Tesis de maestría

On the involutive Banach algebra associated to a topological dynamical system

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Abstract

The crossed product is a construction that serves as a bridge between the theories of operator algebras and that of dynamical systems. Given the action of a countable discrete group G on a compact Hausdorff space X, the crossed product associated to the action is a Banach algebra whose operations encode the action of the group on the space. As such, the crossed product is equipped with a norm that makes it a Banach *-algebra. Typically, the chosen norm for the crossed product is one that turns it into a C^*-algebra, but there are many other possible norms, and this choice crucially determines the resulting crossed product. In particular, the amount of information retained in the crossed product may vary depending on the type of norm chosen.

In this work, we study the crossed product obtained by considering an ℓ^1-type norm, which yields a Banach *-algebra. We denote this crossed product by $\ell^1(G \curvearrowright X)$. We characterize various dynamical properties of the action $G \curvearrowright X$ as analytic-algebraic properties of $\ell^1(G \curvearrowright X)$, generalizing known results for actions of $\mathbb{Z}$. We determine when the action is topologically free in terms of $\ell^1(G \curvearrowright X)$, and, assuming topological freeness, we characterize minimality, topological transitivity, and residual topological freeness of the action as different properties of the ideal structure of $\ell^1(G \curvearrowright X)$. We also show that, for abelian torsion-free groups, $\ell^1(G \curvearrowright X)$ detects freeness of the action in a way that cannot be done in the much more studied C^*-crossed product. This result suggests that the crossed product retains more dynamical information about $G \curvearrowright X$ than its C^*-counterpart.

Finally, we answer the question of exactly how much information about the action is retained by $\ell^1(G \curvearrowright X)$. The answer, perhaps surprisingly, is that all the information of the action is remembered by the crossed product, in the following sense: if two actions $G \curvearrowright X$ and $H \curvearrowright Y$ are such that $\ell^1(G \curvearrowright X)$ is isometrically isomorphic to $\ell^1(H \curvearrowright Y)$, then the actions are conjugate. This shows a sharp contrast between $\ell^1(G \curvearrowright X)$ and its C^*-counterpart since, while the precise amount of dynamical information retained by the latter is not fully understood, it is generally related to the orbital equivalence class of the action rather than its conjugacy class.

Resumen

El producto cruzado es una construcción que actúa como puente entre la teoría de álgebras de operadores y los sistemas dinámicos. Dada la accción de un grupo discreto y numerable G en un espacio compacto Hausdorff X, el producto cruzado asociado a la acción es un álgebra de Banach cuyas operaciones codifican la acción del grupo en el espacio. Como tal, el producto cruzado está equipado con una norma que hace de este una *-álgebra de Banach. Usualmente, el tipo de norma elegida para el producto cruzado suele ser una que hace de este una C^*-álgebra, pero hay multitud de otras normas que pueden elegirse, y esta elección determina de manera crucial el producto cruzado obtenido. En particular, la cantidad de información retenida por el producto cruzado puede variar en función del tipo de norma elegida.

En este trabajo, nos ocuparemos del estudio del producto cruzado obtenido al considerar una norma de tipo ℓ^1, lo cual da como resultado una *-álgebra de Banach. Denotamos a este producto cruzado $\ell^1(G \curvearrowright X)$. Caracterizaremos distintas propiedades dinámicas de la acción $G \curvearrowright X$ como propiedades analítico-algebraicas de $\ell^1(G \curvearrowright X)$, generalizando resultados conocidos para acciones de $\mathbb{Z}$. Determinaremos precisamente cuándo la acción es topológicamente libre en términos de $\ell^1(G \curvearrowright X)$ y, asumiendo libertad topológica de la acción, también caracterizaremos minimalidad, transitividad topológica y libertad topológica residual de $G \curvearrowright X$ en función de propiedades sobre la estructura de ideales de $\ell^1(G \curvearrowright X)$. Probaremos además que, para grupos abelianos y libres de torsión, $\ell^1(G \curvearrowright X)$ detecta libertad de la acción de una forma que no puede ser detectada en el mucho más estudiado producto cruzado C^*. Este resultado es un indicio de que el producto cruzado $\ell^1(G \curvearrowright X)$ retiene más información dinámica de $G \curvearrowright X$ que su contraparte C^*-algebraica.

Finalmente, responderemos la interrogante de exactamente cuánta información de la acción es retenida por $\ell^1(G \curvearrowright X)$. La respuesta, quizás sorprendentemente, es que toda la infomación de la acción es recordada por el producto cruzado en el siguiente sentido: si dos acciones $G \curvearrowright X$ y $H \curvearrowright Y$ son tales que $\ell^1(G \curvearrowright X)$ es isométricamente isomorfo a $\ell^1(H \curvearrowright Y)$, entonces las acciones son conjugadas. Esto muestra un contraste marcado entre $\ell^1(G \curvearrowright X)$ y su contraparte C^*, dado que, si bien no se sabe con exactitud cuál es la información dinámica de la acción que es retenida en el producto cruzado C^*, esta suele estar relacionada con la clase de equivalencia orbital de la acción más que con su clase de conjugación.

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Introduction

We arrived at this thesis with the idea of trying to work both on dynamical systems and operator algebras, and as such, we ended up working with *crossed products.* The crossed product construction yields a large class of examples of operator algebras, having become an essential part of the theory by now.

Crossed products first appeared in an algebraic setting more than a hundred years ago and, inspired by it, Murray and Von Neumann developed an analytic counterpart. This was done in the context of *Von Neumann algebras*, as they were trying to construct examples of the so called *factors*. Von Neumann algebras are usually associated with the measurable setting, and the standard (and essentially unique) example of a commutative Von Neumann algebra is $L^\infty(X,\mu)$, where (X,μ) is a measure space. Under fitting hypotheses, the action of a group G via measurable transformations on (X,μ) gives rise to a crossed product that is a Von Neumann algebra. The crossed product is a noncommutative algebra, and the idea behind it is that its operations somehow codify the action of G on (X,μ).

Von Neumann algebras are a particular type of C^**-algebras*, and crossed product of C^*-algebras have become commonplace by now. These are constructed using as input a C^*-*dynamical system*: that is the action of a locally compact group G by automorphisms on a C^*-algebra A. Gelfand duality says that commutative C^*-algebras are essentially locally compact Hausdorff spaces, where, if X is a locally compact and Hausdorff space, then it corresponds to the commutative algebra $C_0(X)$, which consists of functions from X to $\mathbb{C}$ vanishing at infinite. So, in the case where the algebra involved in the C^*-dynamical system is commutative, we can think of it as a dynamical system in the usual sense, given that the action of G on A is equivalent to the action of G on some locally compact Hausdorff space X. As the action is over $C_0(X)$ and this algebra captures the topology of X, these dynamical systems are topological in nature.

We are going to work with G a discrete group and X a compact Hausdorff space such that there is an action, that we denote by σ, of G on X via homeomorphisms. We sum up the information of this action via the following notation: $G \curvearrowright X$. In order to construct the C^*-algebraic crossed product, what one usually does is to take the completion of an *algebraic skeleton* with respect to a suitable C^*-norm. Once that this has been done, one can try to characterize dynamical properties of $G \curvearrowright X$ as analytic-algebraic properties of the crossed product, and vice versa. This can prove to be a useful tool as it may allow one to construct examples of C^*-algebras with some desired properties: if the crossed product having property P is proved to be equivalent to the action from which it arises having property Q, then in order to construct C^*-algebras with property P it is enough to find dynamical systems with property Q. Examples of dynamical properties of this kind that have been studied are topological freeness, minimality, topological transitivity, among others.

The choice to work on the C^*-algebraic setting is not a whim, as C^*-algebras have really good properties, but there are other frameworks that could be used. The choice of the framework lies in what kind of norm is picked in order to complete the algebraic skeleton of the crossed product and, in that sense, other types of norms have been proposed. One of such norms is the L^p*-operator norm*, which gives a crossed product that is an L^p*-operator algebra*, a class of Banach algebras that has received a great degree of attention in later years (see, for example, the survey paper [**Gar21**]). Another type of norm that can be taken is a sort of ℓ^1*-norm*, this being the norm that we focus on in this work.

Let us be a little more concrete. When we mention an algebraic skeleton, we refer to the following:

$$C_c(G \curvearrowright X) = \{f : G \to C(X) \mid f(s) = 0 \text{ for all but finitely many } s \in G\}.$$

This space is a complex vector space with the natural pointwise operations, and we would like to endow it with a product that would somehow *implement the action of* G *on* X. As the whole information of X is contained in $C(X)$, the algebra of continuous functions from X to $\mathbb{C}$, we have to translate the action of G on X to an action on $C(X)$. That is not difficult to do: for every $s \in G$ the action is given by the map $\alpha_s : C(X) \to C(X)$ satisfying $\alpha_s(f) = f \circ \sigma_s^{-1}$ for every $f \in C(X)$. The vector space $C_c(G \curvearrowright X)$ contains canonically both the group G and the algebra $C(X)$. The group is contained in the following way: for every $s \in G$, we consider δ_s the map from G to $C(X)$ that maps s to the constant function of value 1 and maps to 0 every other element of the group. The algebra $C(X)$ can be seen sitting inside of $C_c(G \curvearrowright X)$ as follows: for every $f \in C(X)$ we identify it with the element of $C_c(G \curvearrowright X)$ that maps the identity of the group to f and everything else to 0. Then, the product is determined by the following two rules:

(1) For every s and $t \in G$, we set $\delta_s \delta_t = \delta_{st}$.
(2) For every $s \in G$ and for every $f \in C(X)$, we set $\delta_s f \delta_s^{-1} = \alpha_s(f)$.

With these rules, we have effectively that $C_c(G \curvearrowright X)$ contains a copy of the group that implements the action of G on $C(X)$, but, as it stands, it has no analytic structure. We endow $C_c(G \curvearrowright X)$ with the following ℓ^1-norm: for every $f \in C(X)$, we set

$$\|f\| = \sum_{s \in G} \|f(s)\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the supremum norm. The completion of $C_c(G \curvearrowright X)$ with respect to this norm yields the following algebra:

$$\ell^1(G \curvearrowright X) = \left\{ f : G \to C(X) : \sum_{s \in G} \|f(s)\|_\infty < \infty \right\}.$$

This algebra is the one that concerns us in this work. We can endow $\ell^1(G \curvearrowright X)$ with an involution such that it becomes a Banach *-algebra (a Banach algebra together with an involution satisfying some natural axioms), and this involution plays a role in detecting some dynamical properties fo the action.

Before going into the details of the work that we are going to develop, let us first give a few arguments in favor of the study of $\ell^1(G \curvearrowright X)$ ahead of other analytic completions of $C_c(G \curvearrowright X)$. The first one comes as Wendel's theorem. Let H and H' be two locally compact Hausdorff groups, we can consider $L^1(H)$ and $L^1(H')$, their algebras of integrable functions (with respect to the Haar measure). It is clear that if H and H' are isomorphic as topological groups, then $L^1(H)$ is isometrically isomorphic to $L^1(H')$. But, maybe surprisingly, Wendel's theorem states that the reciprocal also holds: if $L^1(H)$ is isometrically isomorphic to $L^1(H')$ then H and H' are isomorphic as topological groups (see [**Wen51**]). This means that the L^1-algebra of the group remembers all the information of the group it arises from, something that does not necessarily happen with other analytic completion. For example, the group C^*-algebra $C^*(\mathbb{Z}_4)$ is (isometrically) isomorphic to $C^*(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$, but it is clear that the groups are not isomorphic.

The second reason in favor of $\ell^1(G \curvearrowright X)$ is of more recent nature. In 2012, de Jeu, Svensson and Tomiyama studied in [**dJST12**], for $G = \mathbb{Z}$, the algebra $\ell^1(\mathbb{Z} \curvearrowright X)$, a work that was later followed by de Jeu and Tomiyama in [**dJST12, dJT12, dJT13, dJT16, dJT18**]. They characterized many dynamical properties of $\mathbb{Z} \curvearrowright X$, one of which we want to particularly highlight: [**dJST12**, Theorem 4.4] states that $\mathbb{Z} \curvearrowright X$ is free if and only if every closed ideal of $\ell^1(\mathbb{Z} \curvearrowright X)$ is self-adjoint (by self-adjoint we mean that the ideal is closed under the involution operation). This is interesting because, in a C^*-algebra, every closed ideal is automatically self-adjoint,which means that we cannot detect freeness of the action by looking at the C^*-crossed product arising from $\mathbb{Z} \curvearrowright X$ or, at the very least, we cannot detect freeness the same way we do in $\ell^1(\mathbb{Z} \curvearrowright X)$.

The initial aim of this master's project was to try to generalize the results of [**dJST12**] to discrete and countable infinite groups. In [**dJST12**], topological freeness (among other dynamical properties) of $\mathbb{Z} \curvearrowright X$ was characterized as the ideal intersection property of $\ell^1(\mathbb{Z} \curvearrowright X)$ (Theorem 4.1 of [**dJST12**]). We managed to do this and some more, as it is detailed in Chapter 2. One of our main concerns laid in trying to generalize [**dJST12**, Theorem 4.4] where, as we just mentioned, freeness of $\mathbb{Z} \curvearrowright X$ is proved to be equivalent to every closed ideal of $\ell^1(\mathbb{Z} \curvearrowright X)$ being self-adjoint. Let us point out something that is quite obvious but relevant to the proof of [**dJST12**, Theorem 4.4]. For an action of $\mathbb{Z}$, the following are equivalent:

(1) The action is free.
(2) The action has no finite orbits.

This means that, if you were to prove or use that a $\mathbb{Z}$-action is free, you can use either of the two characterizations of freeness given above. In [**dJST12**, Theorem 4.4], in order to prove that $\mathbb{Z} \curvearrowright X$ being free implies that every closed ideal is self-adjoint, they used (1), but for the other implication they used (2). As this does not hold for arbitrary groups, we cannot adapt the same proof in a more general context.

Does the above result hold for discrete and countable infinite groups? The short answer is no: we show a simple example of an action $G \curvearrowright S^1$ of a nonabelian group G on the circle S^1 that is not free but such that every closed ideal of $\ell^1(G \curvearrowright S^1)$ is self-adjoint. Regardless, the situation changes when working with abelian groups under some extra hypotheses. In general, freeness of the action always implies that every closed ideal of the ℓ^1-crossed product is self-adjoint and, for abelian torsion-free groups, every closed ideal being self-adjoint implies that the action is free. We interpret this result as reinforcing the idea that the ℓ^1-crossed product retains more information about the action than its C^*-counterpart.

The last matter that concerns us is exactly how much information of the action the ℓ^1-crossed product retains. We go into the details later in this work, but let us briefly mention here some precedents when working with other norms. In the (reduced) C^*-algebraic context, given two actions $G \curvearrowright X$ and $H \curvearrowright Y$ (of discrete groups on adequate topological spaces) and their associated C^*-crossed products, an isomorphism between the algebras by itself does not actually give you much information about the relation between the actions. But, if the actions are topologically free, then an isomorphism between the algebras that preserves a canonical commutative subalgebra is equivalent to the actions *having the same orbits in a continuous sense* (see [**Li18**]). This is inspired by an analogous phenomenon occurring on Von Neumann crossed products associated to measurable actions. The situation is more rigid when working with L^p-operator algebras crossed products, for $p \in [1, \infty)\setminus\{2\}$: any isometric isomorphism must automatically preserve the canonical commutative subalgebra, so, if the actions are topologically free, an isometric isomorphism between the algebras is equivalent to the actions having the same orbits (see [**CGT24**]). It would seem then that the key factors playing a role here are the notions of topological freeness and of *orbit equivalence*. But, surprisingly, the ℓ^1-crossed product seems to be even more rigid than the other examples listed above! As we are going to see, an isometric isomorphism between the two ℓ^1-crossed products associated to two actions is equivalent to the action being *conjugated*, in a sense that we make precise later. Moreover, topological freeness of the action is not required. This means that the ℓ^1-crossed product retains all the information of the action up to the notion of conjugation so, essentially, all dynamical information of the action is preserved in the ℓ^1-crossed product. We can think of this result as an analogous to Wendel's theorem for ℓ^1-crossed products arising from topological actions.

The above introduction, in short, sums up the work in this thesis. Let us now briefly describe its structure. In Chapter 1, we introduce, without proofs, the framework that we deem as necessary to develop the rest of the work. We introduce some basic theory about topology, dynamics and Banach algebras. In Chapter 2, which is based mostly on [**Rol25**], we introduce the ℓ^1-crossed product and start to characterize different dynamical properties of the action (topological freeness, minimality, topological transitivity, and residual topological freeness) as analytic-algebraic properties of the ℓ^1-crossed product. In the last section of this chapter we prove that, for abelian torsion-free groups, freeness of the action is equivalent to every closed ideal of the algebra being self-adjoint, and we also see an example of an action of a nonabelian group with torsion acting non-freely but such that every closed ideal of the algebra is self-adjoint. In Chapter 3, which is mostly based on yet unpublished joint work with Eusebio Gardella, we tackle the question of exactly how much information of the action can be recovered from the ℓ^1-crossed product. First of all, we do some preliminary work on general Banach algebras in order to introduce the objects that later allow us to recover both the group and the action (up to some conjugation equivalence) from the ℓ^1-crossed product. We also introduce the (reduced) L^p-operator algebra crossed product, as it is a useful tool for later proving the main theorem of this chapter. In the last section of this chapter, we prove the rigidity result of the ℓ^1-crossed product, as we see that every isometric isomorphism between these kinds of algebras must have quite an explicit formula, and from this follows that the existence of an isometric isomorphism implies that the actions must be conjugated.

CHAPTER 1

Preliminaries

The aim of this chapter is to introduce the necessary background material for the rest of this work. It is divided in two sections. The first one is devoted to briefly giving some context on topology and topological dynamical systems. In the second section, which is a little longer than the previous one, we introduce different classes of Banach algebras and state basic results that we are going to use later on this work.

1.1. Topology and dynamics

We are going to work with topological spaces and topological actions. In consequence, here we give the necessary background for these topics that are needed in this work. All of the results that we mention here are standard and can be seen in any introductory book of topology, as for example, [**Mun00**].

We are going to work with compact Hausdorff spaces, and there are many properties of these spaces that are useful. The first one is that every compact and Hausdorff space is a *Baire space*. We recall that a Baire space is a topological space such that a countable union of closed sets with empty interior has empty interior, or, equivalently, a countable intersection of dense open sets is a dense set.

Compact Hausdorff spaces are *normal spaces*. A normal space is a topological space satisfying that, for every pair of disjoint closed sets D and C, there are disjoint open sets U, V such that $C \subset U$ and $D \subset V$. When working with normal spaces we can use both *Urysohn's lemma* and *Tietze extension theorem*. Urysohn's lemma states that a topological space X is normal if and only if for any pair of closed and nonempty disjoints subsets C and D of X there is a continuous function $f : X \to [0, 1]$ such that $f|_C = 0$ and $f|_D = 1$. Tietze extension theorem says that, if X is a normal space, D is a closed subset of X, and $f : D \to \mathbb{R}$ is a continuous function, then we can extend f to a continuous function $F : X \to \mathbb{R}$, preserving boundedness of f if needed. It is easy to check that this theorem also works for continuous functions $f : D \to \mathbb{C}$ by decomposing f into its real and imaginary parts and applying the statement for real functions twice.

We use the following notation: if Y is a subset of a topological space X, we denote by $\overline{Y}$ its closure on X.

Now, let us briefly shift our attention to topological dynamical systems. In this work, we understand a topological dynamical system as the action of a discrete group via homeomorphisms on a topological space, which in our case is usually going to be compact and Hausdorff. If G is a discrete group and X is a topological space, we denote by $G \curvearrowright^{\sigma} X$ the action of G on X, where $\sigma : G \to \mathrm{Homeo}(X)$ is the group homomorphism between G and the group of homeomorphisms of X that gives rise to the action. We usually drop the σ from the notation and simply write $G \curvearrowright X$ for the action. We call $G \curvearrowright X$ a *topological dynamical system* or simply a *dynamical* *system*, as we only work with actions in the topological setting. Given $G \curvearrowright X$, we denote by $\sigma_s : X \to X$ the homeomorphism corresponding to the action of s on X, for every $s \in G$.

