Tesis de maestría

Fractional Operators: Mixed Formulations and Applications

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Resumen

El objetivo de esta tesis es estudiar problemas de ecuaciones en derivadas parciales que involucran operadores no locales. En particular, nos concentramos en dos problemas principales.

El primero de ellos consiste en una formulación mixta para el Laplaciano fraccionario. Probamos el buen planteo de esta formulación mediante la verificación de la condición inf-sup. Una discretización directa del problema no parece posible, ya que las construcciones usuales conformes en $H(\mathrm{div})$ (por ejemplo, los elementos de Raviart–Thomas) no parecen tener un análogo natural en el contexto no local. Por este motivo, adaptamos el enfoque utilizado en [45] para obtener un problema equivalente y coercivo. La coercividad implica que cualquier discretización por elementos finitos que sea conforme resultará estable. Además, probamos el buen planteo del problema discreto, obtenemos tasas de convergencia y discutimos aspectos relacionados con la implementación del método. Finalmente, realizamos diversos experimentos numéricos que validan los resultados teóricos obtenidos.

El segundo problema consiste en una relajación del modelo clásico de Oseen–Frank para cristales líquidos. Una característica destacada de estos materiales es la presencia de defectos, que corresponden a singularidades en el campo de orientaciones. El modelo clásico de Oseen–Frank no es capaz de capturar este fenómeno, ya que las singularidades de codimensión menor o igual a dos tienen energía infinita. Con el fin de abordar esta dificultad, proponemos relajar la energía de Oseen–Frank disminuyendo los requerimientos de diferenciabilidad del campo de orientaciones mediante el uso de operadores fraccionarios. Esta idea fue explorada en [4] para la versión más simple del modelo, conocida como la aproximación a una constante. En este trabajo probamos la existencia de minimizantes para el problema y adaptamos el método presentado en [36] para una versión simplificada de la energía. Por último, realizamos varios experimentos numéricos con el objetivo de explorar cualitativamente el comportamiento de los minimizantes.

Abstract

The goal of this thesis is to study partial differential equation problems involving nonlocal operators. In particular, we focus on two main problems.

The first one concerns a mixed formulation for the fractional Laplacian. We prove the well-posedness of this formulation by verifying the inf-sup condition. A direct discretization of the problem does not seem feasible, since the usual constructions that are conforming in $H(\text{div})$ (for instance, Raviart–Thomas elements) do not appear to have a natural counterpart in the nonlocal setting. To overcome this difficulty, we adapt the approach introduced in [45] in order to obtain an equivalent and coercive problem. Coercivity implies that any conforming finite element discretization of the formulation is stable. We further prove the well-posedness of the discrete problem, derive convergence rates, and discuss aspects related to its implementation. Finally, we present several numerical experiments that validate the theoretical results.

The second problem addresses a relaxation of the classical Oseen–Frank model for nematic liquid crystals. A distinctive feature of these materials is the presence of defects, which correspond to singularities in the orientation field. The classical Oseen–Frank model is unable to capture this phenomenon, since singularities of codimension less than or equal to two have infinite energy. To address this limitation, we propose a relaxation of the Oseen–Frank energy by weakening the differentiability requirements on the orientation field through the use of fractional operators. This idea was explored in [4] for the simplest version of the model, known as the one-constant approximation. We prove the existence of minimizers for the problem and adapt the method proposed in [36] to a simplified version of the energy. Finally, we perform several numerical experiments to qualitatively investigate the behavior of the minimizers.

Notation and some definitions

- Given $a, b > 0$, we write $a \lesssim b$ if $a \leq Cb$ for some constant $C > 0$ that does not depend on a, b. We also write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

- We denote the norm of a Banach space X by $\|\cdot\|_X$. Similarly, we denote the inner product of a Hilbert space H by $\langle \cdot, \cdot \rangle_H$.

- Given $d \in \mathbb{N}$, we denote by $\mathbb{R}^d$ the Euclidean space, that is, the set of d-tuples $(x_1, \dots, x_d)$ with $x_1, \dots, x_d \in \mathbb{R}$, endowed with the usual inner product:

$$x \cdot y := x_1 y_1 + \dots + x_d y_d \quad \text{if } x = (x_1, \dots, x_d) \text{ and } y = (y_1, \dots, y_d).$$

  We also denote by

$$|x| := \sqrt{x \cdot x}$$

  the Euclidean norm of $x \in \mathbb{R}^d$.

- Given $n, m \in \mathbb{N}$, we denote by $\mathbb{R}^{n \times m}$ the space of matrices with n rows and m columns with real entries. Given a matrix $A \in \mathbb{R}^{n \times m}$, we will also denote by A the linear transformation $x \mapsto Ax$ (by a slight abuse of notation).

