det\left(A_1(z)\Big|\cdots\Big|A_{r-1}(z)\Big|A_r'(z)\right).$$

In conclusion, the right-hand side of (5.14) is

$$\mathbb{E}\left(\left|\det((A(z)|A'(z))_r)\right|^2 \log\left|\det((A(z)|A'(z))_r)\right| \mid A_r(z) = 0\right).$$

Let us do a Gaussian regression, as in Proposition 1 of Wschebor [W]. Write $A_r'(z) = \eta(z) + \beta A_r(z)$ such that $\eta(z)$ is a Gaussian random vector independent of $A_r(z)$.

It is easy to check that $\mathbb{E}(\eta(z)\eta(z)^*) = \Delta\left(d_i(1+|z|^2)^{d_i-2}\right)$ and the last conditional expectation is equal to,

$$\mathbb{E}\left(\left|\det\left(A_1(z)\Big|\cdots\Big|A_{r-1}(z)\Big|\eta(z)\right)\right|^2 \log\left|\det\left(A_1(z)\Big|\cdots\Big|A_{r-1}(z)\Big|\eta(z)\right)\right|\right).$$

Lastly, the last expectation is equal to

$$\mathbb{E}\left(|\det MAD|^2 \log |\det MAD|\right),$$

where A is a random matrix whose entries are all independent and the first $r-1$ columns are complex standard Gaussian vectors and the last column is a centred

Gaussian random vector with covariance matrix $\Delta(d_i)$, M is the diagonal matrix whose entries are $(1+|z|^2)^{\frac{d_i}{2}}$, and D is the diagonal matrix whose first $r-1$ entries are 1 and the last one is $(1+|z|^2)^{-1}$.

The result follows from applying Corollary 3.2.9 and taking into account Remark 3.2.8. $\square$

So, Theorem 5.4.1 follows from the fact that instead of having $\log(d)$ one has $\log(N/r)$ in the previous proposition.

Bibliography

[AAL] Diego Armentano, Jean-Marc Azaïs, and José Rafael León, *On a general Kac-Rice formula for the measure of a level set*, Ann. Appl. Probab. **35** (2025), no. 3, 1828–1851. MR4921344

[ABB$^+$1] Diego Armentano, Carlos Beltrán, Peter Bürgisser, Felipe Cucker, and Michael Shub, *Condition length and complexity for the solution of polynomial systems*, Found. Comput. Math. **16** (2016), no. 6, 1401–1422. MR3579713

[ABB$^+$2] ———, *A stable, polynomial-time algorithm for the eigenpair problem*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 6, 1375–1437. MR3801817

[ABS] Diego Armentano, Carlos Beltrán, and Michael Shub, *Minimizing the discrete logarithmic energy on the sphere: the role of random polynomials*, Trans. Amer. Math. Soc. **363** (2011), no. 6, 2955–2965. MR2775794

[ACF] Diego Armentano, Federico Carrasco, and Marcelo Fiori, *On the logarithmic energy of solutions to the polynomial eigenvalue problem.*, Constructive Approximation (Accepted) (2025).

[AGZV] Vladimir Igorevich Arnold, Sabir Medgidovich Gusein-Zade, and Alexander Varchenko, *Singularities of differentiable maps. Volume 1*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2012. Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition. MR2896292

[AS] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables. Reprint of the 1972 ed.*, 1984 (English).

[AT] Robert J. Adler and Jonathan E. Taylor, *Random fields and geometry*, Springer Monographs in Mathematics, Springer, New York, 2007. MR2319516

[AW] Jean-Marc Azaïs and Mario Wschebor, *On the roots of a random system of equations. The theorem on Shub and Smale and some extensions*, Found. Comput. Math. **5** (2005), no. 2, 125–144. MR2149413

[AZ] Kasra Alishahi and Mohammadsadegh Zamani, *The spherical ensemble and uniform distribution of points on the sphere*, Electron. J. Probab. **20** (2015), no. 23, 27. MR3325094

[BC] Peter Bürgisser and Felipe Cucker, *Condition*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 349, Springer, Heidelberg, 2013. The geometry of numerical algorithms. MR3098452

[BCSS] Lenore Blum, Felipe Cucker, Michael Shub, and Steve Smale, *Complexity and real computation*, Springer-Verlag, New York, 1998. With a foreword by Richard M. Karp. MR1479636

[BE] Carlos Beltrán and Ujué Etayo, *The diamond ensemble: a constructive set of spherical points with small logarithmic energy*, J. Complexity **59** (2020), 101471, 22. MR4099902

[BEMOC] Carlos Beltrán, Ujué Etayo, Jordi Marzo, and Joaquim Ortega-Cerdá, *A sequence of polynomials with optimal condition number*, J. Am. Math. Soc. **34** (2021), no. 1, 219–244 (English).