Let $G \curvearrowright X$ be an action of G on X. We denote by e the identity element of G. Given $x \in X$, we denote by G_x the *stabilizer subgroup of* x, which is defined as follows:

$$G_x = \{s \in G : \sigma_s(x) = x\}.$$

For every $s \in G$, we define

$$\mathrm{Fix}(s) = \{x \in X : \sigma_s(x) = x\} = \{x \in X : s \in G_x\}.$$

We say that $x \in X$ is *aperiodic* or that it *has free orbit* if $G_x = \{e\}$. We denote the set of aperiodic points of the action by $\mathrm{Aper}(G \curvearrowright X)$. We denote by $\mathrm{Per}(G \curvearrowright X)$ the complement of $\mathrm{Aper}(G \curvearrowright X)$. Given $x \in X$, we denote its orbit by $\mathcal{O}(x) = \{\sigma_s(x) : s \in G\}$.

Some known dynamical properties that play a role later on this work are *freeness, topological freeness, minimality, topological transitivity*, and *residual topological transitivity.* We are going to introduce them later, once we are ready to work with them.

1.2. Banach algebras

We introduce here different classes of Banach algebras that are of interest to us in the work that follows, and we also state results from the literature that are going to be needed. For Banach *-algebras and C^*-algebras, standard introductory books are [**Dix83, Mur90**]. For an introduction to L^p-operator algebras, one can look at the survey paper [**Gar21**]. For an introduction to abstract harmonic analysis and, in particular, the Haar measure, see [**Fol16**]. We remark, though, that the theory on these subjects that we need is quite basic, all of it being stated in this section.

Unless explicitly stated otherwise, all Banach spaces and algebras considered are assumed to be over the complex numbers $\mathbb{C}$, and all ideals are assumed to be two-sided.

1.2.1. Banach and C^*-algebras. We recall that a *Banach space* is a vector space equipped with a norm such that it becomes a complete topological space with the topology induced by it. An *algebra* is a vector space equipped with an associative bilinear product. As in the theory of operator algebras is usual to work with algebras that are nonunital, we do not require the unit as part of the definition of an algebra. A *normed algebra* is an algebra endowed with a submultiplicative norm. For a norm $\|\cdot\|$, to be submultiplicative means that $\|ab\| \leq \|a\|\|b\|$ for every pair of elements of the algebra a and b.

Definition 1.2.1. A *Banach algebra* is an algebra A together with a norm that turns it into both a Banach space and a normed algebra. If A is unital, we also require the unit 1_A to have norm one.

Though the definition of algebra we gave does not require the algebra to be unital, we did so as to be consistent with the literature on operator algebras. Regardless, the algebras we work with in this thesis all have a unit.

Proposition 1.2.2. Let A be a unital Banach algebra, and let I be a closed ideal of A. Then, the quotient A/I is a Banach algebra with the canonical sum and multiplication, and endowed with the following norm

$$\|a + I\| = \inf_{b \in I} \|a + b\| \text{ for every a in } A.$$

Three classes of Banach algebras are of particular interest to us: *Banach *-algebras*, *C^*-algebras* and *L^p-operator algebras*. We give definitions of all of them in this chapter. We begin with Banach *-algebras.

Definition 1.2.3. A Banach *-algebra is a Banach algebra A together with a self-map $a \mapsto a^*$, that we call an *involution*, such that satisfies the following properties:

(1) For every $a, b \in A$ and $\mu \in \mathbb{C}$ we have that $(\mu a + b)^* = \overline{\mu} a^* + b^*$.
(2) For every $a \in A$ we have that $(a^*)^* = a$.
(3) For every $a, b \in A$ we have that $(ab)^* = b^* a^*$.
(4) For every $a \in A$ we have that $\|a^*\| = \|a\|$.

A particular type of Banach *-algebras, which has been extensively studied for a long time, are C^*-algebras. A C^*-algebra is a Banach *-algebra A that verifies the following identity:

$$\|a^* a\| = \|a\|^2 \text{ for every } a \in A.$$

The above condition is called the C^*-identity and, though it may seem as an innocent condition to have, it allows us to relate the algebraic structure with the analytic structure of the algebra. As a result, C^*-algebras have a really deep theory and enjoy a wide array of nice properties.

Let us make a few remarks about C^*-algebras. First of all, the following is an important example that plays a key role in the the rest of the work.

Example 1.2.4. Let X be a compact and Hausdorff space, we can consider the algebra

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous}\}.$$

This is an algebra with addition and product defined pointwise, and the involution is likewise defined as pointwise conjugation over $\mathbb{C}$. We endow $C(X)$ with the following norm: for every $f \in C(X)$, we define

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

With these operations, $C(X)$ is a unital and commutative C^*-algebra.

The example above is the prototype example of a commutative and unital C^*-algebra and is, essentially, the only example: for any commutative and unital C^* algebra, there exists a unique (up to homeomorphism) compact Hausdorff space X such that A is isometrically *-isomorphic to $C(X)$ (see [**Mur90**, Theorem 2.1.10]). This result is known as the *Gelfand duality*.

Theorem 1.2.5 ([**Mur90**, Theorem 2.1.10]). Let A be a unital commutative C^*-algebra. Then, there exists a unique (up to homeomorphism) compact and Hausdorff space X such that A is *-isomorphic to $C(X)$.

In the above result, for two C^*-algebras to be *-isomorphic means that there is a bijective homomorphism of algebras that respects the involution. Gelfand duality can be extended to not necessarily unital commutative C^*-algebras by taking $C_0(X)$ instead of $C(X)$, where X is a locally compact Hausdorff space. We won't work with these non-unital C^*-algebras, so we don't go into the details of this statement.

With the above theorem at hand, it is quite easy to characterize *-isomorphisms between commutative C^*-algebras.

Proposition 1.2.6. Let $\varphi : C(X) \to C(Y)$ be an *-isomorphism between $C(X)$ and $C(Y)$. Then, there exists an homeomorphism $\theta : Y \to X$ such that $\varphi(f) = f \circ \theta$ for every $f \in C(X)$.

It is clear from the above statement that φ must be isometric. This is not a coincidence, as all injective *-homomorphisms between C^*-algebras are automatically isometric. We are going to state this in a short while. We want to point out that we can slightly strengthen the result above by means of the following result, which says that it is enough for a map φ to be a contractive homomorphism for it to respect the involution.

Proposition 1.2.7 ([**BLM04**, Proposition A.5.8]). Let $\varphi : A \to B$ be a homomorphism between C^*-algebras. Then, φ is contractive if and only if it is a *-homomorphism.

Remark 1.2.8. The above proposition together with Proposition 1.2.6 means that, given a contractive isomorphism $\varphi : C(X) \to C(Y)$ between the two algebras, it must automatically respect the involution and, consequently, there exists an homeomorphism $\theta : Y \to X$ such that $\varphi(f) = f \circ \theta$ for every $f \in C(X)$. We use this precise statement later in this work.

For commutative C^*-algebras, the following is a well known complete description of their closed ideals. We state it here for later use.

Proposition 1.2.9 ([**Dav96**, Exercise I.14]). Let X be a compact Hausdorff space, and consider the commutative C^*-algebra $C(X)$. If I is a closed ideal of $C(X)$, then there exists a closed subset D of X such that

$$I = \{f \in C(X) : f|_D = 0\}\,.$$

We point out that the closed ideals described in the above result are all self-adjoint. This is actually true for every C^*-algebra, not only for the commutative ones. This is a quite standard result, and we state it below.

Proposition 1.2.10 ([**Mur90**, Theorem 3.1.3]). Let A be a C^*-algebra. Then, every closed ideal of A is self-adjoint.

We will recurrently mention the result above in order to highlight the difference between C^*-algebras and arbitrary Banach *-algebras, where it is not always true that every closed ideal is self-adjoint.

Definition 1.2.11. Let A be a unital Banach algebra with unit 1_A and let $a \in A$. We define the *spectrum of* a as

$$\rho(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_A \text{ is not invertible}\}$$

We briefly use the spectrum in the last chapter of the thesis. We also take this opportunity to mention that, for an element $f \in C(X)$, with X a compact and Hausdorff space, the spectrum of f is simply its image (which is contained in $\mathbb{C}$).

To finish our preliminaries about C^*-algebras, we state the following (again, standard) proposition. In what follows, by a homomorphism between C^*-algebras we mean an algebra homomorphism that respects the involution.

Proposition 1.2.12 ([**Mur90**, Theorems 2.1.7 and 3.1.5]). Let A and B be C^*-algebras, and let $\varphi : A \to B$ be a homomorphism. Then

(1) φ is contractive, i.e., $\|\varphi(a)\| \leq \|a\|$ for every $a \in A$. In particular, every homomorphism between C^*-algebras is continuous.
(2) If φ is injective, then it is also isometric.

In the above proposition, (2) implies that every isomorphism between C^*-algebras is automatically isometric. We point out that this is not necessarily true for general Banach *-algebras.

1.2.2. L^p-operator algebras. There is another important class of Banach algebras that is of interest to us. Let us give some motivation before introducing this class. Above, we gave an abstract definition of C^*-algebras, but this characterization was not the first one to be used. Originally, C^*-algebras were defined as norm-closed *-subalgebras of $B(H)$, where $B(H)$ denotes the bounded operators in a Hilbert space H. This last definition is a more concrete notion of a C^*-algebra and as such, C^*-algebras given this way are sometimes called concrete C^*-algebras. The fact that every norm-closed subalgebra of $B(H)$ is a C^*-algebra with the operator norm (in the abstract sense we first described) is a routine check. So, there are these two notions of C^*-algebras, one abstract and one concrete. The Gelfand-Naimark theorem ([**Mur90**, Theorem 3.4.1]) states that every abstract C^*-algebra is isometrically *-isomorphic to a norm-closed subalgebra of $B(H)$. This means that every abstract C^*-algebra can be realized as a concrete C^*-algebra sitting inside the bounded operators on a Hilbert space. It is well known that every Hilbert space is isomorphic to $L^2(X, \mu)$, where (X, μ) is a measure space and $L^2(X, \mu)$ denotes the integrable functions on X with respect to the measure μ. So, we can think of Hilbert spaces as L^2-spaces, and C^*-algebras as subalgebras of bounded operator on L^2-spaces. From this identification, one arrives at *L^p-operator algebras* by going from L^2-spaces to L^p-spaces.

Definition 1.2.13. For $p \in [1, \infty)$, an L^p-operator algebra is a Banach algebra A such that there exists an isometric isomorphism $\varphi : A \to B(L^p(X, \mu))$, where $B(L^p(X, \mu))$ denotes the algebra of bounded operators on the Banach space $L^p(X, \mu)$.

There are a few comments that we should make about L^p-operator algebras. First, as $L^p(X, \mu)$ is not a normed space for $p < 1$, we only consider L^p-operator algebras for $p \geq 1$. Another fact that we point out is that an L^2-operator algebra is not necessarily a C^*-algebra, because it does not necessarily have an involution (but every C^*-algebra is indeed an L^2-operator algebra).

Let us explain our interest in these classes of Banach algebras. In this work, we are interested in studying actions of groups on compact Hausdorff spaces via homeomorphisms. Given an action of this kind, we can naturally associate to it Banach algebras of any of the three classes described above. These Banach algebras are usually called *crossed products*. So, we can associate Banach *-algebras, C^*-algebras, and L^p-operator algebras to the action, and these algebras retain different amounts of information of the action they arise from.

Our main object of study in this work is a crossed product Banach *-algebra that is not a C^*-algebra nor an L^p-operator algebra: it has some *sort of ℓ^1-norm*. The C^*-algebraic crossed product associated to the action serves as a role model for the type of results that we can aim to prove for the ℓ^1-crossed product, given that, historically, they have received a lot more attention than other kinds of Banach algebras. In more recent years, L^p-operator algebras crossed products have also become quite popular, so they can also serve as role models for the ℓ^1-crossed product, but they also serve importantly as a tool for us to prove the main theorem of Chapter 3. This stems from the fact that, for $p \in (1, \infty) \setminus \{2\}$, we can make use of Lamperti's theorem in order to work with the L^p-operator algebra crossed product. We give more details about crossed products in the following chapter.

1.2.3. Algebras of integrable functions. To finish this chapter, we state some basic facts about the algebra of integrable functions of a discrete group.

If H is a locally compact and Hausdorff topological group, it acts on itself via left translation. In this situation, there exists a unique (up to a positive multiplicative constant) measure defined on the Borel sets of H that is invariant under left translation. This measure is called the *Haar measure* of H. With the Haar measure at hand, one can consider $L^1(H)$ the algebra of integrable functions in H with respect to the Haar measure.

We are interested in this algebra when H is a discrete group. In this setting, every subset of H is measurable and the Haar measure is simply the counting measure, i.e., the measure of a subset A of H is $|A|$. It follows that the algebra of integrable functions in H is

$$\ell^1(H) = \left\{ \xi : H \to \mathbb{C} \mid \|\xi\|_1 = \sum_{s \in H} |\xi(s)| < \infty \right\}.$$

We denote this algebra by $\ell^1(H)$ instead of $L^1(H)$ to put emphasis in the discrete topology in H and, as such, the integral with respect to the Haar measure is described as a sum, as the measure is the counting measure. The map $\xi \mapsto \|\xi\|_1$ is a norm in $\ell^1(H)$ that turns it into a Banach space. The sum and the scalar multiplication in $\ell^1(H)$ are defined pointwise, and the product is given by convolution: if ξ and η belong to $\ell^1(H)$, their convolution is the function $\xi * \eta$ given by

$$(\xi * \eta)(s) = \sum_{t \in H} \xi(t)\eta(t^{-1}s) \text{ for every } s \in H.$$

With this structure, $\ell^1(H)$ is a Banach algebra, and we can endow it with an involution in order to turn it into a Banach *-algebra. For every $\xi \in \ell^1(H)$, the involution is defined as $\xi^*(s) = \overline{\xi(s^{-1})}$ for every $s \in H$.

Given that H is endowed with the discrete topology, it is clear that the characteristic function of {s} is in $\ell^1(H)$ for any $s \in H$. We denote this characteristic function by $\mathbb{1}_s$. A direct computation shows that $(\mathbb{1}_s * \xi)(t) = \xi(s^{-1}t)$ for every $\xi \in \ell^1(H)$ and every $t \in H$.

CHAPTER 2

Crossed products and dynamical characterizations

The crossed product is a major construction in the theory of operator algebras that allows one to, given an action of a group on a C^*-algebra, obtain an algebra whose operations somehow codify the action. One can have, for instance, an action $G \curvearrowright X$ of a discrete group G on a compact Hausdorff space X, which is equivalent to an action of G on $C(X)$, and from there obtain an *algebraic skeleton*, that is, an algebra that canonically contains both a copy of $C(X)$ and a copy of G. From there, what one usually does is to endow the algebraic skeleton with a norm and take the completion with respect to it, in order to obtain an algebra of analytical nature. This completion is called the *crossed product*.

The crossed product obtained via the process we have just described depends fundamentally on the type of norm chosen for the algebraic skeleton. Crossed products endowed with C^*-norms have been studied at length through the years and are an essential part of the theory of C^*-algebras by now. Yet, C^*-norms are one of the many choices one can take, and in this work we mean to study the crossed product obtained via another type of norm, one we refer as an ℓ^1*-norm*. We are going into the details of it shortly, but, for now, let us refer to this crossed product completion with an ℓ^1-norm as $\ell^1(G \curvearrowright X)$. There are a few arguments to make in favor of choosing an L^1-norm instead of a C^*-norm for the crossed product. The first one is Wendel's theorem ([**Wen51**]), that says that the L^1-algebra of a group retains all the information of the group it arises from, in the sense that if two locally compact and Hausdorff groups have isometrically isomorphic L^1-algebras, then the groups are actually isomorphic as topological groups (it is clear that the converse for this is also true). The analogous result for C^*-algebras of groups does not hold, as there are non-isomorphic groups whose group C^*-algebras are isomorphic. This means that, in this setting, information of the group may be lost when passing to its group C^*-algebra, whereas that cannot happen when passing to its group L^1-algebra. It is worthwhile to point out that, for a discrete group G, its group L^1-algebra coincides with the ℓ^1-crossed product associated to the trivial action of G on a point.

Another reason to study $\ell^1(G \curvearrowright X)$, of more recent nature and the main inspiration for the results presented in this chapter, is a result present in [**dJST12**]. There, for actions of $\mathbb{Z}$, it was studied how different dynamical properties of $\mathbb{Z} \curvearrowright X$ are detected in $\ell^1(\mathbb{Z} \curvearrowright X)$. The main result from there that we want to focus our attention on says the following: the action $\mathbb{Z} \curvearrowright X$ is free if and only if every two-sided closed ideal of $\ell^1(G \curvearrowright X)$ is self-adjoint (closed under the involution). We mentioned in the preliminaries that in C^*-algebras all closed ideals are self-adjoint, which means that the analogous result to the one from [**dJST12**] for the C^*-algebra crossed product does not hold. As such, we can draw the following conclusion supporting that the ℓ^1-norm retains more information from the action: the ℓ^1-crossed product detects freeness of the action, whereas the C^*-crossed product does not (or at least does not detect it in the same way). Moreover, any dynamical property of $G \curvearrowright X$ that is detected in the C^*-crossed product associated to it must also be detected in the ℓ^1-crossed product, as the first one can be realized as a completion of the latter with a different norm.

We deal in the next chapter with the question of exactly how much information of the action is retained in $\ell^1(G \curvearrowright X)$. As of this chapter, our goal is to generalize to arbitrary countable and discrete groups the results presented in [**dJST12**] for actions of $\mathbb{Z}$. The first landmark of this chapter is the characterization of topological freeness of $G \curvearrowright X$ in terms of the ideal structure of $\ell^1(G \curvearrowright X)$. Once we have proven this and assuming topological freeness of the action, minimality, topological transitivity and residual topological freeness of $G \curvearrowright X$ are proved to be equivalent to different analytic-algebraic properties of $\ell^1(G \curvearrowright X)$. Finally we deal with the problem of detecting freeness of the action by looking at $\ell^1(G \curvearrowright X)$. We prove that, for G abelian and torsion-free, $G \curvearrowright X$ is free if and only if every closed ideal of $\ell^1(G \curvearrowright X)$ is self-adjoint, generalizing the result presented in [**dJST12**]. We also see an example showing that this result does not hold beyond the abelian and torsion-free case.

This chapter is based on [**Rol25**], which is in turn heavily inspired by the results presented in [**dJST12**] and, as such, we present generalizations of many of them. We also mention that some of the results here can be seen as a consequence of [**BK24**]; we mention them as they come up.

2.1. The ℓ^1-crossed product

We recall that all ideals considered are two-sided. For the rest of this chapter, we fix an action $G \curvearrowright X$ of a discrete countable infinite group G on a compact Hausdorff X space via homeomorphisms. We denote by e the identity element of G. For every $s \in G$, we denote by $\sigma_s : X \to X$ the homeomorphism associated to the action of s on X. We first introduce the ℓ^1-crossed product, as it is our main object of study in this work.

Let $C(X)$ be the algebra of continuous functions from X to $\mathbb{C}$. As we have said in Example 1.2.4, it is a unital commutative C^*-algebra when endowed with pointwise operations and the supremum norm $\|\cdot\|_\infty$. The action of G on X induces an action of G on $C(X)$, that we denote by $\alpha : G \to \mathrm{Aut}(C(X))$, given by

$$\alpha_s(f)(x) = f(\sigma_s^{-1}(x))$$

for every $s \in G$, every $f \in C(X)$ and every $x \in X$. It is clear that, for every $s \in G$, the map $\alpha_s : C(X) \to C(X)$ is an automorphism that preserves the involution in $C(X)$, i.e, $\alpha_s(\overline{f}) = \overline{\alpha_s(f)}$ for every $f \in C(X)$.

We define

$$\ell^1(G \curvearrowright X) = \left\{ f : G \to C(X) : \sum_{s\in G} \|f(s)\|_\infty < \infty \right\}.$$

We define $\|f\| = \sum_{s\in G} \|f(s)\|_\infty$ for every $f \in \ell^1(G \curvearrowright X)$. We have that $\ell^1(G \curvearrowright X)$ is a Banach space with this norm and with pointwise sum and scalar multiplication. We endow $\ell^1(G \curvearrowright X)$ with the following product: given f and g belonging to $\ell^1(G \curvearrowright X)$, we define

$$(fg)(s) = \sum_{t\in G} f(t)\alpha_t(g(t^{-1}s)) \text{ for every } s \in G.$$

We also define an involution $* : \ell^1(G \curvearrowright X) \to \ell^1(G \curvearrowright X)$ as follows: given $f \in \ell^1(G \curvearrowright X)$, we set

$$(f^*)(s) = \alpha_s(\overline{f(s^{-1})}) \text{ for every } s \in G.$$

With these product and involution, $\ell^1(G \curvearrowright X)$ becomes a Banach *-algebra. We can canonically embed $C(X)$ in $\ell^1(G \curvearrowright X)$: given $g \in C(X)$, we identify it with the element of $\ell^1(G \curvearrowright X)$ that maps e to g and every other element of G to the zero function. This mapping from $C(X)$ to $\ell^1(G \curvearrowright X)$ is an isometry. Given $g \in C(X)$ we also denote by g its corresponding element in $\ell^1(G \curvearrowright X)$.