- Let $A \in \mathbb{R}^{n \times m}$. We denote its transpose by A^t and define its Frobenius norm by

$$|A| := \sqrt{\operatorname{tr}(AA^t)}.$$

- Given an open set $\Omega \subset \mathbb{R}^d$ and $k \in \mathbb{N}$, we denote by $C^k(\Omega)$ the set of functions that are k times differentiable with continuous k-th derivative.

- Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be a multi-index. We define

$$|\alpha| := \alpha_1 + \dots + \alpha_d, \quad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}, \quad \alpha! := \alpha_1! \alpha_2! \dots \alpha_d!.$$

- Let $d, m, k \in \mathbb{N}$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ be a multi-index, $\Omega \subset \mathbb{R}^d$ an open set, $f \in C^k(\Omega)$ and $F = (f_1, \dots, f_m) \in [C^k(\Omega)]^m$. We write

$$\frac{\partial_\alpha}{\partial x} = \partial_\alpha f := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots x_d^{\alpha_d}} f$$

  and

$$\partial_\alpha F := (\partial_\alpha f_1, \dots, \partial_\alpha f_m).$$

  Classical and weak derivatives will be denoted in the same way.

- We denote by dx the Lebesgue measure in $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be measurable and $p \in [1, \infty]$. We denote by $L^p(\Omega)$ the Lebesgue spaces.

- Let $\Omega \subset \mathbb{R}^d$ be an open set. Given $p \geq 1$ and $k \geq 1$, we define the Sobolev space

$$W^{k,p}(\Omega) := \{v \in L^p(\Omega) : \partial_\alpha v \in L^p(\Omega) \text{ for every multi-index } |\alpha| \leq k\},$$

  which is a Banach space with the norm

$$\|v\|_{W^{k,p}(\Omega)}^p := \|v\|_{L^p(\Omega)}^p + \sum_{|\alpha| \leq k} \|\partial_\alpha v\|_{L^p(\Omega)}^p.$$

  For $p = 2$, we denote $H^k(\Omega) := W^{k,2}(\Omega)$.

- Given $x \in \mathbb{R}^d$ and $r > 0$, we denote by $B(x, r)$ the ball with center x and radius r. Given a set $\Omega \subset \mathbb{R}^d$, we denote its volume by $|\Omega| := \int_\Omega dx$.
- Let $\Omega \subset \mathbb{R}^d$ be open and let $f : \Omega \setminus \{x\} \to \mathbb{R}$ be locally integrable away from x. Whenever the limit exists, we define the Cauchy principal value of the integral of f at x by

$$\text{p.v.} \int_\Omega f(y)\, dy := \lim_{\varepsilon \to 0} \int_{\Omega \setminus B(x,\varepsilon)} f(y)\, dy.$$

Contents

Introduction

In recent years, the study of nonlocal operators has been an active area of research in different branches of mathematics. Integer-order differential operators (or simply differential operators) are local, that is, the derivative of a function at a given point depends only on the values of the function in a neighborhood of it. On the other hand, fractional order derivatives are nonlocal, integro-differential operators. Nonlocal models have been increasingly used in different areas of science such as machine learning [43, 48], finance [19], image processing [16, 35, 42], magnetohydrodynamics [50], among others. In particular, the fractional Laplacian has been considered in many applications, including, for example, diffusion-reaction problems [57], quasi-geostrophic flows [23], transport in porous media [25], and ultrasound [55].

The operators arising in these applications may vary, and several discrete approximation strategies have been proposed to treat them. Accordingly, offering a unified discretization of all these operators is a too ambitious goal. This thesis mainly focuses on finite element discretizations schemes for PDEs problems involving the fractional Laplacian $(-\Delta)^s$ and the fractional operators introduced by [39] and interpreted as fractional calculus operators in [52]. Namely, non-local counterparts of the differential operators ∇, div and $\mathbf{curl}$, which we will call fractional *gradient*, *divergence* and *curl* respectively, and denote as ∇^s, div^s and $\mathbf{curl}^s$ for an arbitrary $s > 0$.

There are several equivalent definitions of such operators. Perhaps, a first definition to justify why they are called as their local counterparts, is the one given by deeming them as pseudo-differential operators. Namely, for functions in the Schwartz class $\mathcal{S}$, these operators are given by

$$\begin{aligned} \mathscr{F}(-\Delta)^s u(\xi) &= (2\pi)^{2s} \mathscr{F}u(\xi), \\ \mathscr{F}\nabla^s u(\xi) &= (2\pi) i |\xi|^{s-1} \xi \mathscr{F}u(\xi), \\ \mathscr{F}\operatorname{div}^s u(\xi) &= (2\pi) i |\xi|^{s-1} \xi \cdot \mathscr{F}u(\xi), \\ \mathscr{F}\mathbf{curl}^s u(\xi) &= (2\pi) i |\xi|^{s-1} \xi \wedge \mathscr{F}u(\xi), \end{aligned}$$

where $\mathscr{F}$ denotes the Fourier transform. Upon inverting the Fourier transform, these operators can be expressed as integral operators, thereby illustrating their nonlocality. In this work we will mainly focus on two problems regarding these operators.