[BHS] Johann Brauchart, Douglas Hardin, and Edward Saff, *The next-order term for optimal Riesz and logarithmic energy asymptotics on the sphere*, Recent advances in orthogonal polynomials, special functions, and their applications, 2012, pp. 31–61. MR2964138

[BL] Carlos Beltrán and Fátima Lizarte, *A lower bound for the logarithmic energy on $\mathbb{S}^2$ and for the green energy on $\mathbb{S}^n$*, Constr. Approx. **58** (2023), no. 3, 565–587. MR4672232

[BP1] Carlos Beltrán and Luis Miguel Pardo, *On the probability distribution of condition numbers of complete intersection varieties and the average radius of convergence of Newton's method in the underdetermined case*, Math. Comp. **76** (2007), no. 259, 1393–1424. MR2299780

[BP2] ———, *Fast linear homotopy to find approximate zeros of polynomial systems*, Found. Comput. Math. **11** (2011), no. 1, 95–129. MR2754191

[BS1] Carlos Beltrán and Michael Shub, *On the geometry and topology of the solution variety for polynomial system solving*, Found. Comput. Math. **12** (2012), no. 6, 719–763. MR2989472

[BS2] Laurent Bétermin and Etienne Sandier, *Renormalized energy and asymptotic expansion of optimal logarithmic energy on the sphere*, Constr. Approx. **47** (2018), no. 1, 39–74. MR3742809

[C] Federico Carrasco, *Average conditioning of underdetermined polynomial systems*, 2025 (https://arxiv.org/abs/2305.05965).

[D1] Jean-Pierre Dedieu, *Condition number analysis for sparse polynomial systems*, Foundations of computational mathematics (Rio de Janeiro, 1997), 1997, pp. 75–101. MR1661973

[D2] ———, *Points fixes, zéros et la méthode de Newton*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 54, Springer, Berlin, 2006. With a preface by Steve Smale. MR2510891

[D3] Jérôme Dégot, *A condition number theorem for underdetermined polynomial systems*, Math. Comp. **70** (2001), no. 233, 329–335. MR1458220

[D4] James Weldon Demmel, *On condition numbers and the distance to the nearest ill-posed problem*, Numer. Math. **51** (1987), no. 3, 251–289. MR895087

[E] Alan Stuart Edelman, *Eigenvalues and condition numbers of random matrices*, ProQuest LLC, Ann Arbor, MI, 1989. Thesis (Ph.D.)–Massachusetts Institute of Technology. MR2941174

[GR] Izrail Solomonovich Gradshteyn and Iosif Moiseevich Ryzhik, *Table of integrals, series, and products*, Seventh, Elsevier/Academic Press, Amsterdam, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX). MR2360010

[H] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original. MR1416564

[HKPV] J. Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág, *Zeros of Gaussian analytic functions and determinantal point processes*, University Lecture Series, vol. 51, American Mathematical Society, Providence, RI, 2009. MR2552864

[K1] Eric Kostlan, *On the distribution of roots of random polynomials*, From Topology to Computation: Proceedings of the Smalefest (Berkeley, CA, 1990), 1993, pp. 419–431. MR1246137

[K2] Manjunath Krishnapur, *From random matrices to random analytic functions*, Ann. Probab. **37** (2009), no. 1, 314–346 (English).

[K3] Manjunath Ramanatha Krishnapur, *Zeros of random analytic functions*, ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–University of California, Berkeley. MR2709142

[L] Asbjørn Bæ kgaard Lauritsen, *Floating Wigner crystal and periodic jellium configurations*, J. Math. Phys. **62** (2021), no. 8, Paper No. 083305, 17. MR4298023

[M1] Jordi Marzo, *An improved lower bound for the logarithmic energy on* $\mathbb{S}^n$, 2025 (https://arxiv.org/pdf/2506.01660).

[M2] David Mumford, *Algebraic geometry. I*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Complex projective varieties, Reprint of the 1976 edition. MR1344216

[MY] Marcus Michelen and Oren Yakir, *Fluctuations in the logarithmic energy for zeros of random polynomials on the sphere*, Probab. Theory Related Fields **191** (2025), no. 1-2, 569–626. MR4869261

[N] Liviu Nicolaescu, *The co-area formula*, `https://www3.nd.edu/~lnicolae/Coarea.pdf`.

[S] Steve Smale, *Mathematical problems for the next century*, Mathematics: frontiers and perspectives, 2000, pp. 271–294. MR1754783

[SS1] Michael Shub and Steve Smale, *Complexity of Bézout's theorem. I. Geometric aspects*, J. Amer. Math. Soc. **6** (1993), no. 2, 459–501. MR1175980

[SS2] ———, *Complexity of Bezout's theorem. II. Volumes and probabilities*, Computational algebraic geometry (Nice, 1992), 1993, pp. 267–285. MR1230872

[SS3] ______, *Complexity of Bezout's theorem. III. Condition number and packing*, 1993, pp. 4–14. Festschrift for Joseph F. Traub, Part I. MR1213484

[SS4] ______, *Complexity of Bezout's theorem. IV. Probability of success; extensions*, SIAM J. Numer. Anal. **33** (1996), no. 1, 128–148. MR1377247

[T] Alan Mathison Turing, *Rounding-off errors in matrix processes*, Quart. J. Mech. Appl. Math. **1** (1948), 287–308. MR28100

[vNG] John von Neumann and H. H. Goldstine, *Numerical inverting of matrices of high order*, Bull. Amer. Math. Soc. **53** (1947), 1021–1099. MR24235

[W] Mario Wschebor, *On the Kostlan-Shub-Smale model for random polynomial systems. Variance of the number of roots*, J. Complexity **21** (2005), no. 6, 773–789. MR2182444