We now describe a useful way to work with the elements of $\ell^1(G \curvearrowright X)$. We think of the following as *decomposing the elements of* $\ell^1(G \curvearrowright X)$ *in coordinates*. For every $s \in G$, we define $\delta_s \in \ell^1(G \curvearrowright X)$ as:

$$\delta_s(s') = \begin{cases} 1 & \text{if } s' = s, \\ 0 & \text{if } s' \neq s, \end{cases}$$

where 1 denotes the function in $C(X)$ taking constant value 1. With this notation, we can describe an element $f \in \ell^1(G \curvearrowright X)$ as:

$$f = \sum_{s \in G} f(s)\delta_s = \sum_{s \in G} f_s \delta_s,$$

where f_s denotes $f(s) \in C(X)$ for every $s \in G$. We usually use this description of the elements of $\ell^1(G \curvearrowright X)$ without explicit mention. With this notation, we have that

$$(f_s\delta_s)(g_t\delta_t) = f_s\alpha_s(g_t)\delta_{st}$$

for every s and $t \in G$ and every f_s and $g_t \in C(X)$. Also, with this notation we have that the involution is given by the rule

$$(f_s\delta_s)^* = \alpha_s^{-1}(\overline{f_s})\delta_{s^{-1}} \text{ for every } s \in G \text{ and } f_s \in C(X).$$

Definition 2.1.1. We define the canonical projection $E : \ell^1(G \curvearrowright X) \to C(X)$ as the map given by $E(f) = f(e)$ for every $f \in C(X)$.

We mention that one can take the C^*-envelope of $\ell^1(G \curvearrowright X)$ in order to obtain a C^*-algebraic crossed product. There are two ways of doing that. Through one of the ways we get the *universal* crossed product C^*-algebra associated to $G \curvearrowright X$, that we denote by $C^*(G \curvearrowright X)$. By the other way we get the *reduced* crossed product C^*-algebra associated to $G \curvearrowright X$, which we denote by $C^*_\lambda(G \curvearrowright X)$. In both cases we have that $\ell^1(G \curvearrowright X)$ is a dense *-subalgebra of the C^*-crossed product. We do not actually study these algebras in this work, but we might mention them in order to compare the situation between $\ell^1(G \curvearrowright X)$ and their C^*-counterparts. We also mention that the universal C^*-envelope has the same representation theory than the Banach *-algebra it envelops (see [**Dix83**, Proposition 2.7.4]). This is useful to obtain representations of $\ell^1(G \curvearrowright X)$, as we see next.

2.1.1. Representations of $\ell^1(G \curvearrowright X)$.

By a representation of a Banach *-algebra we mean a bounded homomorphism of the algebra into the bounded operators on a Hilbert space that respects the *-operation. As such, the kernel of a representation is always a closed and self-adjoint ideal.

We introduce a family of representations of $\ell^1(G \curvearrowright X)$ that are useful in the following section. In [**Tom92**, Chapter 4], given $x \in X$ and a unitary representation of the isotropy subgroup G_x, it is shown how to obtain an induced representation of the universal crossed product $C^*(G \curvearrowright X)$. As $C^*(G \curvearrowright X)$ and $\ell^1(G \curvearrowright X)$ have the same theory of representations, we also obtain (via restriction) a representation of $\ell^1(G \curvearrowright X)$. We are interested in the induced representation obtained when the unitary representation of G_x we start with is the trivial representation (on a Hilbert space of dimension 1, that is, the complex numbers $\mathbb{C}$), that we denote by π_x. We concretely describe the representation π_x in this case.

Fix $x \in X$ and consider the isotropy subgroup G_x. We can describe the left quotient space as $G/G_x = \{s_\beta G_x : \beta \in J_x\}$, for the set of representatives $\{s_\beta : \beta \in J_x\}$. We consider the Hilbert space H_x with orthonormal basis $\{v_\beta : \beta \in J_x\}$. Then, we can describe the representation $\pi_x : \ell^1(G \curvearrowright X) \to B(H_x)$ as follows:

- Given $f \in C(X)$ and $\beta \in J_x$, we have that $\pi_x(f)v_\beta = f(\sigma_{s_\beta}(x))v_\beta$.
- Given $s \in G$ and $\beta \in J_x$, there exists a unique $\gamma \in J_x$ such that ss_β belongs to the coset $s_\gamma G_x$. Then, we have that $\pi_x(\delta_s)v_\beta = v_\gamma$.

As π_x is multiplicative (and linear), this determines the induced representation. Thus, given an arbitrary element $f = \sum_{s\in G} f_s\delta_s \in \ell^1(G \curvearrowright X)$, we have that

$$\pi_x\left(\sum_{s\in G} f_s\delta_s\right) = \sum_{s\in G} \pi_x(f_s)\pi_x(\delta^s) \in B(H_x).$$

We note that the induced representation is finite dimensional if and only if x has finite orbit.

2.2. The commutant of $C(X)$ and some technical results

In this section, we introduce the commutant of $C(X)$ inside $\ell^1(G \curvearrowright X)$, and we characterize when the communtant is equal to $C(X)$: it happens if and only if the action is topologically free. Following that, we prove a couple of technical results that are used in the next section, where we study in more depth how topological freeness of the action gets reflected on $\ell^1(G \curvearrowright X)$ and, eventually, characterize more dynamical properties such as minimality, topological transitivity and residual topological freeness.

We define the *commutant* of $C(X)$ as

$$C(X)' = \left\{f \in \ell^1(G \curvearrowright X) : fg = gf \text{ for every } g \in C(X)\right\}.$$

It is clear that $C(X) \subset C(X)'$, as $C(X)$ is a commutative algebra. The commutant has a quite nice description that we give in the following proposition. We recall that, for every $g \in C(X)$, we denote its support by $\operatorname{supp}(g) = \overline{\{x \in X : g(x) \neq 0\}}$.

PROPOSITION 2.2.1. We have that

$$C(X)' = \left\{f = \sum_{s\in G} f_s\delta_s \in \ell^1(G \curvearrowright X) : \operatorname{supp}(f_s) \subset \operatorname{Fix}(s) \text{ for every } s \in G\right\}.$$

PROOF. Let $f = \sum_{s\in G} f_s\delta_s \in C(X)'$. This means that, for every $g \in C(X)$, we have that $fg = gf$. We compute:

$$fg = \left(\sum_{s\in G} f_s\delta_s\right) g = \sum_{s\in G} f_s\alpha_s(g)\delta_s = \sum_{s\in G} f_s(g \circ \sigma_s^{-1})\delta_s,$$

$$gf = g\left(\sum_{s\in G} f_s\delta_s\right) = \sum_{s\in G} gf_s\delta_s.$$

If follows that $f_s g \circ \sigma_s^{-1}$ is equal to $f_s g$ for every $s \in G$ and for every $g \in C(X)$. Consider $s \in G$ and let $x \in X$ such that $f_s(x) \neq 0$. Then, we have that $g(\sigma_s^{-1}(x)) = g(x)$ for every $g \in C(X)$. This implies that $x = \sigma_s^{-1}(x)$, which is equivalent to $\sigma_s(x) = x$, i.e., x belongs to $\mathrm{Fix}(s)$. We conclude that $\mathrm{supp}\,(f_s) \subset \mathrm{Fix}(s)$. □

DEFINITION 2.2.2. We say that $G \curvearrowright X$ is *free* if $\mathrm{Aper}(G \curvearrowright X) = X$. We say that $G \curvearrowright X$ is *topologically free* if $\mathrm{Aper}(G \curvearrowright X)$ is dense in X, i.e., if $\overline{\mathrm{Aper}(G \curvearrowright X)} = X$.

The notion of topological freeness is a key property in order to work with $\ell^1(G \curvearrowright X)$, as we see in the next section.

REMARK 2.2.3. If X is finite, then every point of X has an infinite stabilizer subgroup, so the action cannot be topologically free. This means that, if the action is topologically free, then X needs to be an infinite space.

REMARK 2.2.4. We can write

$$\mathrm{Aper}(G \curvearrowright X) = \bigcap_{s \in G \setminus \{e\}} \mathrm{Fix}(s)^c.$$

As X is compact and Hausdorff, it is a Baire space and, as G is countable, we have that $\mathrm{Aper}(G \curvearrowright X)$ is dense if and only if $\mathrm{Fix}(s)$ has empty interior for every $s \in G \setminus \{e\}$.

COROLLARY 2.2.5. The action $G \curvearrowright X$ is topologically free if and only if $C(X) = C(X)'$.

PROOF. We just note that a nonzero continuous function has support with nonempty interior. Then, the result follows from the above remark and Proposition 2.2.1. □

We now delve into the technical work we need for the next section.

LEMMA 2.2.6. Let $x \in X$ and $k \in G$ such that $x \neq \sigma_k(x)$. Then, there exists an open neighborhood U of x and a unimodular function $\theta \in C(X)$ such that θ is equal to 1 in U and

$$\theta(y)\overline{\theta(\sigma_k^{-1}(y))} = i \in \mathbb{C} \text{ for every } y \in U.$$

PROOF. We have that $\sigma_k^{-1}(x) \neq x$, so there exists disjoint open neighborhoods U and U' and closed disjoint subsets C and C' such that $x \in U$, $\sigma_k^{-1}(x) \in U'$, $U \subset C$ and $U' \subset C'$. We can assume that $\sigma_k^{-1}(U)$ is included in C' by replacing U with $U \cap \sigma_k(U')$. Using Urysohn's Lemma, we get a function $\widetilde{\theta} \in C(X)$ such that $\widetilde{\theta}$ is equal to 1 in C' and that it is equal to 0 in C. We define

$$\theta = \exp\left(\frac{-i\pi\widetilde{\theta}}{2}\right) \in C(X),$$

which satisfies what the lemma says. □

The following lemma has a quite technical statement. The idea behind it is to obtain, from an element $f \in \ell^1(G \curvearrowright X)$, another element $f' \in \ell^1(G \curvearrowright X)$ satisfying some desirable properties, and such that the procedure used to obtain f' from f preserves ideals. In particular, f' has the same component at the identity as f.

LEMMA 2.2.7. Let $x \in X$, and let $k_1, \dots, k_N \in G$ such that $\sigma_{k_i}(x) \neq x$ for every $i = 1, \dots, N$. Then, there exists U an open neighborhood of x and unimodular functions $\theta_1, \dots, \theta_{2^N} \in C(X)$ that verify the following property: if $f = \sum_{s \in G} f_s \delta_s \in \ell^1(G \curvearrowright X)$, we consider

$$f' = \frac{1}{2^N} \sum_{l=1}^{2^N} \theta_l f \overline{\theta_l};$$

then, if we write $f' = \sum_{s \in G} f'_s \delta_s$, it verifies that:

1. $f'_e = f_e$.
2. $f'_{k_l}(y) = 0$ for every $y \in U$ and every $l = 1, \dots, N$.

PROOF. Let $f = \sum_{s \in G} f_s \delta_s \in \ell^1(G \curvearrowright X)$ and $h \in C(X)$. We compute the following:

$$\begin{aligned} \frac{1}{2}\left(hf\overline{h} + \overline{h}fh\right) &= \frac{1}{2}\left(h\left(\sum_{s\in G} f_s\delta_s\right)\overline{h} + \overline{h}\left(\sum_{s\in G} f_s\delta_s\right)h\right) \\ &= \frac{1}{2}\left(\sum_{s\in G} hf_s\alpha_s(\overline{h})\delta_s + \sum_{s\in G}\overline{h}f_s\alpha_s(h)\delta_s\right) \\ &= \frac{1}{2}\sum_{s\in G} f_s\left(h\alpha_s\left(\overline{h}\right) + \overline{h\alpha_s\left(\overline{h}\right)}\right)\delta_s \\ &= \frac{1}{2}\sum_{s\in G} f_s 2\mathrm{Re}\left(h\alpha_s\left(\overline{h}\right)\right)\delta_s = \sum_{s\in G} f_s\mathrm{Re}\left(h\alpha_s\left(\overline{h}\right)\right)\delta_s. \end{aligned}$$

So, for every $f \in \ell^1(G \curvearrowright X)$ and $h \in C(X)$ we have:

$$\frac{1}{2}\left(hf\overline{h} + \overline{h}fh\right) = \sum_{s\in G} f_s\mathrm{Re}\left(h\alpha_s\left(\overline{h}\right)\right)\delta_s. \tag{1}$$

Let us now prove the lemma. We prove it by induction on N. Let $N = 1$, we have $k_1 \in G$ such that $x \neq \sigma_{k_1}(x)$. By the previous lemma we have an open neighborhood U of x and a unimodular function $\theta_0 \in C(X)$ that is equal to 1 in U and

$$\theta_0(y)\overline{\theta_0}\left(\sigma_{k_1}^{-1}(y)\right) = i \in \mathbb{C} \text{ for every } y \in U.$$

We set $\theta_1 = \theta_0$ and $\theta_2 = \overline{\theta_0}$. Let $f = \sum_{s\in G} f_s\delta_s$ in $\ell^1(G \curvearrowright X)$. By Equation (1), we have that

$$f' = \frac{1}{2}\left(\theta_1 f\overline{\theta_1} + \theta_2 f\overline{\theta_2}\right) = \frac{1}{2}\left(\theta_0 f\overline{\theta_0} + \overline{\theta_0}f\theta_0\right) = \sum_{s\in G} f_s\mathrm{Re}\left(\theta_0\alpha_s\left(\overline{\theta_0}\right)\right)\delta_s.$$

Then, it follows that:

1. $f'_e = f_e\mathrm{Re}\left(\theta_0\alpha_e\left(\overline{\theta_0}\right)\right) = f_e\mathrm{Re}\left(\theta_0\overline{\theta_0}\right) = f_e\mathrm{Re}\left(|\theta_0|^2\right) = f_e$ because θ_0 is unimodular.
2. For every $y \in U$:

$$f'_{k_1}(y) = f_{k_1}(y)\mathrm{Re}\left(\theta_0(y)\alpha_{k_1}\left(\overline{\theta_0}(y)\right)\right) = f_{k_1}(y)\mathrm{Re}\left(\theta_0(y)\overline{\theta_0}\left(\sigma_{k_1}^{-1}(y)\right)\right) = f_{k_1}(y)\mathrm{Re}(i) = 0.$$

Now, we do the inductive step. Suppose we have $\sigma_{k_1}(x), \dots, \sigma_{k_{N-1}}(x)$, all distinct from x, and we have unimodular functions $\widetilde{\theta}_1, \dots, \widetilde{\theta}_{2^{N-1}} \in C(X)$ and an open set $\widetilde{U}$ of x such that if

$f = \sum_{s\in G} f_s\delta_s$ and we denote

$$\widetilde{f} = \frac{1}{2^{N-1}}\sum_{l=1}^{2^{N-1}} \widetilde{\theta}_l f \overline{\widetilde{\theta}_l} = \sum_{s\in G} \widetilde{f}_s \delta_s,$$

then:

1. $\widetilde{f}_e = f_e$;
2. $\widetilde{f}_{k_l}(y) = 0$ for every $y \in \widetilde{U}$ and every $l = 1, \ldots, N-1$.

We have $\sigma_{k_N}(x) \neq x$, so by the previous lemma we have a unimodular function θ_0 and an open neighborhood V of x such that θ_0 is equal to 1 in V and

$$\theta_0(y)\overline{\theta_0}\left(\sigma_{k_N}^{-1}(y)\right) = i \in \mathbb{C} \text{ for every } y \in V.$$

We set $U = \widetilde{U} \cap V$. We have

$$\begin{aligned}\frac{1}{2}\left(\theta_0\widetilde{f}\overline{\theta}_0 + \overline{\theta}_0\widetilde{f}\theta_0\right) &= \frac{1}{2}\left(\theta_0\left[\frac{1}{2^{N-1}}\sum_{l=1}^{2^{N-1}}\widetilde{\theta}_l f\overline{\widetilde{\theta}_l}\right]\overline{\theta_0} + \overline{\theta_0}\left[\frac{1}{2^{N-1}}\sum_{l=1}^{2^{N-1}}\widetilde{\theta}_l f\overline{\widetilde{\theta}_l}\right]\theta_0\right)\\ &= \frac{1}{2^N}\left(\sum_{l=1}^{2^{N-1}}(\theta_0\widetilde{\theta}_l)f\overline{(\theta_0\widetilde{\theta}_l)} + \sum_{l=1}^{2^{N-1}}(\overline{\theta_0}\widetilde{\theta}_l)f\overline{(\overline{\theta_0}\widetilde{\theta}_l)}\right).\end{aligned}$$

Setting $\theta_l = \theta_0\widetilde{\theta}_l$ for $l = 1, \ldots, 2^{N-1}$ and $\theta_l = \overline{\theta_0}\widetilde{\theta}_{l-2^{N-1}}$ for $l = 2^{N-1}+1, \ldots, 2^N$, we get from the above equation that

$$f' = \frac{1}{2^N}\sum_{l=1}^{2^N}\theta_l f\overline{\theta_l} = \frac{1}{2}\left(\theta_0\widetilde{f}\overline{\theta}_0 + \overline{\theta}_0\widetilde{f}\theta_0\right).$$

Now, using Equation (1), we have that

$$f' = \frac{1}{2}\left(\theta_0\widetilde{f}\overline{\theta}_0 + \overline{\theta}_0\widetilde{f}\theta_0\right) = \sum_{s\in G}\widetilde{f}_s \mathrm{Re}\left(\theta_0\alpha_s\left(\overline{\theta_0}\right)\right)\delta_s.$$

Finally, writing $f' = \sum_{s\in G} f'_s\delta_s$, we obtain:

1. $f'_e = \widetilde{f}_e \mathrm{Re}\left(\theta_0\overline{\theta_0}\right) = f_e$ as θ_0 is unimodular.
2. If $y \in U$ and $l = 1, \ldots, N-1$, we have that
$$f'_{k_l}(y) = \widetilde{f}_{k_l}(y)\mathrm{Re}\left(\theta_0(y)\alpha_{k_l}\left(\overline{\theta_0}\right)(y)\right) = 0$$
given that if $y \in U \subset \widetilde{U}$ then $\widetilde{f}_{k_l}(y) = 0$.
3. If $y \in U$, as $U \subset V$, we have
$$f_{k_N}(y) = \widetilde{f}_{k_N}(y)\mathrm{Re}\left(\theta_0\overline{\theta_0}\left(\sigma_{k_N}^{-1}(y)\right)\right) = \widetilde{f}_{k_N}(y)\mathrm{Re}\,(i) = 0.$$

This ends the proof. □

We need the following theorem, which can be seen in [**Sak71**].

THEOREM 2.2.8 ([**Sak71**, Theorem 1.2.4]). If $\|\cdot\|$ is a norm on $C(X)$ that turns it into a normed algebra, then $\|\cdot\|_\infty \leq \|\cdot\|$.

The following is the last lemma we need before going into the next section.

LEMMA 2.2.9. Suppose that I is a closed ideal of $\ell^1(G \curvearrowright X)$ such that $I \cap C(X) = \{0\}$. Then, for every $f = \sum_{s\in G} f_s\delta_s \in I$, we have that f_s vanishes on $\mathrm{Aper}(G \curvearrowright X)$ for every $s \in G$.

PROOF. Let $f = \sum_{s\in G} f_s\delta_s \in I$. We want to prove that $f_s(x) = 0$ for every $x \in \mathrm{Aper}(G \curvearrowright X)$ and for every $s \in G$. It is enough to prove that $f_e(x) = 0$ for every $x \in \mathrm{Aper}(G \curvearrowright X)$ as, given that I is an ideal, $f\delta_{s^{-1}}$ belongs to I for every $s \in G$. We fix $x \in \mathrm{Aper}(G \curvearrowright X)$.

We consider the quotient $\ell^1(G \curvearrowright X)/I$. It is a Banach algebra with the following norm:

$$\|h + I\| = \inf_{j\in I}\|h + j\| \text{ for every } h \in \ell^1(G \curvearrowright X).$$

Let $q : \ell^1(G \curvearrowright X) \to \ell^1(G \curvearrowright X)/I$ be the canonical projection, which is clearly a contraction. We have that $I \cap C(X) = \{0\}$, so we can define a norm in $C(X)$ using the norm in the quotient: we define $\|\cdot\|' = \|\cdot\| \circ q$. This norm turns $C(X)$ into a normed algebra, so by the previous theorem we have that $\|\cdot\|_\infty \le \|\cdot\|'$. But, as q is contractive, we also have that $\|\cdot\|' \le \|\cdot\|_\infty$, so the two norms must coincide. We have, then, that the restriction of q to $C(X)$ is an isometry.