Note that $(-\Delta)^s = \operatorname{div}^s \circ \nabla^s$. Thus, we propose to study a mixed formulation for the fractional Poisson problem, which we will refer to as the *fractional Darcy problem*. Namely, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, let $f : \Omega \to \mathbb{R}$ be sufficiently smooth, and let $s \in (0, 1)$. We seek $(p, \mathbf{\Phi})$

such that

$$\begin{cases} \boldsymbol{\Phi} + \nabla^s p = 0 & \text{in } \mathbb{R}^d, \\ \operatorname{div}^s \boldsymbol{\Phi} = f & \text{in } \Omega, \\ p = 0 & \text{in } \Omega^c. \end{cases} \tag{D}$$

We study this problem in Chapter 2 and develop a finite element scheme that allows the simultaneous approximation of the solution and its fractional gradient. We prove the well-posedness of the formulation via an inf–sup condition and, following [45], construct a stabilized and equivalent formulation that allows the use of standard finite element discretizations. We conclude by establishing a priori error bounds and performing several numerical experiments. The content of this analysis is largely based on [9]. Despite its paramount importance in the modeling of phenomena with long-range interactions, to the best of our knowledge [9] is the first work to address the fractional Laplacian through the mixed formulation (D) and to approximate the associated nonlocal fluxes. The stable numerical computation of such fluxes is particularly relevant in view of the scarcity of explicit solutions to the fractional Poisson problem: the few available examples are restricted to problems posed on balls [29], and we are not aware of any closed-form expressions in the literature for the corresponding fluxes.

The other problem that we study in this thesis is the minimization of a fractional version of the Oseen–Frank energy arising in the modeling of liquid crystals. A classical continuum theory describing the equilibrium configurations of nematic liquid crystals is the *Oseen–Frank model*. In this framework, the orientation of the molecules is represented by a unit-length vector field $u : \Omega \subset \mathbb{R}^d \to S^{d-1}$, called the *director field*. The equilibrium states correspond to minimizers of the Oseen–Frank energy

$$E(u) := \frac{1}{2} \int_\Omega k_1 (\operatorname{div} u)^2 + k_2 (u \cdot \mathbf{curl}\, u)^2 + k_3 |u \times \mathbf{curl}\, u|^2 + (k_2 + k_4)(\operatorname{tr}([\nabla u]^2) - (\operatorname{div} u)^2)\, dx,$$

which penalizes distortions of the director field through different deformation modes known as *splay* (k_1), *twist* (k_2), *bend* (k_3), and *saddle-splay* ($k_2 + k_4$).

A distinctive feature of nematic liquid crystals is the presence of *defects*, which are singularities in the orientation field where the director becomes ill-defined. From a mathematical point of view, such singularities pose significant challenges. In particular, the classical Oseen–Frank energy is unable to fully capture this phenomena. In this thesis, we consider a fractional version of the Oseen–Frank energy in which the classical differential operators are replaced by their fractional counterparts. This modification lowers the regularity requirements on the admissible configurations. On the analytical side, we investigate the well-posedness of the corresponding variational problems and analyze the properties of the fractional differential operators that arise in the model. In particular, we study continuity and compactness properties that allow us to establish the existence of minimizers under suitable assumptions. On the numerical side, we develop and analyze a discretization strategy for a simplified version of the fractional model, which is based on [36]. Finally, we present a series of computational experiments in two space dimensions illustrating the behavior of the method and the qualitative properties of the resulting equilibrium configurations.

The thesis is organized as follows. In Chapter 1 we set the notation, introduce the functional analytic framework, and review the main properties of fractional Sobolev spaces and fractional differential operators used throughout the work.

In Chapter 2 we develop a sufficiently general framework to study mixed formulations in Hilbert spaces. We then study the fractional Darcy problem and establish its well-posedness. Since a direct discretization of this problem appears to be out of reach, we adapt the stabilized approach of [45], which yields a coercive and well-posed formulation. The coercivity ensures the stability of any conforming finite element discretization. We further prove the convergence of this discretization, derive convergence rates, and discuss implementation aspects. Finally, we present numerical experiments that highlight both the importance of stabilization and the accuracy of the theoretical results.

In Chapter 3 we introduce the fractional Oseen–Frank model and study the associated variational problem. Using the direct method of the calculus of variations, we establish the existence of minimizers. We also present a numerical discretization for a simplified version of the model based on an operator-splitting strategy and describe its finite element approximation.

Finally, Appendix A is devoted to the implementation of the matrices involved in the discrete problems considered throughout the thesis.

Chapter 1

Preliminaries

This chapter collects all the necessary results that we shall need for a finite element analysis of problems involving the fractional calculus operators treated in this thesis. Our main goal is to construct a rigorous and coherent framework that connects fractional differential operators with appropriate variational spaces, embedding results, and compactness and continuity properties, which will later be essential for the analysis of nonlocal variational models.

There are several nonequivalent notions of nonlocal differential generalizations of the classical calculus operators: ∇, div and **curl**. In this work, we shall stick with the fractional operators introduced in [39] and interpreted as fractional calculus operators in [52]. These definitions are presented in Section 1.1.

The second part of the chapter is devoted to the functional analytic structure of the variational spaces associated with the operators defined in the first section. In particular, we study the Hilbert case and its relation with Bessel potential spaces, showing how fractional Sobolev spaces arise naturally as variational spaces for the fractional Laplacian and fractional vector operators. This provides a precise link between nonlocal operators, Fourier-based definitions, and classical Sobolev theory.

Finally, in Section 1.3 we establish several connections between fractional vector calculus operators, Sobolev spaces, and Bessel potential spaces. We extend the operators to suitable Sobolev spaces by density and derive their mapping properties as pseudo-differential operators. Moreover, we establish compactness results, weak convergence characterizations, and weak continuity properties. These results are specifically designed to support the study of nonlocal energies, and in particular to enable the analysis of fractional versions of vector-valued energies such as the Oseen–Frank functional.

1.1 Fractional vector calculus

Fractional differential operators arise naturally in the mathematical modeling of nonlocal phenomena, anomalous diffusion, and long-range interactions, and play a fundamental role in modern analysis and partial differential equations. In contrast with their classical local counterparts, these operators are intrinsically nonlocal and cannot be characterized through pointwise differential expressions, but rather through integral formulations and Fourier representations.

We begin this chapter by defining a fractional vector calculus, including fractional versions of the

Laplacian, gradient, divergence, and curl operators. We discuss two equivalent perspectives on these operators: one based on singular integral representations and another based on pseudo-differential operators defined through the Fourier transform. All operators are first defined on spaces of sufficiently smooth functions, namely the Schwartz class $\mathcal{S}$. This framework allows us to establish precise Fourier symbols, structural identities, and composition rules for the fractional operators, laying the foundation for their extension to variational spaces.

1.1.1 The fractional Laplacian

We shall define first the fractional Laplacian, as it is probably the most well known of the fractional operators.

Definition 1.1.1. *Let $s \in (0,1)$. Given $w\colon \mathbb{R}^d \to \mathbb{R}$ in $\mathcal{S}$ we define its fractional Laplacian as*

$$(-\Delta)^s w(x) := \nu(d,s)\ p.v. \int_{\mathbb{R}^d} \frac{w(x)-w(y)}{|x-y|^{d+2s}} dy, \quad x \in \mathbb{R}^d, \tag{1.1.1}$$

where

$$\nu(d,s) := \frac{2^{2s} s\Gamma(s+\frac{d}{2})}{\pi^{d/2}\Gamma(1-s)}. \tag{1.1.2}$$

One can get rid of the principal value writing the operator a as the integral of a weighted second order differential quotient; see [26, Lemma 3.2].

Proposition 1.1.2. *Let $s \in (0,1)$, then for any $w \in \mathcal{S}$,*

$$(-\Delta)^s w(x) = -\frac{\nu(d,s)}{2} \int_{\mathbb{R}^d} \frac{u(x+z)-2u(x)+u(x-z)}{|z|^{d+2s}}\, dz \tag{1.1.3}$$

The constant $\nu(d,s)$, provides the adequate scaling in the limits $s \to 0^+$ and $s \to 1^-$,

$$\nu(d,s) \sim s(s-1) \quad \text{when } s \to 0^+ \text{ or } s \to 1^-,$$

see Proposition 1.2.16. Furthermore, it ensures the consistence of Definition 1.1.1 and the one defined via Fourier transform. We define the Fourier transform of a function $u \in L^1(\mathbb{R}^d)$ as

$$\mathscr{F}u(\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} u(x)\, dx.$$

For the sake of completeness, we state here some basic results that will be useful in what follows.