Let $\varepsilon > 0$. We can take a finite subset $F \subset G \setminus \{e\}$ such that

$$\|f - (f_e + \sum_{s\in F} f_s\delta_s)\| < \varepsilon.$$

We set b equal to $\sum_{s\in F} f_s\delta_s$ and set c equal to $f - (f_e + b)$, so we can write $f = f_e + b + c$ with $\|c\| < \varepsilon$. As F is finite, we have $F = \{k_1 \ldots, k_N\}$, and as x belongs to $\mathrm{Aper}(G \curvearrowright X)$, we have that $\sigma_{k_i}(x) \neq x$ for every $i = 1\ldots, N$. Using Lemma 2.2.7 we obtain unimodular functions $\theta_1, \ldots, \theta_M \in C(X)$ and an open neighborhood U of x such that if we write

$$f' = \frac{1}{M}\sum_{l=1}^{M} \theta_l f \overline{\theta_l} = \sum_{s\in G} f'_s\delta_s$$

then $f'_e = f_e$ and $f'_{k_i}(y) = 0$ for every $y \in U$ and every $i = 1, \ldots, N$. It is clear that f' also belongs to I. We take $\varphi \in C(X)$, $0 \le \varphi \le 1$, supported in U and such that $\varphi(x) = 1$. We set $\widetilde{f} = \varphi f'$ and, again, we have that $\widetilde{f}$ belongs to I. We can write $\widetilde{f} = \sum_{s\in G} \widetilde{f}_s\delta_s$, and we have that $\widetilde{f}_e(x) = \varphi(x)f'_e(x) = f_e(x)$.

Now, we have that

$$\begin{aligned}
\widetilde{f} = \varphi f' &= \frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l f\overline{\theta_l} \\
&= \frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l f_e\overline{\theta_l} + \frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l b\overline{\theta_l} + \frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l c\overline{\theta_l} \\
&= \varphi f_e + \frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l b\overline{\theta_l} + \frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l c\overline{\theta_l},
\end{aligned}$$

as we are using that the funtions θ_l are unimodular in order to get the first summand in the last equality. We have that $b = \sum_{s\in F} f_s\delta_s$, so, using that φ is supported in U and that the functions f'_{k_i} vanish in U for every $i = 1, \ldots, N$, we obtain

$$\frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l b\overline{\theta_l} = \sum_{s\in F} \varphi f'_s\delta_s = \sum_{i=1}^{N} \varphi f'_{k_i}\delta_{k_i} = 0.$$

So, we have that

$$\widetilde{f} = \varphi f_e + \frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l c\overline{\theta_l}.$$

We set $\widetilde{c} = \frac{1}{M}\sum_{l=1}^{M} \varphi\theta_l c\overline{\theta_l}$. Then

$$\|\widetilde{c}\| = \|\frac{\varphi}{M}\sum_{l=1}^{M} \theta_l c\overline{\theta_l}\| \leq \frac{1}{M}\sum_{l=1}^{M} \|\theta_l\|\|c\|\|\overline{\theta_l}\| = \frac{M}{M}\|c\| < \varepsilon.$$

Now, as $\widetilde{f}$ is in I, we have $q(\widetilde{f}) = 0$. This implies that $q(\varphi f_e) = -q(\widetilde{c})$. Then, using that q is a contraction and that its restriction to $C(X)$ is an isometry, we obtain that

$$|f_e(x)| = |\varphi(x) f_e(x)| \leq \|\varphi f_e\|_\infty = \|q(\varphi f_e)\| = \|q(\widetilde{c})\| \leq \|\widetilde{c}\| < \varepsilon.$$

So, we obtain that $|f_e(x)| < \varepsilon$. As ε was arbitrary, we conclude that $f_e(x) = 0$. □

2.3. The ideal intersection property and consequences

We can now reap the rewards of our work done in the last section. We start by proving Theorem 2.3.1, where we relate the action being topologically free with the ideal structure of $\ell^1(G \curvearrowright X)$. With this result at our disposal we can, assuming topological freeness of the action, quite easily characterize minimality, topological transitivity and residual topological freeness of $G \curvearrowright X$ as analytic-algebraic properties of $\ell^1(G \curvearrowright X)$. Equivalence between (1) and (3) of Theorem 2.3.1 can be seen as an interpretation for $\ell^1(G \curvearrowright X)$ of the more general context [**BK24**, Theorem 5.9], but the proof we include here follows the ideas from [**dJST12**, Theorem 4.1], of which this result is a generalization to arbitrary countable (infinite) discrete groups.

In order to prove Theorem 2.3.1, we make use of the family of representations

$$\{\pi_x : \ell^1(G \curvearrowright X) \to B(H_x) \mid x \in X\}$$

that we introduced in the preliminaries (see Section 2.1.1 of this chapter).

Theorem 2.3.1. The following are equivalent:

(1) For every closed nonzero ideal I of $\ell^1(G \curvearrowright X)$, the intersection $I \cap C(X)$ is nonzero.
(2) For every closed and self-adjoint nonzero ideal of $\ell^1(G \curvearrowright X)$, the intersection $I \cap C(X)$ is nonzero.
(3) The action $G \curvearrowright X$ is topologically free.

Proof. It is evident that (1) implies (2). We prove first that (2) implies (3). Suppose that $G \curvearrowright X$ is not topologically free. Then, by Remark 2.2.4, there exists some $s_0 \in G \setminus \{e\}$ such that $\mathrm{Fix}(s_0)$ has nonempty interior. We can then take a nonzero $f \in C(X)$ such that $\mathrm{supp}(f)$ is contained in $\mathrm{Fix}(s_0)$. Define I as the closed and self-adjoint ideal of $\ell^1(G \curvearrowright X)$ generated by $f - f\delta_{s_0}$.

Let $x \in X$, and consider $\pi_x : \ell^1(G \curvearrowright X) \to B(H_x)$ the representation associated to x. We remember that H_x is the Hilbert space with orthonormal basis $\{v_\beta : \beta \in J_x\}$, where $\{s_\beta : \beta \in J_x\}$ are representatives of $G/G_x = \{s_\beta G_x : \beta \in J_x\}$. We will prove that π_x vanishes at $f - f\delta_{s_0}$, which implies that π_x vanishes on I. Let $\beta \in J_x$, we have that there are unique $\gamma \in J_x$ and

$s' \in G_x$ such that $s_0 s_\beta = s_\gamma s' \in s_\gamma G_x$. Then:

$$\pi_x(f - f\delta_{s_0})v_\beta = \pi_x(f)v_\beta - \pi_x(f)\pi_x(\delta_{s_0})v_\beta = f(\sigma_{s_\beta}(x))v_\beta - \pi_x(f)v_\gamma$$
$$= f(\sigma_{s_\beta}(x))v_\beta - f(\sigma_{s_\gamma}(x))v_\gamma.$$

Suppose first that $\sigma_{s_\beta}(x)$ is in $\mathrm{Fix}(s_0)$. Then, we have that

$$\sigma_{s_\beta}(x) = \sigma_{s_0}(\sigma_{s_\beta}(x)) = \sigma_{s_o s_\beta}(x) = \sigma_{s_\gamma}(\sigma_{s'}(x)) = \sigma_{s_\gamma}(x),$$

and as $\{s_\beta : \beta \in J_x\}$ are representatives of G/G_x, it must be that β equals γ. Then, it is clear that $\pi_x(f - f\delta_{s_0})v_\beta = 0$. Suppose now that $\sigma_{s_\beta}(x)$ is not in $\mathrm{Fix}(s_0)$. As f is supported on $\mathrm{Fix}(s_0)$, we have that $f(\sigma_{s_\beta}(x)) = 0$. We can write $s_\gamma = s_0 s_\beta (s')^{-1}$. Then

$$f(\sigma_{s_\gamma}(x)) = f(\sigma_{s_0}(\sigma_{s_\beta}(\sigma_{s'}^{-1}(x)))) = f(\sigma_{s_0}(\sigma_{s_\beta}(x))).$$

As $\mathrm{Fix}(s_0)$ is invariant under σ_{s_0}, we have that $\sigma_{s_0}(\sigma_\beta(x))$ is not in $\mathrm{Fix}(s_0)$ and, again, as f is supported on $\mathrm{Fix}(s_0)$, it must be that $f(\sigma_{s_0}(\sigma_{s_\beta}(x))) = 0$. It follows that $\pi_x(f - f\delta_{s_0})v_\beta = 0$ in both cases. As this stands true for every $\beta \in J_x$, we have that π_x vanishes at $f - f\delta_{s_0}$, and as a consequence it also vanishes on I.

Let $g \in I \cap C(X)$. Then, for every $x \in X$ we have that $0 = \pi_x(g)v_e = g(x)v_e$, so $g(x) = 0$. This means that $g = 0$, so I is a closed and self-adjoint nonzero ideal such that $I \cap C(X) = \{0\}$. This contradicts (2).

Let's prove that (3) implies (1). Let I be a closed ideal such that $I \cap C(X) = \{0\}$. Let $f = \sum_{s \in G} f_s \delta_s \in I$. Because of Lemma 2.2.9 we have that f_s vanishes on $\mathrm{Aper}(G \curvearrowright X)$ for every $s \in G$. As $G \curvearrowright X$ is topologically free, we have that $\mathrm{Aper}(G \curvearrowright X)$ is dense in X, so we actually have that $f_s = 0$ for every $s \in G$, so $f = 0$. It follows that $I = \{0\}$. □

Points (1) and (2) of the above theorem differ only in the fact that, in one of them, we take closed ideals of $\ell^1(G \curvearrowright X)$, whereas in the other we take closed and self-adjoint ideals. Distinction between closed and closed and self-adjoint is meaningful, contrary to what happens with C^*-algebras, as for general Banach *-algebras there may be closed ideals that are not self-adjoint. Regardless, we later see that many of the dynamical properties of the action that we are able to detect through $\ell^1(G \curvearrowright X)$ admit characterizations both for only closed and for closed and self-adjoint ideals, as happens with topological freeness. We study in the next section under which dynamical assumptions we have that every closed ideal of $\ell^1(G \curvearrowright X)$ is self-adjoint.

With Theorem 2.3.1 at our disposal, we can relatively easily characterize different dynamical properties of $G \curvearrowright X$ as analytic-algebraic properties of $\ell^1(G \curvearrowright X)$ and vice versa, because it allows us to relate the closed ideals of $\ell^1(G \curvearrowright X)$ with invariant closed sets of the action. Following the ideas from [**dJST12**], we now characterize minimality and topological transitivity of the action. Now that we have already proved Theorem 2.3.1, the arguments are quite similar.

DEFINITION 2.3.2. We say that the action $G \curvearrowright X$ is *minimal* if there are no proper closed subsets of X that are invariant under the action of G. This is equivalent to the orbit of every element of X being dense.

Our next objective is to characterize $G \curvearrowright X$ being minimal as an analytic-algebraic property of $\ell^1(G \curvearrowright X)$. In order to do so, we have to understand how G-invariant and closed subsets of X give rise to closed (and self-adjoint) ideals of $\ell^1(G \curvearrowright X)$.

Let $D \subset X$ be a closed subset invariant under the action of G. We define:

$$I_D = \left\{ f = \sum_{s \in G} f_s \delta_s \in \ell^1(G \curvearrowright X) : f_s|_D = 0 \text{ for every } s \in G \right\}.$$

It is straightforward to check that I_D is a closed (and self-adjoint) ideal of $\ell^1(G \curvearrowright X)$. The following propositions tell us how to go the other way around, that is, to obtain a closed G-invariant subset of X from a closed ideal of $\ell^1(G \curvearrowright X)$.

Proposition 2.3.3. Let I be a closed ideal of $\ell^1(G \curvearrowright X)$. Then, there exists a closed and G-invariant subset D of X such that

$$I \cap C(X) = \{f \in C(X) : f|_D = 0\}.$$

Proof. As I is a closed ideal of $\ell^1(G \curvearrowright X)$, it is clear that $I \cap C(X)$ is a closed ideal of $C(X)$. By Proposition 1.2.9 in the preliminaries chapter, we know that every closed ideal of $\ell^1(G \curvearrowright X)$ consists of the continuous functions that vanish on some closed subset of X. So, there exists a closed subset D of X such that

$$I \cap C(X) = \{f \in C(X) : f|_D = 0\} .$$

We just have to check that D is invariant under the action of G. Suppose that there is some $s \in G$ such that $\sigma_s(D)$ is not included in D. This means that there is some $x \in D$ such that $\sigma_s(x) \notin D$. By Urysohn's lemma, we can take $g \in C(X)$ such that g vanishes on D and $g(\sigma_s(x)) \neq 0$. As I is an ideal, $(\delta_s)^{-1} g \delta_s = g \circ \sigma_s$ belongs to I, so, in particular, it belongs to $I \cap C(X)$. It follows that $g(x) = 0$, which is a contradiction. □

A few remarks about the proposition above. In general, it could happen that, given a closed ideal I of $\ell^1(G \curvearrowright X)$, we get D equal to the whole space X through the proposition above. But when $G \curvearrowright X$ is topologically free, Theorem 2.3.1 tells us that $I \cap C(X)$ is nonzero, so the closed G-invariant subset we obtain through the proposition must be a proper closed G-invariant subset of X. Also, it is not clear if given two different closed ideals I and J of $\ell^1(G \curvearrowright X)$ we can get the same closed G-invariant subset. This has to do with the ability of $C(X)$ to distinguish different ideals through the intersection with it. Later in this section we characterize exactly when this happens.

The following theorem is a generalization of [**dJST12**, Theorem 4.2], where this was proven for $\mathbb{Z}$-actions. It characterizes simplicity of the algebra as minimality of the action. Equivalence between (1) and (3) can be seen as a consequence of the more general [**BK24**, Theorem 6.13].

Theorem 2.3.4. Suppose that the action $G \curvearrowright X$ is topologically free. Then, the following are equivalent:

(1) There are no proper closed ideals in $\ell^1(G \curvearrowright X)$.
(2) There are no proper closed and self-adjoint ideals in $\ell^1(G \curvearrowright X)$.
(3) The action $G \curvearrowright X$ is minimal.

Proof. It is clear that (1) implies (2). Let's prove that (2) implies (3). Suppose that $G \curvearrowright X$ is not minimal, then there is some $x \in X$ such that its orbit is not dense in X. Take D as the closure of the orbit of x. Then, D is a closed and G-invariant proper subset of X, so

$$I_D = \left\{ f = \sum_{s \in G} f_s \delta_s \in \ell^1(G \curvearrowright X) : f_s|_D = 0 \text{ for every } s \in G \right\}$$

is a proper closed and self-adjoint ideal of $\ell^1(G \curvearrowright X)$. This contradicts (2).

We prove now that (3) implies (1). Let I be a closed nonzero ideal of $\ell^1(G \curvearrowright X)$. As $G \curvearrowright X$ is topologically free, using Theorem 2.3.1 we get that $I \cap C(X)$ is nonzero. Then, by Proposition 2.3.3, there is a closed G-invariant subset D that is not the whole space. But, as the action is minimal, the only G-invariant and closed subsets of X are X itself and the empty set. It follows that D is the empty set, so $I \cap C(X) = C(X)$. This implies that $I = \ell^1(G \curvearrowright X)$. □

Definition 2.3.5. We say that the action $G \curvearrowright X$ is *topologically transitive* if for every pair of nonempty open subsets U and V of X there is some $s \in G$ such that $\sigma_s(U) \cap V$ is nonempty.

We now characterize when $G \curvearrowright X$ is topologically transitive in terms of the properties of the algebra. In order to do so, we first need the following lemma.

Lemma 2.3.6. The following are equivalent:

(1) There exist two nonempty disjoint and G-invariant open subsets O_1 and O_2 of X such that $X = \overline{O_1} \cup \overline{O_2}$.
(2) The action $G \curvearrowright X$ is not topologically transitive.

Proof. That (1) implies (2) is trivial. Let's prove that (2) implies (1). As $G \curvearrowright X$ is not topologically transitive, there are two nonempty and open subsets U and V of X such that $\sigma_s(U) \cap V$ is empty for every $s \in G$. Take

$$O_1 = \bigcup_{s \in G} \sigma_s(U).$$

It is clear that O_1 is an open G-invariant subset of X. As $V \subset \overline{O_1}^c$, we can take $O_2 = \overline{O_1}^c$. These sets verify what we need. □

The following theorem generalizes [**dJST12**, Theorem 4.10], which was proved for $\mathbb{Z}$-actions. It characterizes a notion of primality of the algebra as the action being topologically transitive. In the proof, given a closed subset D of X, we use the notation

$$\ker(D) = \{g \in C(X) : g|_D = 0\}.$$

We also recall that E denotes the canonical projection from $\ell^1(G \curvearrowright X)$ to $C(X)$ introduced in Definition 2.1.1.

Theorem 2.3.7. Suppose that the $G \curvearrowright X$ is topologically free. Then, the following are equivalent:

(1) For every pair of nonzero closed ideals I and J of $\ell^1(G \curvearrowright X)$, we have that $I \cap J$ is nonzero.
(2) For every pair of nonzero closed and self-adjoints ideals I and J of $\ell^1(G \curvearrowright X)$, we have that $I \cap J$ is nonzero.
(3) The action $G \curvearrowright X$ is topologically transitive.

Proof. It is evident that (1) implies (2). Let's see that (2) implies (3). Suppose that the action $G \curvearrowright X$ is not topologically transitive. Then, by Lemma 2.3.6, we have two nonempty open sets O_1 and O_2, disjoint and G-invariant, such that $X = \overline{O_1} \cup \overline{O_2}$. We define

$$I_i = \left\{ \sum_{s \in G} f_s \delta_s : f_s \in \ker(\overline{O_i}) \text{ for every } s \in G \right\}$$

for $i = 1, 2$. Both are closed and self-adjoint ideals of $\ell^1(G \curvearrowright X)$. Then

$$E(I_1 \cap I_2) \subset E(I_1) \cap E(I_2) = \ker(\overline{O_1}) \cap \ker(\overline{O_2}) = \ker(\overline{O_1} \cup \overline{O_2}) = \ker(X) = \{0\},$$

so $I_1 \cap I_2 = \{0\}$. This contradicts (2).

Let's now prove that (3) implies (1). Let I_1 and I_2 be two proper and closed ideals of $\ell^1(G \curvearrowright X)$ such that $I_1 \cap I_2 = \{0\}$. As $G \curvearrowright X$ is topologically free, Theorem 2.3.1 tells us that $I_1 \cap C(X)$ and $I_2 \cap C(X)$ are both proper ideals of $C(X)$, so by Proposistion 2.3.3 we get two proper closed and G-invariant subsets D_1 and D_2 of X such that $I_1 \cap C(X) = \ker(D_1)$ and $I_2 \cap C(X) = \ker(D_2)$. As $I_1 \cap I_2 = \{0\}$ we also have that $I_1 \cap I_2 \cap C(X) = \{0\}$. Then

$$\{0\} = I_1 \cap I_2 \cap C(X) = \ker(D_1) \cap \ker(D_1) = \ker(D_1 \cup D_2),$$

so it must be $D_1 \cup D_2 = X$. As both are proper and closed, they must both have nonempty interior, so if we denote by U the interior of D_1 and by V the complement of D_1 we have that $\sigma_s(U) \cap V = \emptyset$ for every $s \in G$. This contradicts the topological transitivity of $G \curvearrowright X$. $\square$

We described a procedure to obtain closed ideals of $C(X)$ from ideals of $\ell^1(G \curvearrowright X)$, that goes as follows: if I is a closed ideal of $\ell^1(G \curvearrowright X)$, then $I \cap C(X)$ is a closed ideal of $C(X)$. In this setting, one may wonder whether $C(X)$ detects all ideals of $\ell^1(G \curvearrowright X)$ thorugh this procedure, i.e., if I is a closed and nonzero ideal then $I \cap C(X)$ is also a nonzero ideal. This is called the *ideal intersection property*, and we have that, according to Theorem 2.3.1, this holds precisely when the action is topologically free. Another similar property is the *ideal separation property*. This holds when $C(X)$ can distinguish ideals of $\ell^1(G \curvearrowright X)$, in the sense that, if I and J are two closed ideals of $\ell^1(G \curvearrowright X)$ such that $I \cap C(X)$ is equal to $J \cap C(X)$, then $I = J$. These properties have been studied in a different context recently in [**AT25, AR23**].