Proposition 1.1.3. *Let $u \in L^1(\mathbb{R}^d)$. The Fourier transform satisfies the following properties:*

1. *If $\hat{u} \in L^1(\mathbb{R}^d)$, then the following Fourier inversion formula holds,*

$$u(x) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \hat{u}(\xi)\, d\xi.$$

   For that reason, we denote

$$\mathscr{F}^{-1}u(x) = \check{u}(x) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} u(\xi)\, d\xi.$$

2. *The Fourier transform can be (uniquely) extended to an unitary isomorphism on* $L^2(\mathbb{R}^d)$*.*

3. *Given* $z \in \mathbb{R}^d$*, define* $\tau_z u(x) := u(x-z)$*. Then,*
$$\mathscr{F}(\tau_z u)(\xi) = e^{-2\pi i \xi \cdot z} \mathscr{F} u(\xi).$$

4. *Let* $k \in \mathbb{N}$*. If* $u \in C^k(\mathbb{R}^d)$ *and* $\partial_\alpha u \in L^1(\mathbb{R}^d)$ *for* $|\alpha| \leq k$*, then*
$$\mathscr{F}(\partial^k u)(\xi) = (2\pi i)^{|\alpha|} \xi^\alpha \mathscr{F} u(\xi), \quad \forall |\alpha| \leq k.$$

The previous proposition implies that
$$\widehat{(-\Delta w)}(\xi) = (2\pi|\xi|)^2 \hat{w}(\xi),$$
an analogous result holds true for the fractional Laplacian.

Theorem 1.1.4 (see [26, Proposition 3.3])**.** *Let* $s \in (0,1)$*. Then, for any* $u \in \mathcal{S}$*, it holds*
$$\widehat{(-\Delta)^s w}(\xi) = (2\pi|\xi|)^{2s} \widehat{w}(\xi). \tag{1.1.4}$$

The proof of this result uses formula (1.1.3) and the following characterization of the constant $\nu(d,s)$:
$$\frac{1}{\nu(d,s)} = \int_{\mathbb{R}^d} \frac{1-\cos(z_1)}{|z|^{d+2s}}\, dz, \tag{1.1.5}$$
see [17, Lemma 3.1.3].

Formula (1.1.4) allows one to extend to extend the definition of the fractional Laplacian to arbitrary values of $s > -d$; see also formula (1.1.10) below and its preceding discussion. In integral form, this extension can be written as
$$(-\Delta)^s w(x) := \begin{cases} -\nu(d,s) \displaystyle\int_{\mathbb{R}^d} \frac{w(x+h)}{|h|^{d+2s}} dy & \text{if } -d < s < 0, \\ (-\Delta)^s w(x) & \text{if } s \in \mathbb{N}, \\ -\nu(d,s) \text{ p.v.} \displaystyle\int_{\mathbb{R}^d} \frac{w(x+h) - \sum_{|\alpha| \leq 2\lfloor s \rfloor} \partial_\alpha u(x) \frac{h^\alpha}{\alpha!}}{|h|^{d+2s}} dh & \text{if } s \in \mathbb{R}^+ \setminus \mathbb{N}. \end{cases} \tag{1.1.6}$$
This representation can be derived, for example, as a consequence of [53, Theorem 4.7]. Note that this extension is consistent with the case $s \in (0,1)$ defined above; it follows from the change of variables $h = y - x$ in Definition (1.1.1).

1.1.2 The operators ∇^s, div^s and curl^s

There are several nonequivalent notions of nonlocal differential operators, which in turn give rise to different ways of writing $(-\Delta)^s$ in divergence form. One customary option is to consider *unweighted* operators [28], namely to regard gradients as two-point operators and the divergence operator as minus the adjoint of the gradient. In our setting, this would reduce to consider, for sufficiently smooth functions $w\colon \mathbb{R}^d \to \mathbb{R}$, $\mathbf{\Psi}\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$,
$$\mathcal{G}^s w\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d, \quad \mathcal{G}^s w(x,y) := \sqrt{\frac{\nu(d,s)}{2}} \frac{(w(x) - w(y))}{|x-y|^{d/2+s}} \frac{(x-y)}{|x-y|},$$
$$\mathcal{D}^s \mathbf{\Psi}\colon \mathbb{R}^d \to \mathbb{R}, \quad \mathcal{D}^s \mathbf{\Psi}(x) := \sqrt{\frac{\nu(d,s)}{2}} \int_{\mathbb{R}^d} \frac{(\mathbf{\Psi}(x,y) + \mathbf{\Psi}(y,x))}{|x-y|^{d/2+s}} \cdot \frac{(x-y)}{|x-y|}\, dy.$$

This yields the identity $(-\Delta)^s = -\mathcal{D}^s \circ \mathcal{G}^s$. Although these notions arise naturally in the nonlocal setting, interpreting the fractional gradient as a two-point operator complicates its physical interpretation and obscures the classical perspective of gradients as indicating the direction of maximal growth. Since the above definition essentially corresponds to a difference quotient, we argue that in applications it may be more meaningful to construct nonlocal gradients by integration of difference quotients. In this vein, the fractional Laplacian can be regarded as a composition of certain *weighted,* non-local, vector calculus operators (cf. Lemma 1.1.15).

Definition 1.1.5. *Let $s \in (0,2)$ and $d, m \in \mathbb{N}$. Given $w\colon \mathbb{R}^d \to \mathbb{R}^m$ in $\mathcal{S}^m$, we define its* fractional gradient of order s*, $\nabla^s w\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$,*

$$\nabla^s w(x) := \begin{cases} \mu(d,s) \displaystyle\int_{\mathbb{R}^d} \frac{(w(y)-w(x))}{|x-y|^{d+s}} \otimes \frac{(y-x)}{|x-y|}\,dy & \text{if } s \in (0,1), \\ \nabla w(x) & \text{if } s = 1, \\ \mu(d,s) \displaystyle\int_{\mathbb{R}^d} \frac{(w(y)-w(x)-\nabla w(x)\cdot(x-y))}{|x-y|^{d+s}} \otimes \frac{(x-y)}{|x-y|}\,dy & \text{if } s \in (1,2). \end{cases} \tag{1.1.7}$$

and, given $\mathbf{\Psi}\colon \mathbb{R}^d \to \mathbb{R}^d$ we define its fractional divergence of order s*, $\operatorname{div}^s \mathbf{\Psi}\colon \mathbb{R}^d \to \mathbb{R}$,*

$$\operatorname{div}^s \mathbf{\Psi}(x) := \begin{cases} \mu(d,s) \displaystyle\int_{\mathbb{R}^d} \frac{(\mathbf{\Psi}(y)-\mathbf{\Psi}(x))}{|x-y|^{d+s}} \cdot \frac{(y-x)}{|x-y|}\,dy & \text{if } s \in (0,1), \\ \operatorname{div} \mathbf{\Psi}(x) & \text{if } s = 1, \\ \mu(d,s) \displaystyle\int_{\mathbb{R}^d} \frac{(\mathbf{\Psi}(y)-\mathbf{\Psi}(x)-\nabla\mathbf{\Psi}(x)(x-y))}{|x-y|^{d+s}} \cdot \frac{(x-y)}{|x-y|}\,dy & \text{if } s \in (1,2). \end{cases}$$

Also, for $d=2$ or $d=3$, we define its fractional curl of order s*, $\mathbf{curl}^s \mathbf{\Psi}\colon \mathbb{R}^d \to \mathbb{R}^{(d-1)d/2}$,*

$$\mathbf{curl}^s \mathbf{\Psi}(x) := \begin{cases} \mu(d,s) \displaystyle\int_{\mathbb{R}^d} \frac{(\mathbf{\Psi}(y)-\mathbf{\Psi}(x))}{|x-y|^{d+s}} \times \frac{(y-x)}{|x-y|}\,dy & \text{if } s \in (0,1), \\ \mathbf{curl}\, \mathbf{\Psi}(x) & \text{if } s = 1, \\ \mu(d,s) \displaystyle\int_{\mathbb{R}^d} \frac{(\mathbf{\Psi}(y)-\mathbf{\Psi}(x)-\nabla\mathbf{\Psi}(x)(x-y))}{|x-y|^{d+s}} \times \frac{(x-y)}{|x-y|}\,dy & \text{if } s \in (1,2). \end{cases}$$

In the three definitions above, we have taken

$$\mu(d,s) := \frac{2^s \Gamma(\frac{d+s+1}{2})}{\pi^{d/2}\Gamma(\frac{1-s}{2})}.$$

Again, the constant $\mu(d,s)$ is there to ensure the consistency between Definitions 1.1.5 and the definitions motivated by the Fourier transform; see Corollary 1.1.9.

Remark 1.1.6 (Consistency)**.** *The definitions of the fractional operators deliver finite numbers and well-defined vectors, and they map the Schwartz class into itself; see for example [53, Section 7] for consistency, and [53, Section 6] for the mapping property.*

Now, we aim to show that the operators 1.1.5 are pseudo-differential operators. Recall the Riesz potentials for $0 < \alpha < d$, defined by

$$\mathcal{I}_\alpha w := I_\alpha * w, \quad w \in \mathcal{S},$$

where I_α is the so-called Riesz kernel given by the formula

$$I_\alpha(x) := \frac{c(d,\alpha)}{|x|^{d-\alpha}}, \tag{1.1.8}$$

with

$$c(d,\alpha) := \frac{\Gamma(\frac{d-\alpha}{2})}{\pi^{\frac{d}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}.$$