It is clear that the ideal separation property is stronger than the ideal intersection property. This means that, for the ideal separation property to hold, we need the action to at least be topologically free. It turns out that the ideal separation property is related to the definition that follows, which is a strengthening of topological freeness. This definition first appeared for groupoids by the name of *essentially principal* in [**Ren80**, Definition 4.3], and more recently has been called *residual topological freeness*, as can be seen, for example, in [**Sie10**, Definition 1.13] and [**BK24**, Definition 6.11].

Definition 2.3.8. We say that the action $G \curvearrowright X$ is *residually topologically free* if for any G-invariant closed subset D of X, the restriction of the action to D is topologically free. For brevity, we sometimes write r.t.f. instead of residually topologically free.

Remark 2.3.9. For abelian groups, the notions of free actions and of r.t.f. actions are equivalent. It is evident that if the action is free then it is also r.t.f. For the other direction, suppose that there is some $x \in X$ such that $G_x \neq \{e\}$. Take D as the closure of the orbit of x. As G is abelian, all points of the orbit of x have the same stabilizer subgroup G_x. Then, given $t \in G_x$, for every $s \in G$ we have that $\sigma_t(\sigma_s(x)) = \sigma_s(\sigma_t(x)) = \sigma_s(x)$. This means that t fixes the whole orbit of x, so it also fixes D, so the restriction of the action to D is not topologically free.

Example 2.3.10. Let G be the free group on two generators $\mathbb{F}_2$, acting canonically on its Gromov boundary $X = \partial\mathbb{F}_2$. It can be seen that this action is topologically free but it is not free. It can also be shown to be minimal, so it has no proper G-invariants subsets and, as such, it is a residually topologically free action.

The main result of the rest of this section is Theorem 2.3.13, where we characterize when the ideal separation property holds. We note that this theorem can be seen as a consequence of the more general [**BK24**, Theorem 6.12]. We present here a self-contained proof for this context, where the ideas draw inspiration from [**Sie10**, Section 1], where something similar is done for C^*-dynamical systems.

Let us fix some notation. Suppose that D is a closed G-invariant subset of X. Then, we can restrict to D the action of G and obtain a new action $G \curvearrowright D$. We also denote by σ the restriction of the action to D. As this is also a dynamical system, we can consider the algebra $\ell^1(G \curvearrowright D)$ and the map $R_D : \ell^1(G \curvearrowright X) \to \ell^1(G \curvearrowright D)$ given by

$$R_D\left(\sum_{s\in G} f_s\delta_s\right) = \sum_{s\in G} f_s|_D\delta_s \text{ for every } \sum_{s\in G} f_s\delta_s \in \ell^1(G \curvearrowright X),$$

where δ_s also denotes the canonical unitary element of $\ell^1(G \curvearrowright D)$ associated to $s \in G$. It is clear that R_D is a surjective and norm-one *-homomorphism, and we can write the kernel of this map as $\ker(R_D) = \{\sum_{s\in G} f_s\delta_s : f_s|_D = 0 \text{ for every } s \in G\}$.

Lemma 2.3.11. Let I be a closed ideal of $\ell^1(G \curvearrowright X)$, and let D be a closed G-invariant subset of X such that $I \cap C(X) = \{f \in C(X) : f|_D = 0\}$ (as in Proposition 2.3.3). Then $R_D(I)$ has zero intersection with $C(D)$.

Proof. Let g be in $R_D(I) \cap C(D)$, so there is some $\sum_{s\in G} f_s\delta_s \in I$ that gets mapped to g through R_D, i.e.,

$$\sum_{s\in G} f_s|_D\delta_s = g.$$

It follows that $f_s|_D = 0$ for every $s \in G \setminus \{e\}$ and $f_e|_D = g$. Then, $\sum_{s\neq e} f_s\delta_s$ is in $\ker(R_D)$. As $\{f \in C(X) : f|_D = 0\}$ is equal to $I \cap C(X)$, which is contained in I, it is clear that $\ker(R_D) \subset I$, so it follows that

$$f_e = \sum_{s\in G} f_s\delta_s - \sum_{s\neq e} f_s\delta_s \in I.$$

Then $f_e \in I \cap C(X)$, so its restriction to D is zero. As g is equal to $f_e|_D$, we conclude that $R_D(I) \cap C(D) = \{0\}$. □

Remark 2.3.12. Given a bounded and surjective linear operator $T : A \to B$ between Banach spaces, and given I a nonempty subset of A, we have that $T(I)$ is closed in B if and only if $I + \ker(T)$ is closed in A. We use this in the following theorem.

In the next proof, given D a closed G-invariant subset of X, we denote by $r_D : C(X) \to C(D)$ the map given by $r_D(f) = f|_D$ for every f in $C(X)$. We denote by $E_D : \ell^1(G \curvearrowright D) \to C(D)$ the map given by $E_D(\sum_{s\in G} f_s\delta_s) = f_e$ for every $\sum_{s\in G} f_s\delta_s$ in $\ell^1(G \curvearrowright D)$. It is clear that $E_D \circ R_D = r_D \circ E$.

Theorem 2.3.13. The following are equivalent:

(1) For every pair of ideals I and J of $\ell^1(G \curvearrowright X)$, if $I \cap C(X)$ is equal to $J \cap C(X)$ then $I = J$.
(2) $E(I)$ is contained in I for every closed ideal I of $\ell^1(G \curvearrowright X)$.
(3) The action $G \curvearrowright X$ is residually topologically free.

Proof. Let's first prove that (1) implies (2). Let I be a closed ideal of $\ell^1(G \curvearrowright X)$. Using Proposition 2.3.3, we have that there is some closed and G-invariant subset D of X such that $I \cap C(X)$ consists of the continuous functions that vanish on D. We define the closed ideal

$$J = \left\{ \sum_{s \in G} f_s \delta_s : f_s|_D = 0 \text{ for every } s \in G \right\}.$$

It is clear that $J \cap C(X) = I \cap C(X)$, so using (1) we have that $J = I$. Now it follows that $E(I) = I \cap C(X) \subset I$.

We prove now that (2) implies (3). Let D be a closed and G-invariant subset of X. In order to prove that $G \curvearrowright D$ is topologically free, we use Theorem 2.3.1. Let J be a closed ideal of $\ell^1(G \curvearrowright D)$ such that $J \cap C(D) = \{0\}$. We have to prove that $J = \{0\}$. We define $I = R_D^{-1}(J)$, which is a closed ideal of $\ell^1(G \curvearrowright X)$. By (2) we have that $E(I) \subset I$ so, in particular, $E(I) \subset I \cap C(X)$. Now, we have that

$$E_D(J) = E_D(R_D(I)) = r_D(E(I)) \subset r_D(I \cap C(X)) \subset J \cap C(D) = \{0\}.$$

As $E_D(J) = \{0\}$ we have that $J = \{0\}$.

That (3) implies (1) is the only implication left to prove. Suppose that I and J are two closed ideals of $\ell^1(G \curvearrowright X)$ such that $I \cap C(X) = J \cap C(X)$. Using Proposition 2.3.3 we get a closed and G-invariant subset D of X such that $I \cap C(X)$ and $J \cap C(X)$ are both equal to $\{f \in C(X) : f|_D = 0\}$. We have that

$$\ker(R_D) = \left\{ \sum_{s \in G} f_s \delta_s : f_s|_D = 0 \text{ for every } s \in G \right\}$$

is included in I and J. As R_D is contractive and surjective, and I and J are closed, we have by Remark 2.3.12 that both $R_D(I)$ and $R_D(J)$ are closed ideals of $\ell^1(G \curvearrowright D)$. Now, using Lemma 2.3.11 we get that both $R_D(I) \cap C(D)$ and $R_D(I) \cap C(D)$ are equal to $\{0\}$. By (3), we have that $G \curvearrowright D$ is topologically free, so Theorem 2.3.1 implies that $R_D(I) = R_D(J) = 0$. This means that I and J are included in $\ker(R_D)$. As we have proved that both inclusions hold, we have that $I = \ker(R_D) = J$. ☐

Remark 2.3.14. The theorem above bears a resemblance to some of the work done in [**dJT13**, Section 3]. There, for $\mathbb{Z}$-actions, two notions of noncommutative *spectral synthesis* are studied for $\ell^1(G \curvearrowright X)$. It is proved there that (1) and (2) are equivalent to the action being free, which makes sense given that $\mathbb{Z}$ is abelian and Remark 2.3.9.

Remark 2.3.15. Theorem 2.3.13 implies that, when the action $G \curvearrowright X$ is r.t.f., then we can completely describe the (closed) ideal structure of $\ell^1(G \curvearrowright X)$. Concretely, we have that every closed ideal I is such that there exists a G-invariant closed subset D of X that satisfies

$$I = \left\{ \sum_{s \in G} f_s \delta_s : f_s|_D = 0 \text{ for every } s \in G \right\}.$$

In this situation, we refer to I as the ideal generated by D. It follows that, when the action is residually topologically free, there is a bijection between the closed ideals of $\ell^1(G \curvearrowright X)$ and the G-invariant closed subsets of X.

An ideal I is *prime* if it is not the whole algebra and $JK \subset I$ implies $J \subset I$ or $K \subset I$, for every pair of ideals J and K. An ideal I is called *semiprime* if $J^2 \subset I$ implies $J \subset I$. Equivalently, an ideal is semiprime if it is an intersection of prime ideals (see [**Lam91**, Theorem 10.11]). In C^*-algebras, every closed ideal is semiprime, given that every closed ideal is the intersection of primitive ideals and primitive ideals are prime. Recently, there has been an interest on the study of non-closed prime and semiprime ideals in C^*-algebras, see [**GT24, GKT24, GKT25**]. With the above remark at our disposal we are able to show that if the action $G \curvearrowright X$ is r.t.f., then every closed ideal of $\ell^1(G \curvearrowright X)$ is semiprime. We can also characterize closed prime ideals of $\ell^1(G \curvearrowright X)$.

A *minimal* set C of the action $G \curvearrowright X$ is a closed and G-invariant subset of X such that it does not contain any proper closed and G-invariant subset. This is equivalent to the restriction of the action to C being minimal.

Corollary 2.3.16. Let $G \curvearrowright X$ be residually topologically free. Then, every closed ideal of $\ell^1(G \curvearrowright X)$ is semiprime. Furthermore, a closed ideal I of $\ell^1(G \curvearrowright X)$ is prime if and only if there exists some closed and minimal G-invariant subset C of X such that

$$I = \left\{ \sum_{s \in G} f_s \delta_s \in \ell^1(G \curvearrowright X) : f_s|_C = 0 \text{ for every } s \in G \right\}.$$

Proof. In order to check that a closed ideal I is prime, it is enough to check that the condition for primeness holds with respect to closed ideals. That is, if J and K are closed ideals such that $JK \subset I$, then $J \subset I$ or $K \subset I$. This follows from the fact that the product ideal $\overline{J}\,\overline{K}$ is contained in $\overline{JK}$ and that, if I is closed, then $\overline{I} = I$. The same argument justifies that, in order to check that a closed ideal I is semiprime, it is enough to prove that if J is a closed ideal such that J^2 is contained in I then J is included in I.

Let I be a closed ideal and, as the action is r.t.f., let C be the closed G-invariant subset of X such that I is generated by C, i.e.:

$$I = \left\{ \sum_{s \in G} f_s \delta_s : f_s|_C = 0 \text{ for every } s \in G \right\}.$$

For every closed G-invariant subset D of X, we denote by J_D the closed ideal of $\ell^1(G \curvearrowright X)$ generated by D, which is defined analogously to I. It is clear that, for every pair of closed G-invariant sets E and D, if C is contained in $E \cup D$ then $J_E J_D$ is contained in I. The reciprocal also holds, that is, if $J_E J_D$ is contained in I, then C is contained in $E \cup D$. This follows from the fact that if $J_E J_D$ is contained in I, then for every f and g belonging to $C(X)$ and vanishing on E and D, respectively, the product fg belongs to I, so it must vanish on C. This happens only if C is contained in $D \cup E$. So, we have that $J_D J_E$ is contained in I if and only if C is contained in $D \cup E$.

Let us observe that J_D is contained in I, for some closed G-invariant set D, if and only if C is contained in D. Now, as the action is residually topologically free, every closed ideal of $\ell^1(G \curvearrowright X)$ is of the form J_D for some closed G-invariant set D. This means that we can easily translate the ideal I being semiprime or prime to a condition on the closed G-invariants sets of the action. First, I is always semiprime because J_D^2 being contained in I is equivalent to C being contained in D, which implies that J_D is contained in I. Finally, the ideal I is prime if and only if for every pair D and E of closed G-invariants subsets of X such that C is contained in $D \cup E$ we have that C is contained in D or C is contained in E. This is equivalent to C being a minimal set for the action. □

2.4. Detecting freeness through self-adjoint ideals

Up until this point we have mentioned multiple times the result [**dJST12**, Theorem 4.4], which states, for $\mathbb{Z}$-actions, that the action is free if and only if every closed ideal of $\ell^1(\mathbb{Z} \curvearrowright X)$ is self-adjoint. This theorem is interesting in the sense that freeness of the action can be detected by $\ell^1(\mathbb{Z} \curvearrowright X)$ through the property of every closed ideal being automatically self-adjoint, this being something that cannot be detected by its C^*-envelope as, in a C^*-algebra, all closed ideals are self-adjoint. We would like to generalize this result to a more general setting, but some problems arise when trying to do this. First of all, the following example shows that [**dJST12**, Theorem 4.4] is not true in general.

Example 2.4.1. Consider the action $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$ of $\mathbb{Z}_2$ on $\mathbb{Z}$, given by $\varphi(0) = \mathrm{Id}$ and $\varphi(1) = -\mathrm{Id}$, and take the semidirect product $G = \mathbb{Z} \rtimes_\varphi \mathbb{Z}_2$. Let θ be an irrational number. Now, let X be the circle $S^1 = \mathbb{R}/\mathbb{Z}$, and let G act on X the following way: $\sigma_{(n,0)}(\beta) = \beta + n\theta$ (modulo $\mathbb{Z}$) and $\sigma_{(0,1)}(\beta) = -\beta$ for every $\beta \in \mathbb{R}/\mathbb{Z}$. It is straightforward to check that this defines an action on X, given by

$$\sigma_{(n,m)}(\beta) = (-1)^m \beta + n\theta.$$

If we identify $\mathbb{R}/\mathbb{Z}$ with the unit circle on the complex plane, what we have is that $(1,0)$ acts as an irrational rotation of angle θ and that $(0,1)$ acts as a symmetry with respect to the real axis. As θ is irrational, we have that the action $G \curvearrowright X$ is minimal. It is easy to verify that $\mathrm{Per}(G \curvearrowright X) = \left\{\beta = \frac{n\theta+m}{2} \in \mathbb{R}/\mathbb{Z} : n, m \in \mathbb{Z}\right\}$, so the action is topologically free but not free. But as the action is topologically free and minimal, by Theorem 2.3.4 we have that $\ell^1(G \curvearrowright X)$ has no proper closed ideals so, trivially, every closed ideal of $\ell^1(G \curvearrowright X)$ is self-adjoint.

We remark that the proof of one of the implications of [**dJST12**, Theorem 4.4], although we do not explicitly write it, generalizes well to this setting: if the action $G \curvearrowright X$ is free, then every closed ideal of $\ell^1(G \curvearrowright X)$ is self-adjoint. It is interesting that, in that proof, what is actually used is that if the action is free then it is also r.t.f. This detail is not relevant when working with $\mathbb{Z}$-actions as a consequence of Remark 2.3.9, but, if we are working with a nonabelian group, we have that being r.t.f (which is weaker than the action being free) is enough to have that every closed ideal is self-adjoint. Given the work we have already done, we can obtain this result easily as a simple corollary of Theorem 2.3.13.

Corollary 2.4.2. If the action $G \curvearrowright X$ is residually topologically free, then every closed ideal of $\ell^1(G \curvearrowright X)$ is self-adjoint.

Proof. Let I be a closed ideal. As $C(X)$ is a C^*-algebra, every closed ideal is self-adjoint, so it must be that $I \cap C(X) = I^* \cap C(X)$. As the action $G \curvearrowright X$ is r.t.f, by Theorem 2.3.13 we obtain that $I = I^*$. □

As we said before, freeness of the action is stronger than it being r.t.f, so the above theorem also works if we change one condition by the other. Our goal now is to try to generalize the reciprocal of the theorem, but we cannot do the same proof that was done in [**dJST12**, Theorem 4.4] because, there, it was used that if an action of $\mathbb{Z}$ is not free then there is a finite orbit, which is not necessarily true in general.

We denote by $\ker(\alpha)$ the set of elements of G that act trivially on X. In what follows, given a discrete group H, we denote by $\ell^1(H)$ the Banach *-algebra of integrable functions with respect to the counting measure, with the usual convolution and involution, and we denote its norm by $\|\cdot\|_1$ (see Section 1.2.3 from Chapter 1 for details). Given any $t \in H$ we write the characteristic function of $\{t\}$ by $\mathbb{1}_t \in \ell^1(H)$, and we recall that $(\mathbb{1}_t * \xi)(h) = \xi(t^{-1}h)$ for every $\xi \in \ell^1(H)$ and $h \in H$, where $*$ denotes the usual convolution in $\ell^1(H)$.

Lemma 2.4.3. Let G be abelian. Suppose that H is a subgroup of G contained in $\ker(\alpha)$ and such that $\ell^1(H)$ has a closed and non-self-adjoint ideal. Then $\ell^1(G \curvearrowright X)$ contains a closed and non-self-adjoint ideal.

Proof. Let J be a closed and non-self-adjoint ideal of $\ell^1(H)$. For every $x \in X$, every $s \in G$ and every $f \in \ell^1(G \curvearrowright X)$, we define $\phi_{x,s}(f)$ as the map from H to $\mathbb{C}$ such that $\phi_{x,s}(f)(h) = f(sh)(x)$ for every $h \in H$. We have that

$$\|\phi_{x,s}(f)\|_1 = \sum_{t\in H} |\phi_{x,s}(f)(t)| = \sum_{t\in H} |f(st)(x)| \leq \sum_{t\in H} \|f(st)\|_\infty \leq \sum_{s\in G} \|f(s)\|_\infty = \|f\| < \infty,$$

so $\phi_{x,s}(f)$ belongs to $\ell^1(H)$ for every $x \in X$, every $s \in G$ and every $f \in \ell^1(G \curvearrowright X)$. We can, then, think of $\phi_{x,s}$ as a contracting linear map from $\ell^1(G \curvearrowright X)$ to $\ell^1(H)$. We define

$$\widetilde{J} = \left\{ f \in \ell^1(G \curvearrowright X) : \phi_{x,s}(f) \in J \text{ for every } x \in X \text{ and } s \in G \right\}.$$

We will prove that $\widetilde{J}$ is closed and non-self-adjoint ideal of $\ell^1(G \curvearrowright X)$.