Properties regarding these potentials can be found in [54, Chapter 5], from which we remark that the operator $\mathcal{I}_\alpha$ satisfies the semi-group property:

$$\mathcal{I}_\alpha \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta}, \quad \text{for } \alpha, \beta > 0 \text{ and } \alpha + \beta < d,$$

that the Laplacian maps Riesz potentials of order α to potentials of order $\alpha - 2$:

$$\Delta \mathcal{I}_{\alpha+2} w = \mathcal{I}_\alpha w,$$

that

$$\widehat{I_\alpha}(\xi) = |2\pi\xi|^{-\alpha}, \tag{1.1.9}$$

and that $\mathcal{I}_\alpha : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a continuous operator for $p \in (1,\infty)$:

$$\|\mathcal{I}_\alpha w\|_{L^p(\mathbb{R}^d)} \leq c\|w\|_{L^p(\mathbb{R}^d)}, \quad \text{for all } w \in L^p(\mathbb{R}^d).$$

By means of analytic continuation, Riesz potentials can be extended to $\alpha \notin 2\mathbb{Z}$ (cf. [40, Chapter 9]) and the previous properties hold for the extension. Therefore, one also deduces the formula

$$(-\Delta)^s w = \mathcal{I}_{-2s} w, \quad w \in \mathcal{S}. \tag{1.1.10}$$

This motivates the idea that Riesz potentials can be used to define fractional derivatives.

Definition 1.1.7 (Riesz gradient). *Let $s \in (0,1)$ and $w \in L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. We define, for $k = 1, \ldots, d$,*

$$\partial_k^s w := \partial_k \mathcal{I}_{1-s} w,$$

in the weak sense:

$$\int_{\mathbb{R}^d} (\partial_k^s w) v \, dx = - \int_{\mathbb{R}^d} (I_{1-s} * w)(\partial_k v) \, dx,$$

for all $v \in C_c^\infty(\mathbb{R}^d)$.

Given the properties of the Riesz kernels, these operators lie between their classical counterparts and the so-called Riesz transforms, which can be defined via Fourier transform as

$$\widehat{\mathcal{R}_K w}(\xi) := i \frac{\xi_k}{|\xi|} \widehat{w}(\xi), \quad k = 1, \ldots, d.$$

We refer to [54, Chapter 2] for an introduction and properties of these operators. The following result merges the results proved in [52, Theorem 1.3 and Theorem 1.2]. We include the proof here for completeness. It heuristically states that one can move the differentiation in the definition of ∂_k^s from one factor of the convolution to the other, and that the Riesz derivatives are pseudo-differential operators.

Theorem 1.1.8. *Let $s \in (0,1)$. Then, for all $w \in C_c^\infty(\mathbb{R}^d)$,*

$$\partial_k^s w = I_{1-s} * \partial_k w,$$

for all $k = 1, \ldots, d$. Therefore, for all $w \in C_c^\infty(\mathbb{R}^d)$,

$$\widehat{\partial_k^s w}(\xi) = (2\pi)^s i |\xi|^{s-1} \xi_k \widehat{w}(\xi).$$

Proof. Let $v \in C_c^\infty(\mathbb{R}^d)$ and $R > 0$ such that $\operatorname{supp} w(x-\cdot) \subset B(0,R)$ for all $x \in \operatorname{supp} v$. By definition,

$$\begin{aligned}
\langle \partial_k^s w, v \rangle &= -\int_{\mathbb{R}^d} (I_{1-s} * w)\partial_k v \, dx \\
&= -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} (I_{1-s} * w) \frac{v(x+\varepsilon e_k) - v(x)}{\varepsilon} \, dx \\
&= -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{w(x-y)v(x+\varepsilon e_k)}{|y|^{d+s-1}} \, dydx - \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{w(x-y)v(x)}{|y|^{d+s-1}} \, dydx \\
&= -\lim_{\varepsilon \to 0} \int_{\operatorname{supp} v} \int_{B(0,R)} \frac{1}{\varepsilon} \frac{w(x-y-\varepsilon e_k) - w(x-y)}{|y|^{d+s-1}} \, dy \, v(x) \, dx.
\end{aligned}$$

Since w is smooth and has compact support, we have that

$$\frac{1}{\varepsilon} \frac{w(x-y-\varepsilon e_k) - w(x-y)}{|y|^{d+s-1}} \leq \frac{c}{|y|^{d+s-1}} \in L^1(B(0,R)),$$

for all $x \in \operatorname{supp} v$ and $y \in B(0,R)$. Thus, by the dominated convergence theorem, we deduce

$$\langle \partial_k^s w, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\partial_k w(x-y)}{|y|^{d+s-1}} v(x) \, dydx = \int_{\mathbb{R}^d} (I_{1-s} * \partial_k w) v \, dx.$$