We can write

$$\widetilde{J} = \bigcap_{\substack{x\in X\\ s\in G}} \phi_{x,s}^{-1}(J)$$

and, as for every $x \in X$ and every $s \in G$ we have that $\phi_{x,s}$ is contractive and J is closed, we have that $\widetilde{J}$ is also closed. We have to check that it is an ideal. We write $G/H = \{s_\beta H : \beta \in I_H\}$ where $\{s_\beta : \beta \in I_H\}$ is a set of representatives of the quotient, so we have that $G = \bigsqcup_{\beta\in I_H} s_\beta H$. Let $f \in \ell^1(G \curvearrowright X)$ and $g \in \widetilde{J}$, and fix $x \in X$ and $s \in G$. Let $h \in H$. We have that

$$\begin{aligned}\phi_{x,s}(fg)(h) = (fg)(sh)(x) &= \left[\sum_{i\in G} f(i)\alpha_i(g(i^{-1}sh))\right](x)\\ &= \left[\sum_{\beta\in I_H}\sum_{t\in H} f(s_\beta t)\alpha_{s_\beta t}(g((s_\beta t)^{-1}sh))\right](x).\end{aligned}$$

Let's fix $\beta \in I_H$. We have that $H \subset \ker(\alpha)$ and that G is abelian, so

$$\begin{aligned}\left[\sum_{t\in H} f(s_\beta t)\alpha_{s_\beta t}(g((s_\beta t)^{-1}sh))\right](x) &= \sum_{t\in H} f(s_\beta t)(x)\alpha_{s_\beta}(g(t^{-1}s_\beta^{-1}sh))(x)\\ &= \sum_{t\in H} f(s_\beta t)(x) g(s_\beta^{-1}st^{-1}h)(\sigma_{s_\beta}^{-1}(x))\\ &= \sum_{t\in H} \phi_{x,s_\beta}(f)(t)\phi_{\sigma_{s_\beta}^{-1}(x),s_\beta^{-1}s}(g)(t^{-1}h)\\ &= (\phi_{x,s_\beta}(f) * \phi_{\sigma_{s_\beta}^{-1}(x),s_\beta^{-1}s}(g))(h),\end{aligned}$$

and it follows that we can express, for every $h \in H$,

$$\phi_{x,s}(fg)(h) = \sum_{\beta \in I_H} (\phi_{x,s_\beta}(f) * \phi_{\sigma_\beta^{-1}(x), s_\beta^{-1}s}(g))(h).$$

So, we write

$$\phi_{x,s}(fg) = \sum_{\beta \in I_H} \phi_{x,s_\beta}(f) * \phi_{\sigma_\beta^{-1}(x), s_\beta^{-1}s}(g),$$

where the sum converges because

$$\begin{aligned} \| \sum_{\beta \in I_H} \phi_{x,s_\beta}(f) * \phi_{\sigma_{s_\beta}^{-1}(x), s_\beta^{-1}s}(g) \|_1 &\leq \sum_{\beta \in I_H} \|\phi_{x,s_\beta}(f)\|_1 \|\phi_{\sigma_{s_\beta}^{-1}(x), s_\beta^{-1}s}(g)\|_1 \\ &\leq \sum_{\beta \in I_H} \left(\sum_{t \in H} |\phi_{x,s_\beta}(f)(t)| \right) \|g\| = \|g\| \sum_{\beta \in I_H} \sum_{t \in H} |f(s_\beta t)(x)| \\ &\leq \|g\| \sum_{\beta \in I_H} \sum_{t \in H} \|f(s_\beta t)\|_\infty = \|g\| \sum_{s \in G} \|f(s)\|_\infty \\ &= \|g\| \|f\| < \infty. \end{aligned}$$

As $f \in \widetilde{J}$, we have that $\phi_{x,s_\beta}(f) \in J$, and it follows that $\phi_{x,s_\beta}(f) * \phi_{\sigma_\beta^{-1}(x), s_\beta^{-1}s}(g)$ also belongs to J for every $\beta \in I_H$. Thus, $\phi_{x,s}(fg)$ belongs to J as it is the limit of elements in J, which is a closed ideal. As this works for every $x \in X$ and every $s \in G$, we conclude that $fg \in \widetilde{J}$.

Similarly, we prove that gf belongs to $\widetilde{J}$. Let $x \in X$ and let $s \in G$, and take $h \in H$. We have that:

$$\begin{aligned} \phi_{x,s}(gf)(h) = (gf)(sh)(x) &= \left[\sum_{i \in G} g(i) \alpha_i (f(i^{-1}sh)) \right] (x) \\ &= \left[\sum_{\beta \in I_H} \sum_{t \in H} g(s_\beta t) \alpha_{s_\beta t}(f((s_\beta t)^{-1} sh)) \right] (x) \\ &= \sum_{\beta \in I_H} \sum_{t \in H} g(s_\beta t)(x) \alpha_{s_\beta}(f(s_\beta^{-1} s t^{-1} h))(x) \\ &= \sum_{\beta \in I_H} \sum_{t \in H} g(s_\beta t)(x) f(s_\beta^{-1} s t^{-1} h)(\sigma_{s_\beta}^{-1}(x)). \end{aligned}$$

Fixing $\beta \in I_H$, we have that

$$\begin{aligned} \sum_{t \in H} g(s_\beta t)(x) f(s_\beta^{-1} s t^{-1} h)(\sigma_{s_\beta}^{-1}(x)) &= \sum_{t \in H} \phi_{x,s_\beta}(g)(t) \phi_{\sigma_{s_\beta}^{-1}(x), s_\beta^{-1}s}(f)(t^{-1}h) \\ &= (\phi_{x,s_\beta}(g) * \phi_{\sigma_{s_\beta}^{-1}(x), s_\beta^{-1}s}(f))(h), \end{aligned}$$

which belongs to J because g belongs to $\widetilde{J}$. Then, we write

$$\phi_{x,s}(gf) = \sum_{\beta \in I_H} \phi_{x,s_\beta}(g) * \phi_{\sigma_{s_\beta}^{-1}(x), s_\beta^{-1}s}(f),$$

where the sum again converges by the same kind of computation as in the case above. Then, $\phi_{x,s}(gf)$ belongs to J, as it is the limit of elements of J. As this works for every $x \in X$ and every $s \in G$, we have that $gf \in \widetilde{J}$. We can conclude then that $\widetilde{J}$ is a closed ideal of $\ell^1(G \curvearrowright X)$.

We only have left to check that $\widetilde{J}$ is not self-adjoint. As J is not self-adjoint, there is some $\xi \in J$ such that $\xi^* \notin J$. We define $\widehat{\xi} : G \to C(X)$ the following way: for every $s \in H$, we set $\widehat{\xi}(s)$ as the constant function $\xi(s)$, and we set is as 0 in any other case. As we have that $\xi \in \ell^1(H)$, it is direct to check that $\widehat{\xi} \in \ell^1(G \curvearrowright X)$. Let's first check that $\widehat{\xi}$ is in $\widetilde{J}$. Let $x \in X$ and let $s \in G$. We have that

$$\phi_{x,s}(\widehat{\xi})(h) = \widehat{\xi}(sh)(x) = \begin{cases} \xi(sh) & \text{if } sh \in H \\ 0 & \text{if } sh \notin H \end{cases} = \begin{cases} (\mathbb{1}_{s^{-1}} * \xi)(h) & \text{if } s \in H \\ 0 & \text{if } s \notin H. \end{cases}$$

So, if $s \notin H$, then $\phi_{x,s}(\widehat{\xi}) = 0$, so it is in J. If $s \in H$, then $\phi_{x,s}(\widehat{\xi}) = \mathbb{1}_{s^{-1}} * \xi$, which again belongs to J because J is an ideal. Then, we conclude that $\widehat{\xi}$ is in $\widetilde{J}$. Let's check that $(\widehat{\xi})^*$ is not in $\widetilde{J}$. Take any $x \in X$ and consider $\phi_{x,e}((\widehat{\xi})^*)$. For every $h \in H$ we have

$$\begin{aligned} \phi_{x,e}((\widehat{\xi})^*)(h) &= ((\widehat{\xi})^*)(h)(x) = \alpha_h\left(\overline{\widehat{\xi}(h^{-1})}\right)(x) = \alpha_h\left(\overline{\xi(h^{-1})}\right)(x) \\ &= \overline{\xi(h^{-1})}(\sigma_h^{-1}(x)) = \overline{\xi(h^{-1})} = \xi^*(h), \end{aligned}$$

so $\phi_{x,e}((\widehat{\xi})^*) = \xi^*$, which does not belong to J. So we obtain that $(\widehat{\xi})^*$ is not in $\widetilde{J}$, and conclude that $\widetilde{J}$ is not self-adjoint. □

We need the following theorem, which is proved in [**Rud62**].

THEOREM 2.4.4 ([**Rud62**, Theorem 7.7.1]). Let H be a discrete and abelian group. If H is infinite, then there is a closed and non-self-adjoint ideal in $\ell^1(H)$.

LEMMA 2.4.5. Let G be abelian and suppose that there is some $x \in X$ such that G_x is infinite. Then, $\ell^1(G \curvearrowright X)$ has a closed and non-self-adjoint ideal.

PROOF. Let $x \in X$ such that G_x is an infinite subgroup of G. By the theorem above, as G_x is abelian, there is a closed and non-self-adjoint ideal in $\ell^1(G_x)$. Let D be the closure of the orbit of x, which is a G-invariant and closed subset of X. Consider the restriction of the action to D, and consider the algebra $\ell^1(G \curvearrowright D)$ arising from the restriction. As G is abelian, we have that every point in the orbit of x has the same stabilizer subgroup, so G_x is contained in the kernel of the action of G on D. Using Lemma 2.4.3, we have that there is a closed and non-self-adjoint ideal $\widetilde{I}$ in $\ell^1(G \curvearrowright D)$. We consider the contractive *-homomorphism $R_D : \ell^1(G \curvearrowright X) \to \ell^1(G \curvearrowright D)$ given by $R_D(\sum_{s\in G} f_s\delta_s) = \sum_{s\in G} f_s|_D\delta_s$. As R_D is surjective, we have that $I = R_D^{-1}(\widetilde{I})$ is a closed and non-self-adjoint ideal of $\ell^1(G \curvearrowright X)$. □

The above proposition basically tells us that, for abelian groups, an infinite point-stabilizer subgroup induces a closed but non-self-adjoint ideal in $\ell^1(G \curvearrowright X)$. This allows us to obtain the following statement.

THEOREM 2.4.6. Let G be an abelian torsion-free group. Then, the action $G \curvearrowright X$ is free if and only if every closed ideal of $\ell^1(G \curvearrowright X)$ is self-adjoint.

PROOF. If $G \curvearrowright X$ is free then it is also r.t.f, so every closed ideal of $\ell^1(G \curvearrowright X)$ is self-adjoint as a consequence of Corollary 2.4.2. Suppose now that every closed ideal of $\ell^1(G \curvearrowright X)$

is self-adjoint. If $G \curvearrowright X$ is not free, then there is some $x \in X$ such that $G_x \neq \{e\}$. As G is torsion-free, G_x must be infinite, so by Lemma 2.4.5 there is a closed and non-self-adjoint ideal in $\ell^1(G \curvearrowright X)$. This is a contradiction. $\square$

CHAPTER 3

Rigidity theorem for the crossed product

In last chapter, we obtained evidence sustaining that, most likely, $\ell^1(G \curvearrowright X)$ retains more information of the action than its C^*-algebraic counterpart as, according to Theorem 2.4.6, it may detect freeness of the action in a way that cannot be done in the C^*-algebra crossed product. In this chapter we want to study exactly how much information of the action is remembered in $\ell^1(G \curvearrowright X)$. We first mention some precedents in other contexts.

By themselves, Von Neumann and C^*-algebraic crossed products do not contain much information of the dynamics. But, together with a canonical commutative algebra and under some conditions on the dynamics, one can recover the *orbital equivalence class* of the action. Roughly speaking, two actions are orbit equivalent if they *have the same orbits*, but the precise definition depends on the context we are working on. Let us be a little more precice in these statements.

When working in the Von Neumann algebraic context, one has to look at measurable dynamics. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be two probability preserving actions of discrete countable groups on standard spaces by Borel automorphisms. In this case, the commutative (Von Neumann) algebra on which the action $G \curvearrowright X$ is reflected is $L^\infty(X)$, and one can consider the Von Neumann algebra crossed product, which we denote by $L^\infty(X) \rtimes G$, and analogously we obtain $L^\infty(Y) \rtimes H$. The actions are *orbit equivalent* (in this measurable setting) if there is an isomorphism of measure spaces $\varphi : X \to Y$ such that $\varphi(\mathcal{O}_G(x)) = \mathcal{O}_H(\varphi(x))$ for a.e. $x \in X$. Here, by the calligraphic letter $\mathcal{O}$, we denote the orbit of an element through the action of the appropriate group. The following theorem holds and can be found in [**Li18**, Theorem 1.1].

THEOREM. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be an *(essentially) free* probability measure preserving actions. The following are equivalent:

(1) The actions $G \curvearrowright X$ and $H \curvearrowright Y$ are orbit equivalent.
(2) There is a Von Neumann algebra isomorphism $\Phi : L^\infty(X) \rtimes G \to L^\infty(Y) \rtimes H$ such that $\Phi(L^\infty(X)) = L^\infty(Y)$.

The action is *essentially free* if the set of points of the space with trivial stabilizer subgroup has total measure. So, an isomorphism by itself is not enough to have orbit equivalence unless it preserves a canonical commutative subalgebra. The situation is similar in the C^*-algebraic context, as seen in the following theorem. In this context, we have the notion of *continuous orbit equivalence*. Given two actions by homeomorphisms $G \curvearrowright X$ and $H \curvearrowright Y$, we say that they are *continuously orbit equivalent* if there is a homeomorphism $\varphi : X \to Y$ and two continuous maps $a : G \times X \to H$ and $b : H \times Y \to G$ such that $\varphi(g.x) = a(g, x).\varphi(x)$ for every $x \in X$ and for every $g \in G$, and $\varphi^{-1}(h.y) = b(h, y).\varphi^{-1}(y)$ for every $y \in Y$ and for every $h \in H$ (for simplicity, here we denote both actions by a dot). We recall that, given an action $G \curvearrowright X$, by $C^*_\lambda(G \curvearrowright X)$ we denote the reduced C^*-algebra crossed product associated to the action.

Theorem ([**Li18**, Theorem 1.2]). Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free actions on second countable spaces X and Y. The following are equivalent:

(1) The actions $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent.
(2) There is an isomorphism of C^*-algebras $\Phi : C^*_\lambda(G \curvearrowright X) \to C^*_\lambda(H \curvearrowright Y)$ such that $\Phi(C(X)) = C(Y)$.

One can also look at the L^p-operator algebra setting. We properly introduce the *reduced L^p-operator algebra crossed product* associated to the action later in this chapter, but, in the meantime, let us state a rigidity result similar to the ones above. Given an action $G \curvearrowright X$ by homeomorphisms, for $p \in [1, \infty)$ we denote by $F^p_\lambda(G \curvearrowright X)$ the reduced L^p-operator algebra crossed product associated to the action. For $p \neq 2$, there is the following rigidity result.

Theorem ([**CGT24**, Theorem 6.7]). Let $p \in [1, \infty) \setminus \{2\}$. Let G and H be discrete groups, X and Y compact Hausdorff spaces, and $G \curvearrowright X$ and $H \curvearrowright Y$ topological free actions. The following are equivalent:

(1) The actions $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent.
(2) There is an isometric isomorphism between $F^p_\lambda(G \curvearrowright X)$ and $F^p_\lambda(H \curvearrowright Y)$.

Surprisingly, point (2) of the above result does not require the isometric isomorphism to preserve the canonical commutative algebra associated to the action. This shows that L^p-operator algebras crossed product exhibit a more rigid behavior than its C^*-algebraic counterpart. This is because, for $p \neq 2$, one can recover $C(X)$ from $F^p_\lambda(G \curvearrowright X)$ as the *core* of the L^p-operator algebra (see [**CGT24**, Corollary 2.20]), and the core of the algebra must be preserved by isometric isomorphisms.

Given these precedents, we would like to understand this problem for $\ell^1(G \curvearrowright X)$. It turns out that $\ell^1(G \curvearrowright X)$ is even more rigid than the L^p-operator algebra case! First of all, as with the L^p-operator algebra crossed product, one can recover $C(X)$ from inside $\ell^1(G \curvearrowright X)$ by an analogous construction to the core of $F^p_\lambda(G \curvearrowright X)$ (see Theorem 3.3.5). It follows that any isometric isomorphism between ℓ^1-crossed product must automatically preserve the canonical commutative algebra, analogous to the what happens in the L^p-operator algebra setting. What is even more surprising is that what is remembered by $\ell^1(G \curvearrowright X)$ is actually the action *up to conjugacy* (in a sense that we make explicit later). Moreover, topological freeness of the action is not required. In order to prove this, we actually make use of the L^p-operator algebra for $p \in (1, \infty) \setminus \{2\}$, as there is an injective map from $\ell^1(G \curvearrowright X)$ to it. That is particularly useful because, for $p \in [1, \infty) \setminus \{2\}$, we can make use of *Lamperti's theorem*, which describes the surjective isometries of an L^p-space.

The aim of this chapter is to prove the rigidity result about $\ell^1(G \curvearrowright X)$ that we have just described. We briefly go over the structure of this chapter. First of all, we do some preliminary work about *Hermitian* and *unitary* elements of general Banach algebras. Hermitian elements are used to define the core of $\ell^1(G \curvearrowright X)$, and unitaries are used to recover the acting group from the algebra. In the following section, we properly introduce $F^p_\lambda(G \curvearrowright X)$, the reduced L^p-operator algebra crossed product associated to the action $G \curvearrowright X$, and see that there is an injective map from $\ell^1(G \curvearrowright X)$ to $F^p_\lambda(G \curvearrowright X)$. Last section is devoted to proving the rigidity of $\ell^1(G \curvearrowright X)$.

All of the original work in this chapter is based on yet unpublished joint work together with Eusebio Gardella. We also use results from [**BD71**], [**CGT24**] and [**BK24**], which are properly stated as we need them.

3.1. Core of a unital Banach algebra

Through the rest of this chapter, groups are assumed only to be discrete. In this section we introduce Hermitian and unitary elements of unital Banach algebras. We also define the *core* of a unital Banach algebra, which, under some assumptions on the algebra, is a C^*-algebra contained in the algebra that can be defined using its Hermitian elements. We collect some facts about these objects from [**BD71**] and [**CGT24**], in order to apply them later to $\ell^1(G \curvearrowright X)$.

We recall that, for a unital Banach algebra A and an element $a \in A$, the exponential of a is given by $e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$.

Definition 3.1.1. Let A be a unital Banach algebra. We say that $a \in A$ is *Hermitian* if $\|e^{iat}\| = 1$ for every $t \in \mathbb{R}$. We denote the set of Hermitian elements of A as A_h.

The above is the usual definition of Hermitian elements of a unital Banach algebra. In any case, we are soon giving two more characterizations of Hermitian elements that are more useful to us. The Hermitian elements of a unital Banach algebra form a real Banach space and, if A is a C^*-algebra, then A_h consists precisely of the self-adjoint elements of A. One can actually give an abstract characterization of C^*-algebras via the Hermitian elements of the algebra (see [**BD71**, Example 5.3]). This is the Vidav-Palmer Theorem (see [**BD71**, Theorem 6.9]), which we state next.

Theorem 3.1.2 (Vidav-Palmer). Let A be a unital Banach algebra. Then, A is a C^*-algebra if and only if $A = A_h + iA_h$.

The Vidav-Palmer theorem suggests a way of finding a C^*-algebra sitting inside a unital Banach algebra. Explicitly, given a unital Banach algebra A, we would like to say that $A_h + iA_h$ is a C^*-subalgebra of A. One can naturally define an involution on $A_h + iA_h$, but the problem lies in the fact that it is not always true that $A_h + iA_h$ is closed under multiplication. But, in case we know that $A_h + iA_h$ is closed under multiplication, it is a unital Banach algebra, and it would follow from the Vidav-Palmer theorem that it is actually a C^*-algebra. This motivates the following definition.

Definition 3.1.3. Let A be a unital Banach algebra such that $A_h + iA_h$ is closed under multiplication. Then, we define $\mathrm{core}(A) = A_h + iA_h$, which is a C^*-subalgebra of A.

In [**CGT24**, Proposition 2.3] it is stated that it is enough to ask for A_h to be closed under multiplication for $A_h + iA_h$ to be an algebra, and that, in such case, we have that $A_h + iA_h$ is a commutative C^*-algebra. We state here this result for later reference.

Proposition 3.1.4 ([**CGT24**, Proposition 2.3]). Let A be a unital Banach algebra. If A_h is closed under multiplication, then $A_h + iA_h$ is a commutative, unital C^*-subalgebra of A.

Let A be a unital Banach algebra. Following [**BD71**, Definition 2.1 and Lemma 2.2], for every $a \in A$ we define its *numerical range* as

$$V(a) = \{f(a) : f \in A', f(1) = 1 = \|f\|\} \subset \mathbb{C},$$

where A' denotes the dual space of A (i.e., the space of continuous linear maps $T : A \to \mathbb{C}$). It satisfies that $V(\lambda a + \mu) = \lambda V(a) + \mu$ for every $a \in A$, every $\lambda \in \mathbb{C}$ and every $\mu \in \mathbb{C}$ (see [**BD71**, Page 15]). With the numerical range at our disposal, we can state another characterization of Hermitian elements, from [**BD71**], that is useful for later.

Lemma 3.1.5 ([**BD71**, Lemma 5.2]). Let A be unital Banach algebra, and let $a \in A$. The following are equivalent:

(1) a is an Hermitian element.
(2) The numerical range $V(a)$ is contained in $\mathbb{R}$.
(3) The limit
$$\lim_{t\to 0} \frac{1}{t} \left(\|1 + ita\| - 1 \right)$$
exists and is equal to 0, where the limit is taken as t approaches 0 through the real numbers.

We now introduce the unitary elements of a unital Banach algebra.

Definition 3.1.6. Let A be a unital Banach algebra. We define its *unitary group* as
$$\mathcal{U}(A) = \{u \in A : u \text{ is invertible and } \|u\| \leq 1 \text{ and } \|u^{-1}\| \leq 1\}.$$
We call any $u \in \mathcal{U}(A)$ a *unitary element of* A.

This definition of unitary element generalizes the notion of a unitary element in a C^*-algebra. It follows from the definition that $\mathcal{U}(A)$ is a group for every unital Banach algebra A. It is also straightforward to check that an invertible $u \in A$ belongs to $\mathcal{U}(A)$ if and only if both $\|u\|$ and $\|u^{-1}\|$ are equal to 1. Also, let us observe that for any contractive homomorphism $\varphi : A \to B$, between two unital Banach algebras A and B, it is clear that $\varphi(\mathcal{U}(A))$ is contained in $\mathcal{U}(B)$.

Lemma 3.1.7. Let A be a unital Banach algebra. If $u \in \mathcal{U}(A)$, then uA_hu^{-1} is contained in A_h and, consequently, $\mathcal{U}(A)$ acts on A_h by conjugation.

Proof. Let $a \in A_h$. For every $t \in \mathbb{R}$. we have that
$$\frac{1}{t}\left(\|1 + it(uau^{-1})\| - 1\right) = \frac{1}{t}\left(\|u(1 + ita)u^{-1}\| - 1\right) \leq \frac{1}{t}\left(\|1 + ita\| - 1\right) \xrightarrow[t\to 0]{} 0,$$
so, using characterization (3) of Lemma 3.1.5, we obtain that $uau^{-1} \in A_h$. □

Remark 3.1.8. As per the lemma above, we have that $\mathcal{U}(A)$ acts on A_h by conjugation, and this implies that it also acts on $A_h + iA_h$. Now, if A_h is closed under multiplication, Proposition 3.1.4 tells us that core(A) is a commutative C^*-algebra, so by Gelfand duality it is isomorphic to $C(X_A)$ for some compact and Hausdorff space X_A, so the action of $\mathcal{U}(A)$ on core(A) is equivalent to an action of $\mathcal{U}(A)$ on X_A via homeomorphisms.

The purpose of the above remark is that, for an action $G \curvearrowright X$, we can apply it to $\ell^1(G \curvearrowright X)$ in order to recover the action from the algebra. To properly do this, there are a few steps that we have to take before actually recovering the action. First of all, we would like to recover the acting group G, and in order to do so we have to compute $\mathcal{U}(\ell^1(G \curvearrowright X))$. It happens that $\mathcal{U}(\ell^1(G \curvearrowright X))$ is actually bigger than G, but G can be recovered as a quotient of it. Then, we can compute core$(\ell^1(G \curvearrowright X))$ in order to recover the space X from the algebra.

We state a lemma from [**CGT24**] and a direct consequence of it, as both are useful for later.

Lemma 3.1.9 ([**CGT24**, Lemma 2.4]). Let A and B be unital Banach algebras, and let $\varphi : A \to B$ be a unital, contractive linear map. Then $\varphi(A_h)$ is contained in B_h.

Remark 3.1.10. The lemma above implies that, given that $\varphi : A \to B$ is a contractive and unital linear map, then $\varphi(A_h + iA_h)$ is included in $B_h + iB_h$. In particular, when both $A_h + iA_h$ and $B_h + iB_h$ are closed under multiplication, it follows that $\varphi(\text{core}(A))$ is included in core(B).

To end this section, we take a little detour to prove Lemma 3.1.14, which is about the value of the limit in (3) of Lemma 3.1.5 for an element $f \in C(X)$, where X is a compact Hausdorff space. It is useful to characterize the Hermitian elements of $\ell^1(G \curvearrowright X)$ later in this chapter (see Theorem 3.3.5). To proceed, we need a few results from [**BD71**].

Lemma 3.1.11 ([**BD71**, Theorem 5.2]). Let A be a unital Banach algebra. For every $a \in A$, we have that

$$\lim_{t\to 0^+} \frac{1}{t}\left(\|1+ta\|-1\right) = \max\{\mathrm{Re}(\lambda) : \lambda \in V(a)\}.$$

In what follows, for every $E \subset \mathbb{C}$ we denote its convex hull by $co(E)$, that is, the smallest convex subset of $\mathbb{C}$ containing E.

Let A be a unital Banach algebra. We say that $a \in A$ is *normal* if it can be written as $a = h + ik$, for some h and k in A_h satisfying $hk = kh$ ([**BD71**, Definition 5.13]). When $a \in A$ is normal, we have the following nice description of the numerical range $V(a)$. We recall that, given an element $a \in A$, we denote its spectrum by $\rho(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_A \text{ is not invertible}\}$, where 1_A denotes the unit of A.

Lemma 3.1.12 ([**BD71**, Corollary 5.11]). Let A be a unital Banach algebra. For every element $a \in A$, let $\rho(a)$ be its spectrum. If $a \in A$ is a normal element of A, then $V(a) = co(\rho(a))$.

The above lemma states that, for normal elements of the algebra, the numerical range is the convex hull of its spectrum. We now specialize the results above in commutative unital C^*-algebras.

Lemma 3.1.13. Let X be a compact Hausdorff space. Then, the numerical range of every element of $C(X)$ is the convex hull of its image. Explicitly, for every f in $C(X)$ we have that $V(f) = co(f(X))$.

Proof. As $C(X)$ is a C^*-algebra, the Vidav-Palmer theorem tells us that every element of $C(X)$ can be written as $h + ik$, where h and k are Hermitian elements of $C(X)$. As $C(X)$ is commutative, it follows that every element of $C(X)$ is normal.

Take $f \in C(X)$, which is a normal element per the paragraph above. As the spectrum of a continuous function is its image, by Lemma 3.1.12 we obtain that the numerical range $V(f)$ is equal to the convex hull $co(f(X))$. □

Lemma 3.1.14. Let X be a compact Hausdorff space, and let f be in $C(X)$. Then:

$$\lim_{t\to 0^+} \frac{1}{t}\left(\|1+itf\|-1\right) = -\min \mathrm{Im}(co(f(X))),$$
$$\lim_{t\to 0^-} \frac{1}{t}\left(\|1+itf\|-1\right) = -\max \mathrm{Im}(co(f(X))).$$

Proof. We first point out that, as $C(X)$ is commutative, every element of the algebra is normal.

Take $f \in C(X)$. By Lemma 3.1.11 we have that

$$\begin{aligned}\lim_{t\to 0^+} \frac{1}{t}\left(\|1+itf\|-1\right) &= \max\{\mathrm{Re}(\lambda) : \lambda \in V(if)\} = \max\{\mathrm{Re}(\lambda) : \lambda \in iV(f)\} \\ &= \max\{-\mathrm{Im}(\lambda) : \lambda \in V(f)\}.\end{aligned}$$

Using Lemma 3.1.13 we have that $V(f) = co(f(X))$, so we get that

$$\begin{aligned}\lim_{t\to 0^+} \frac{1}{t}\left(\|1+itf\|-1\right) &= \max\{-\mathrm{Im}(\lambda) : \lambda \in co(f(X))\}\\ &= -\min\{\mathrm{Im}(\lambda) : \lambda \in co(f(X))\}\\ &= -\min \mathrm{Im}(co(f(X))).\end{aligned}$$

Similarly, we get that

$$\begin{aligned}\lim_{t\to 0^-} \frac{1}{t}\left(\|1+itf\|-1\right) &= -\lim_{t\to 0^-} \frac{1}{-t}\left(\|1+(-t)(-if)\|-1\right) = -\lim_{t\to 0^+} \frac{1}{t}\left(\|1+t(-if)\|-1\right)\\ &= -\max\{\mathrm{Re}(\lambda) : \lambda \in V(-if)\} = -\max\{\mathrm{Re}(\lambda) : \lambda \in -iV(f)\}\\ &= -\max\{\mathrm{Im}(\lambda) : \lambda \in V(f)\} = -\max\{\mathrm{Im}(\lambda) : \lambda \in co(f(X))\}\\ &= -\max \mathrm{Im}(co(f(X))).\end{aligned}$$

□

3.2. The L^p-operator algebra crossed product

Now, we return to the setting of the last chapter. We fix an action $G \curvearrowright X$ of a discrete group G on a compact and Hausdorff space X. We now describe the construction of the *reduced L^p-operator algebra crossed product* arising from $G \curvearrowright X$. The reduced L^p-operator algebra crossed product is a useful tool to prove the main theorem of this chapter, as there is an injective map from $\ell^1(G \curvearrowright X)$ to the reduced L^p-operator algebra. This allows us to take advantage of the theory known about the L^p-operator algebras, such as Lamperti's theorem, in order to obtain information about $\ell^1(G \curvearrowright X)$.

We define the reduced L^p-operator algebra crossed product following [**BK24**, Section 2]. We remember that an *L^p-operator algebra* is a Banach algebra A such that there exists an L^p-space E and an isometric homomorphism $\varphi : A \to B(E)$.

Given the action $G \curvearrowright X$, we can obtain an L^p-space by taking the counting measure on $X \times G$, which is the space in which we represent the L^p-operator algebra crossed product. We can define a map $\Lambda_p : \ell^1(G \curvearrowright X) \to B(\ell^p(X \times G))$ as follows: for every $f = \sum_{s\in G} f_s \delta_s$ in $\ell^1(G \curvearrowright X)$, we set

$$\Lambda_p(f)\xi(x,t) = \sum_{s\in G} f_s(x)\xi(\sigma_s^{-1}(x), s^{-1}t) \tag{2}$$

for every ξ in $\ell^p(X \times G)$ and for every (x,t) in $X \times G$. It is straightforward to check that Λ_p is a contractive map and, according to [**BK24**, Lemma 2.11], it is an injective representation of $\ell^1(G \curvearrowright X)$, for every $p \in [1,\infty)$. We state this lemma for later reference.

Lemma 3.2.1 ([**BK24**, Lemma 2.11]). The map $\Lambda_p : \ell^1(G \curvearrowright X) \to B(\ell^p(X \times G))$ given by Equation (2) is an injective representation of $\ell^1(G \curvearrowright X)$.

By a representation we mean that the map is an algebra homomorphism. For $p \in [1,\infty)$, this representation Λ_p allows us to define the reduced L^p-operator algebra crossed product of $G \curvearrowright X$, which we denote by $F^p_\lambda(G \curvearrowright X)$, as the completion of the image of Λ_p sitting inside $B(\ell^p(X \times G))$. Explicitly, that is

$$F^p_\lambda(G \curvearrowright X) = \overline{\Lambda_p(\ell^1(G \curvearrowright X))}^{\|\cdot\|_{B(\ell^p(X\times G))}}.$$

This is not the usual way to construct the reduced L^p-operator algebra crossed product. Usually, the norm in $F^p_\lambda(G \curvearrowright X)$ is defined by means of taking the supremum over the norm of the image of all regular representations of $\ell^1(G \curvearrowright X)$ (or of $C_c(G \curvearrowright X)$, which is the algebra of functions $f : G \to C(X)$ such that $f(s) = 0$ except for a finite number of $s \in G$). See [**Gar21**, Section 7] or [**CGT24**, Paragraph 2.17] for details. Proposition 3.1 of [**BK24**] states that both constructions yield the same algebra.

Given the action $G \curvearrowright X$, we can naturally obtain an action of G on $X \times G$ as follows: given s in G, we consider the bijection $\widehat{\sigma}_s : X \times G \to X \times G$ given by

$$\widehat{\sigma}_s(x, t) = (\sigma_s(x), st) \text{ for every } (x, t) \text{ in } X \times G.$$

Similarly, this action induces an action of G on $\ell^p(X \times G)$: every s in G acts on $\ell^p(X \times G)$ via the map $\widehat{\alpha}_s : \ell^p(X \times G) \to \ell^p(X \times G)$ given by

$$\widehat{\alpha}_s\xi(x, t) = \xi(\widehat{\sigma}_s^{-1}(x, t)) = \xi(\sigma_s^{-1}(x), s^{-1}t),$$

for every ξ in $\ell^p(X \times G)$ and for every (x, t) in $X \times G$.

Every function in $\ell^\infty(X \times G)$ acts by multiplication on $\ell^p(X \times G)$. We recall that $\ell^\infty(X \times G)$ is defined as

$$\ell^\infty(X \times G) = \left\{ f : X \times G \to \mathbb{C} \mid \sup_{(x,t) \in X \times G} |f(x, t)| < \infty \right\}.$$

Explicitly, for every f in $\ell^\infty(X \times G)$, we have an operator m_f in $B(\ell^p(X \times G))$ given by

$$m_f(\xi) = f\xi \text{ for every } \xi \text{ in } \ell^p(X \times G).$$

In particular, we have that every f in $C(X)$ gives rise to an operator m_f in $B(\ell^p(X \times G))$ satisfying $m_f(\xi)(x, t) = f(x)\xi(x, t)$ for every ξ in $\ell^p(X \times G)$ and for every (x, t) in $X \times G$. Here, we are implicitly identifying f with the function $\widehat{f} : X \times G \to \mathbb{C}$ given by $\widehat{f}(x, t) = f(x)$ for every (x, t) in $X \times G$. We use this identification throughout this chapter without explicit mention of it. Furthermore, we write f in place of m_f, following the usual convention for L^p-operator algebra crossed products. Now, if for every s in G we denote $\Lambda_p(\delta_s)$ by λ_s, then we can, for every $f = \sum_{s \in G} f_s\delta_s$ in $\ell^1(G \curvearrowright X)$, write

$$\Lambda_p\left(\sum_{s \in G} f_s\delta_s\right) = \sum_{s \in G} f_s\lambda_s \in F^p_\lambda(G \curvearrowright X).$$

With this notation, we have that $\lambda_s\xi(t, s) = \xi(\widehat{\alpha}_s(x, t)) = \xi(\sigma_s^{-1}(x), t^{-1}s)$ for every s in G, every ξ in $\ell^p(X \times G)$ and every (x, t) in $X \times G$.

We use the description of the surjective linear isometries of $\ell^p(X \times G)$, for p in $(1, \infty)$ and different than 2, that follows from the general Lamperti's theorem, which appears in [**GT22**, Theorem 3.7]. We must introduce some notation in order to state this theorem. Suppose that $\varphi : X \times G \to X \times G$ is a bijection, then it induces a surjective linear isometry of $\ell^p(X \times G)$, that we denote by $u_\varphi : \ell^p(X \times G) \to \ell^p(X \times G)$, that is given by

$$u_\varphi\xi = \xi \circ \varphi \text{ for every } \xi \text{ in } \ell^p(X \times G).$$

Another way of obtaining linear surjective isometries is via multiplication by unimodular functions. Concretely, if f in $\ell^\infty(X \times G)$ is such that $|f| \equiv 1$, then $m_f \in B(\ell^p(X \times G))$ is a surjective linear isometry. Lamperti's theorem says that, for $p \in (1, \infty) \setminus \{2\}$, all surjetive isometries are obtained from the isometries that we have just described. We state the theorem in the generality needed for our context.

Theorem 3.2.2 ([**GT22**, Theorem 3.7]). Let $p \in (1, \infty)$ with $p \neq 2$, and let Y be a set endowed with the counting measure. Consider the space $\ell^p(Y)$ and let $T : \ell^p(Y) \to \ell^p(Y)$ be a surjective linear isometry. Then, there exists a bijection $\varphi : Y \to Y$ and a function $f : Y \to \mathbb{T}$ such that $T = m_f u_\varphi$.

Remark 3.2.3. To make the theorem above a little more explicit in our context, it tells us that for every surjective linear isometry T of $\ell^p(X \times G)$ there exists a bijection φ of $X \times G$ and a function $f : X \times G \to \mathbb{T}$ such that

$$T\xi(x,t) = m_f(u_\varphi \xi)(x,t) = f(x,t)\xi(\varphi(x,t))$$

for every ξ in $\ell^p(X \times G)$ and every (x,t) in $X \times G$.

Remark 3.2.4. Every u in $\mathcal{U}(F^p_\lambda(G \curvearrowright X))$ is a surjective linear isometry of $\ell^p(X \times G)$. That u is surjective follows from the fact that elements of $\mathcal{U}(F^p_\lambda(G \curvearrowright X))$ are invertible. To see that u is an isometry, for every ξ in $\ell^p(X \times G)$ we compute

$$\|u(\xi)\| \leq \|\xi\| = \|u^{-1}u(\xi)\| \leq \|u^{-1}\|\|u(\xi)\| = \|u(\xi)\|,$$

so we obtain that $\|u(\xi)\| = \|\xi\|$ and, as such, u is an isometry. This means that we can apply Lamperti's theorem to elements of $\mathcal{U}(F^p_\lambda(G \curvearrowright X))$.

3.3. Rigidity of the ℓ^1-crossed product

The objective of this section is to prove the main result of this chapter, namely, Theorem 3.3.7. Let us briefly motivate this result. Suppose that we have two actions $G \curvearrowright^\sigma X$ and $H \curvearrowright^\tau Y$ of discrete groups on compact and Hausdorff spaces, and consider the induced actions $G \curvearrowright^\alpha C(X)$ and $H \curvearrowright^\beta C(Y)$ on their algebras of continuous functions. A notion of equivalence between the two actions that we can naturally consider is that of the existence of a group isomorphism $\rho : G \to H$ and a homeomorphism $\theta : X \to Y$ such that $\theta(\sigma_s(x)) = \tau_{\rho(s)}(\theta(x))$ for every s in G and every x in X. This condition is equivalent to $\alpha_s(f) \circ \theta^{-1}$ being equal to $\beta_{\rho(s)}(f \circ \theta^{-1})$ for every s in G and every f in $C(X)$. In these conditions, one can easily verify that the map $\Phi : \ell^1(G \curvearrowright X) \to \ell^1(H \curvearrowright Y)$ given by

$$\Phi\left(\sum_{s \in G} f_s \delta_s\right) = \sum_{s \in G} f_s \circ \theta^{-1} \delta_{\rho(s)}, \tag{3}$$

for every $\sum_{s \in G} f_s \delta_s$ in $\ell^1(G \curvearrowright X)$, is an isometric isomorphism between the two ℓ^1-crossed products. On the other hand, if we have an isometric isomorphism Φ satisfying Equation (3) for some homeomorphism θ and some isomorphism ρ, it is clear that Φ restricts to an (isometric) isomorphism between $C(X)$ and $C(Y)$ given by precomposition with the inverse of θ and, for every s in G and every f in $C(X)$, we have that

$$\alpha_s(f) \circ \theta^{-1} = \Phi(\alpha_s(f)) = \Phi(\delta_s f \delta_s^{-1}) = \delta_{\rho(s)}(f \circ \theta^{-1})\delta_{\rho(s)}^{-1} = \beta_{\rho(s)}(f \circ \theta^{-1}),$$

which implies that the actions are equivalent. This means that equivelence between the actions can be stated as the existence of a homomorphism Φ between the crossed products $\ell^1(G \curvearrowright X)$ and $\ell^1(H \curvearrowright Y)$ that satisfies Equation (3), which is, evidently, an isometric isomorphism.

We will prove that any isometric isomorphism $\Phi : \ell^1(G \curvearrowright X) \to \ell^1(H \curvearrowright Y)$ must be of the form given in Equation (3). This means that the equivalence class of the action is completely captured by the ℓ^1-crossed product, as any isometric isomorphism between the ℓ^1-crossed products implies that the actions are equivalent.

Let us briefly describe how we proceed. First of all, given an action $G \curvearrowright X$, we describe the unitary elements of $\ell^1(G \curvearrowright X)$. In order to do so, we make use of the injective map $\Lambda_p : \ell^1(G \curvearrowright X) \to F^p_\lambda(G \curvearrowright X)$ described in the previous section, which allows us to use Lamperti's theorem to describe the unitary elements of the algebra (see Lemma 3.3.1 and Proposition 3.3.3). Once we know the unitary elements in $\ell^1(G \curvearrowright X)$, we are able to recover the group from the algebra (see Corollary 3.3.4). Then, we prove that the core of $\ell^1(G \curvearrowright X)$ is exactly $C(X)$ (see Theorem 3.3.5). It follows that the action described in Remark 3.1.8 allows us to recover the action $G \curvearrowright X$ from the algebra $\ell^1(G \curvearrowright X)$ (see Proposition 3.3.6). All of this together implies the main theorem of the chapter.

In what follows, we fix an action $G \curvearrowright X$. When we write $\mathbb{1}_{(x,t)}$, for (x,t) in $X \times G$, we mean the characteristic function of $\{(x,t)\}$, which is an element of $\ell^p(X \times G)$. Similarly, if Y is contained in $X \times G$, we denote the characteristic function of Y by $\mathbb{1}_Y$.

Lemma 3.3.1. Let $p \in (1,\infty)$ with $p \neq 2$, consider both crossed products $\ell^1(G \curvearrowright X)$ and $F^p_\lambda(G \curvearrowright X)$, and let $\Lambda_p : \ell^1(G \curvearrowright X) \to F^p_\lambda(G \curvearrowright X)$ be the map from Lemma 3.2.1. Then, $\mathcal{U}(\Lambda_p(\ell^1(G \curvearrowright X)))$ is contained in the following set:

$$\left\{ \sum_{s\in G} f \mathbb{1}_{X_s} \lambda_s \in B(\ell^p(X \times G)) : f \text{ is a function from } X \text{ to } \mathbb{T} \text{ and } X = \bigsqcup_{s\in G} X_s \right\}.$$

Proof. For every $s \in G$, let ρ_s denote the element of $B(\ell^p(X \times G))$ given by

$$\rho_s \xi(x,t) = \xi(x,ts) \text{ for every } \xi \in \ell^p(X \times G) \text{ and } (x,t) \in X \times G.$$

Claim: For every $g \in G$ and for every $a \in F^p_\lambda(G \curvearrowright X)$, we have that $\rho_g a = a \rho_g$.

To prove the claim, as $F^p_\lambda(G \curvearrowright X)$ is the closure of $\Lambda_p(\ell^1(G \curvearrowright X))$ inside $B(\ell^p(X \times G))$, it is enough to check that it holds for every element of $\Lambda_p(\ell^1(G \curvearrowright X))$. Let $\sum_{s\in G} f_s \delta_s$ be in $\ell^1(G \curvearrowright X)$ and fix g in G. We want to prove that $\Lambda_p(f)$ and ρ_g commute. Take $\xi \in \ell^p(X \times G)$ and $(x,t) \in X \times G$. We compute:

$$\begin{aligned} \rho_g\left(\Lambda_p\left(\sum_{s\in G} f_s\delta_s\right)\right)\xi(x,t) &= \rho_g\left(\sum_{s\in G} f_s\lambda_s\right)\xi(x,t) = \rho_g\left(\sum_{s\in G} f_s(\lambda_s(\xi))\right)(x,t) \\ &= \sum_{s\in G} f_s(\lambda_s(\xi))(x,tg) = \sum_{s\in G} f_s(x)\lambda_s(\xi)(x,tg) \\ &= \sum_{s\in G} f_s(x)\xi(\sigma_s^{-1}(x), s^{-1}tg), \end{aligned}$$

$$\Lambda_p\left(\sum_{s\in G} f_s\delta_s\right)\rho_g(\xi)(x,t) = \left(\sum_{s\in G} f_s\lambda_s\right)\rho_g(\xi)(x,t) = \sum_{s\in G} f_s(\lambda_s(\rho_g(\xi)))(x,t)$$
$$= \sum_{s\in G} f_s(x)\lambda_s(\rho_g(\xi))(x,t) = \sum_{s\in G} f_s(x)\rho_g(\xi)(\sigma_s^{-1}(x), s^{-1}t)$$
$$= \sum_{s\in G} f_s(x)\xi(\sigma_s^{-1}(x), s^{-1}tg).$$

As both equations yield the same value, we conclude that the operators commute. This proves the claim.

Now, take $u \in \mathcal{U}(\Lambda_p(\ell^1(G \curvearrowright X)))$. According to Remark 3.2.4, u is a surjective linear isometry of $\ell^p(X\times G)$, so it follows from Remark 3.2.3 that there is a bijection $\varphi : X\times G \to X\times G$ and a function $f : X \times G \to \mathbb{T}$ such that $u = m_f u_\varphi$. We can write $\varphi = (\varphi_1, \varphi_2)$, where φ_1 and φ_2 are functions from $X \times G$ to X and to G, respectively. Given the claim above, we have, for every $s \in G$, that $\rho_s u = u\rho_s$. If we take $\xi \in \ell^p(X \times G)$ and $(x, t) \in X \times G$, we can compute:

$$u\rho_s(\xi)(x,t) = u(\rho_s(\xi))(x,t) = m_f u_\varphi(\rho_s(\xi))(x,t) = f(x,t)\rho_s(\xi)(\varphi(x,t))$$
$$= f(x,t)\xi(\varphi_1(x,t), \varphi_2(x,t)s),$$
$$\rho_s u(\xi)(x,t) = \rho_s(u(\xi))(x,t) = u(\xi)(x,ts) = m_f u_\varphi(\xi)(x,ts) = f(x,ts)\xi(\varphi(x,ts))$$
$$= f(x,ts)\xi(\varphi_1(x,ts), \varphi_2(x,ts)).$$

Then, for every ξ in $\ell^p(X \times G)$, every s in G and every (x, t) in $X \times G$, it holds that

$$f(x,t)\xi(\varphi_1(x,t), \varphi_2(x,t)s) = f(x,ts)\xi(\varphi_1(x,ts), \varphi_2(x,ts)). \tag{4}$$

Using that f never vanishes, by taking $\xi = \mathbb{1}_{\varphi(x,ts)}$ in Equation (4) we get that

$$(\varphi_1(x,t), \varphi_2(x,t)s) = (\varphi_1(x,ts), \varphi_2(x,ts)),$$

and it follows that $f(x, t) = f(x, ts)$. As this works for every x in X and for every t and s in G, we get that f is actually a function on X (i.e. $f(x, t) = f(x, e)$ for every $t \in G$). Similarly, φ_1 is actually a function on X, and we also deduce that

$$\varphi_2(x,t) = \varphi_2(x,e)t \text{ for every } t \in G.$$

This means that, if we define s_x as the inverse of $\varphi_2(x, e)$, then for every $x \in X$ and every $t \in G$ we get that $\varphi_2(x, t) = s_x^{-1}t$. As the map $(x, t) \mapsto (\varphi_1(x), s_x^{-1}t)$, from $\{x\} \times G$ to $\{\varphi_1(x)\} \times G$, is a bijection and coincides with the restriction of φ to $\{x\} \times G$, it follows that φ_1 must be a bijection when thinking of it as a function from X to itself.

As u belongs to $\Lambda_p(\ell^1(G \curvearrowright X))$, we can write $u = \sum_{s\in G} f_s\lambda_s$. Then, for every $\xi \in \ell^p(X\times G)$ and for every $(x, t) \in X \times G$, we have that

$$f(x)\xi(\varphi_1(x), s_x^{-1}t) = \sum_{s\in G} f_s\lambda_s(\xi)(x,t) = \sum_{s\in G} f_s(x)\xi(\sigma_s^{-1}(x), s^{-1}t).$$

Taking in the above equation $\xi = \mathbb{1}_{(\varphi_1(x), s_x^{-1}t)}$, we obtain that

$$f(x) = \sum_{s\in G} f_s(x)\mathbb{1}_{(\varphi_1(x), s_x^{-1}t)}(\sigma_s^{-1}(x), s^{-1}t) = f_{s_x}(x)\mathbb{1}_{(\varphi_1(x), s_x^{-1}t)}(\sigma_{s_x}^{-1}(x), s_x^{-1}t).$$

We have that f takes values in $\mathbb{T}$, which means that $f(x)$ must be nonzero, so it follows that $\varphi_1(x) = \sigma_{s_x}^{-1}(x)$ and $f_{s_x}(x) = f(x)$. For every s in G we define $X_s = \{x \in X : s_x = s\}$, so we get a partition

$$X = \bigsqcup_{s \in G} X_s.$$

It follows that, if x is in X_s, then for every t in G we have that $\varphi(x,t) = (\sigma_s^{-1}(x), s^{-1}t)$, so it coincides with the action of s on (x,t). Now, we can directly compute that

$$u = \sum_{s \in G} f_s \lambda_s = \sum_{s \in G} f \mathbb{1}_{X_s} \lambda_s.$$

□

REMARK 3.3.2. If we have that $\sum_{s \in G} m_{f_s} \lambda_s = 0$, with f_s in $\ell^\infty(X \times G)$ for every $s \in G$, then it must be that $f_s = 0$ for every s. In order to see this, note that for every ξ in $\ell^p(X \times G)$ and for every (x,t) in $X \times G$, we have that

$$\sum_{s \in G} f_s(x,t) \xi(\sigma_s^{-1}(x), s^{-1}t) = 0,$$

so, if for every $s \in G$ we take $\xi = \mathbb{1}_{(\sigma_s^{-1}(x), s^{-1}t)}$, we get that $f_s(x,t) = 0$. As this works for every (x,t) in $X \times G$, it follows that $f_s = 0$ for every s in G.

The following proposition is a description of the unitary elements of $\ell^1(G \curvearrowright X)$, which we can obtain rather easily thanks to the last lemma and the fact that the map Λ_p is injective.

PROPOSITION 3.3.3. The group of unitary elements of $\ell^1(G \curvearrowright X)$ has the following description:

$$\mathcal{U}(\ell^1(G \curvearrowright X)) = \{ f\delta_s : s \in G \text{ and } |f(x)| = 1 \text{ for every } x \in X \}.$$

PROOF. Let $u = \sum_{s \in G} f_s \delta_s$ be in $\mathcal{U}(\ell^1(G \curvearrowright X))$. Take $p \in (1, \infty)$, different from 2, and consider the map $\Lambda_p : \ell^1(G \curvearrowright X) \to F^p_\lambda(G \curvearrowright X)$ as in Lemma 3.2.1, which is an injective and contractive representation. As a consequence, it is clear that $\Lambda_p(\mathcal{U}(\ell^1(G \curvearrowright X)))$ is included in $\mathcal{U}(\Lambda_p(\ell^1(G \curvearrowright X)))$, so it follows from Lemma 3.3.1 that there exists a function $f : X \to \mathbb{T}$ and some partition $\bigsqcup_{s \in G} X_s$ of X such that

$$\sum_{s \in G} f \mathbb{1}_{X_s} \lambda_s = \Lambda_p \left(\sum_{s \in G} f_s \delta^s \right) = \sum_{s \in G} f_s \lambda_s.$$

Then, we have that

$$\sum_{s \in G} (f \mathbb{1}_{X_s} - f_s) \lambda_s = 0,$$

so it follows from Remark 3.3.2 that f_s equals $f \mathbb{1}_{X_s}$ for every s in G and, as f_s is a continuous function, X_s must be a closed and open subset of X and f is also continuous. We have then that $u = \sum_{s \in G} f \mathbb{1}_{X_s} \delta_s$, so we get that

$$\|u\| = \sum_{s \in G} \|f \mathbb{1}_{X_s}\|_\infty = \#\{ s \in G : X_s \neq \emptyset \}.$$

As u belongs to $\mathcal{U}(\ell^1(G \curvearrowright X))$, we have that $\|u\| = 1$. It follows that there is a unique s in G such that $X_s = X$, and X_t is empty for every other t in G. In other words, we have that $u = f\delta_s$. □

With the work we have done until now we are able to recover the group from the algebra $\ell^1(G \curvearrowright X)$. It is straightforward to check that:

$$\mathcal{U}(C(X)) = \{f \in C(X) : |f(x)| = 1 \text{ for every } x \in X\}.$$

We recall that we denote by α the action of G on $C(X)$, which restricts to an action on $\mathcal{U}(C(X)$ that we also denote by α.

COROLLARY 3.3.4. The unitary group of $\ell^1(G \curvearrowright X)$ is isomorphic to a semidirect product. Explicitly, we have that

$$\mathcal{U}(\ell^1(G \curvearrowright X)) \cong \mathcal{U}(C(X)) \rtimes_\alpha G.$$

Consequently, G is isomorphic to the quotient of $\mathcal{U}(\ell^1(G \curvearrowright X))$ with $\mathcal{U}(C(X))$.

PROOF. Given the description of $\mathcal{U}(\ell^1(G \curvearrowright X))$ we obtained in Proposition 3.3.1, it is directly verified that the map from $\mathcal{U}(\ell^1(G \curvearrowright X))$ to $\mathcal{U}(C(X)) \rtimes_\alpha G$ given by $f\delta_s \mapsto (f, s)$, for every $f\delta_s \in \mathcal{U}(\ell^1(G \curvearrowright X))$, is an isomorphism of groups. It follows that G is isomorphic to the quotient of $\mathcal{U}(\ell^1(G \curvearrowright X))$ with $\mathcal{U}(C(X))$. □

We now want to recover the topological space X from $\ell^1(G \curvearrowright X)$, which is equivalent to recovering the algebra $C(X)$. We tackle this in the following theorem, in which we prove that the core of $\ell^1(G \curvearrowright X)$ (as in Definition 3.1.3) is precisely $C(X)$.

THEOREM 3.3.5. The space of Hermitian elements of $\ell^1(G \curvearrowright X)$ is $C(X, \mathbb{R})$, that is, the continuous functions from X to $\mathbb{R}$. Consequently, we have that the core of $\ell^1(G \curvearrowright X)$ is $C(X)$.

PROOF. Let $f = \sum_{s\in G} f_s\delta_s$ be an Hermitian element of $\ell^1(G \curvearrowright X)$. By Lemma 3.1.5, we have that $\lim_{t\to 0} \frac{1}{t}(\|1 + itf\| - 1) = 0$. We compute

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t}(\|1 + itf\| - 1) &= \lim_{t\to 0} \frac{1}{t}\left(\|(1 + itf_e)\delta_e + \sum_{s\in G\setminus\{e\}} itf_s\delta_s\| - 1\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left(\|1 + itf_e\|_\infty + \sum_{s\in G\setminus\{e\}} \|itf_s\|_\infty - 1\right) \\
&= \lim_{t\to 0} \frac{1}{t}(\|1 + itf_e\|_\infty - 1) + \frac{|t|}{t}\left(\sum_{s\in G\setminus\{e\}} \|f_s\|_\infty\right).
\end{aligned}$$

As f_e belongs to $C(X)$, by Lemma 3.1.14 we get that

$$\lim_{t\to 0^+} \frac{1}{t}(\|1 + itf\| - 1) = -\min \operatorname{Im}(co(f_e(X))) + \sum_{s\in G\setminus\{e\}} \|f_s\|_\infty, \tag{5}$$

$$\lim_{t\to 0^-} \frac{1}{t}(\|1 + itf\| - 1) = -\max \operatorname{Im}(co(f_e(X))) - \sum_{s\in G\setminus\{e\}} \|f_s\|_\infty. \tag{6}$$

As f is Hermitian, we have that both Equation (5) and Equation (6) are equal to zero. Suppose that $\sum_{t\in G\setminus\{e\}} \|f_s\|_\infty > 0$, from Equation (5) we obtain that

$$\min \operatorname{Im}(co(f_e(X))) = \sum_{s\in G\setminus\{e\}} \|f_s\|_\infty.$$

But this implies that Equation (6) is strictly negative, contradicting that $\lim_{t\to 0} \frac{1}{t}\left(\|1+itf\|-1\right)$ is zero. Then, it must be that $\sum_{s\in G\setminus\{e\}}\|f_s\|_\infty = 0$. It follows that $f = f_e\delta_e$, which belongs to $C(X)$, and as the norm in $C(X)$ coincides with the restriction of the norm in $\ell^1(G \curvearrowright X)$, we have that f_e is an Hermitian element of $C(X)$. As Hermitian elements of $C(X)$ are the continuous functions from X to $\mathbb{R}$, we obtain that f must be one of those.

As $C(X,\mathbb{R})$ is closed under multiplication, it follows from Proposition 3.1.4 that the core of $\ell^1(G \curvearrowright X)$ is well defined and is equal to $C(X,\mathbb{R}) + iC(X,\mathbb{R})$. That is, $\operatorname{core}(\ell^1(G \curvearrowright X))$ is equal to $C(X)$. □

We mentioned in Remark 3.1.8 that the unitaries of a unital Banach algebra act on its core by conjugation. We already know the unitary group of $\ell^1(G \curvearrowright X)$, and we also know the core of $\ell^1(G \curvearrowright X)$. It is clear that $\mathcal{U}(C(X))$, which is contained in $\mathcal{U}(\ell^1(G \curvearrowright X))$, acts trivially on $\operatorname{core}(\ell^1(G \curvearrowright X))$, as the core is equal to $C(X)$, which is commutative, and the action is by conjugation. As such, we actually have an action of the quotient $\mathcal{U}(\ell^1(G \curvearrowright X))/\mathcal{U}(C(X))$ on X. This action of $\mathcal{U}(\ell^1(G \curvearrowright X))/\mathcal{U}(C(X))$ on X is equivalent to the action of G on X in the sense we described at the start of this section, that is, there exists a group isomorphism ρ from G to the quotient $\mathcal{U}(\ell^1(G \curvearrowright X))/\mathcal{U}(C(X))$ such that, for every $s \in G$ and for every f in $C(X)$, it holds that

$$\alpha_s(f) = \rho(s)\cdot f = \rho(s) f \rho(s)^{-1}.$$

Proposition 3.3.6. The action of G on $C(X)$ is conjugated to the action of the quotient $\mathcal{U}(\ell^1(G \curvearrowright X))/\mathcal{U}(C(X))$ on $\operatorname{core}(\ell^1(G \curvearrowright X))$, in the sense described in the paragraph above.

Proof. For every s in G we have that δ_s belongs to $\mathcal{U}(\ell^1(G \curvearrowright X))$, and we also denote its projection to the quotient group $\mathcal{U}(\ell^1(G \curvearrowright X))/\mathcal{U}(C(X))$ by the same symbol δ_s. By Corollary 3.3.4, it is clear that every element of $\mathcal{U}(\ell^1(G \curvearrowright X))/\mathcal{U}(C(X))$ is represented by a unique element of the form δ_s, and the map $s \mapsto \delta_s$ is an isomorphism between G and the quotient group. We call this isomorphism ρ. As the action is by conjugation, for every f in $C(X)$ we have that

$$\rho(s)\cdot f = \rho(s) f \rho(s)^{-1} = \delta_s f \delta_s^{-1} = \alpha_s(f).$$

□

The proposition above tells us that $G \curvearrowright X$ is conjugated to the action of the quotient $\mathcal{U}(\ell^1(G \curvearrowright X))/\mathcal{U}(C(X))$ on X, so now it is clear that both actions yield isomorphic ℓ^1-crossed products. This allows us to finally prove the main theorem of this chapter.

Theorem 3.3.7. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be two actions of discrete groups G and H on compact Hausdorff spaces X and Y, respectively. Consider the ℓ^1-crossed products $\ell^1(G \curvearrowright X)$ and $\ell^1(H \curvearrowright Y)$ associated with each action, and let $\Phi : \ell^1(G \curvearrowright X) \to \ell^1(H \curvearrowright Y)$ be a homomorphism between the two algebras. Then, Φ is an isometric isomorphism if and only if there exists a homeomorphism $\theta : X \to Y$ and a group isomorphism $\rho : G \to H$ such that

$$\Phi(f\delta_s) = f \circ \theta^{-1}\delta_{\rho(s)} \text{ for every } f \text{ in } C(X) \text{ and every } s \text{ in } G.$$

Proof. As we discussed at the start of this section, it is clear that if there is a homeomorphism $\theta : X \to Y$ and a group isomorphism $\rho : G \to H$, then the map Φ, defined as in the statement of this theorem, is an isometric isomorphism between $\ell^1(G \curvearrowright X)$ and $\ell^1(H \curvearrowright Y)$.

Now, suppose that $\Phi : \ell^1(G \curvearrowright X) \to \ell^1(H \curvearrowright Y)$ is an isometric isomorphism between the two algebras. Then, Φ restricts to an isometric isomorphism between $\operatorname{core}(\ell^1(G \curvearrowright X))$ and core($\ell^1(H \curvearrowright Y)$), that is, an isometric isomorphism between $C(X)$ and $C(Y)$. By Remark 1.2.8 in the Preliminaries, this is equivalent to the existence of a homeomorphism $\theta : X \to Y$ such that $\Phi(f) = f \circ \theta^{-1}$ for every f in $C(X)$. Finally, as Φ is isometric, we have that $\mathcal{U}(\ell^1(G \curvearrowright X))$ and $\mathcal{U}(\ell^1(H \curvearrowright Y))$ are isomorphic and, by Corollary 3.3.4, it induces an isomorphism $\rho : G \to H$, which, by Proposition 3.3.6, is such that $\Phi(\delta_s) = \delta_{\rho(s)}$ for every s in G. $\square$

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