Now, using (1.1.9), a direct calculation shows

$$\widehat{\partial_k^s w}(\xi) = \widehat{I_{1-s}} \, \widehat{\partial_k w}(\xi) = (2\pi)^{s-1} |\xi|^{s-1} 2\pi i \xi_k \widehat{w}(\xi) = (2\pi)^s i |\xi|^{s-1} \xi_k \widehat{w}(\xi).$$

□

We can define a fractional gradient D^s in the same manner as the classical one, using the operators ∂_k^s, and one deduces the formulas

$$D^s w = I_{1-s} * \nabla w, \quad \widehat{D^s w}(\xi) = (2\pi)^s i |\xi|^{s-1} \xi \widehat{w}(\xi).$$

We conclude the discussion on the definition of our fractional derivatives with the following fact: D^s and ∇^s are the same object. It is a direct consequence of Theorem 1.1.8.

Corollary 1.1.9. *Let $s \in (0,1)$ and $w \in C_c^\infty(\mathbb{R}^d)$. It holds*

$$\nabla^s w = D^s w.$$

Proof. First, note that

$$\widehat{D^s w}(\xi) = m_s(\xi)\widehat{w}(\xi), \text{ with } m_s(\xi) = (2\pi i \xi)|2\pi\xi|^{s-1}.$$

Hence, by (1.1.9), we deduce

$$\mathscr{F}^{-1}(m_s)(x) = \nabla \left[\mathscr{F}^{-1} \left(|2\pi\xi|^{s-1} \right) \right](x) = -(d+s-1)c(d,1-s) \frac{x}{|x|^{d+s+1}}.$$

Defining the kernel

$$K_s(x) = \mathscr{F}^{-1}(m_s)(x),$$

we deduce that

$$D^s w(x) = (K_s * w)(x),$$

in the sense of distributions, since $K_s \notin L^1_{loc}(\mathbb{R}^d)$. Note that

$$\int_{B(0,\varepsilon)^c} K_s(x)\,dx = 0$$

for all $\varepsilon > 0$. Therefore, we can rewrite

$$D^s w(x) = p.v. \int_{\mathbb{R}^d} K_s(x-y)(w(y)-w(x))\,dy = (d+s-1)c(d,1-s)\,p.v. \int_{\mathbb{R}^d} \frac{(w(y)-w(x))}{|x-y|^{d+s+1}}(y-x)\,dy.$$

Finally, we justify our choice of the constant $\mu(d,s)$ in Definition 1.1.5. Indeed, using the identity $\Gamma(z)z = \Gamma(z+1)$, we obtain

$$(d+s-1)c(d,1-s) = \frac{(d+s-1)}{2}\Gamma\left(\frac{d+s-1}{2}\right)\frac{2^s}{\pi^{\frac{d}{2}}\Gamma(\frac{1-s}{2})} = \frac{2^s\Gamma(\frac{d+s+1}{2})}{\pi^{d/2}\Gamma(\frac{1-s}{2})} = \mu(d,s),$$

thus concluding that

$$D^s w = \nabla^s w.$$

□

We want to point out that Theorem 1.1.8 and Corollary 1.1.9 remain valid for vector fields $w : \mathbb{R}^d \to \mathbb{R}^m$. It is a matter of applying Theorem 1.1.8 coordinate-wise, from which one deduces

$$\widehat{\nabla^s w}(\xi) = (2\pi)^s i|\xi|^{s-1}\xi \otimes \widehat{w}(\xi).$$

We finish this part with a Fourier characterization of the operators operators div^s and $\mathbf{curl}^s$. This is, again, immediate from Theorem 1.1.8 and Corollary 1.1.9.

Proposition 1.1.10 (Fourier representation of div^s and $\mathbf{curl}^s$)**.** *Let $s \in (0,1)$. Then, for all $\mathbf{\Psi} = (\mathbf{\Psi}_1, \dots, \mathbf{\Psi}_d) \in [C_c^\infty(\mathbb{R}^d)]^d$,*

$$\operatorname{div}^s \mathbf{\Psi} = I_{1-s} * \operatorname{div} \mathbf{\Psi}.$$

Furthermore, if $d = 2$ or $d = 3$ we have

$$\mathbf{curl}^s \mathbf{\Psi} = I_{1-s} * \mathbf{curl}\, \mathbf{\Psi}.$$

Therefore,

$$\widehat{\operatorname{div}^s \mathbf{\Psi}}(\xi) = (2\pi)^s i|\xi|^{s-1}\xi \cdot \widehat{\mathbf{\Psi}}(\xi),$$
$$\widehat{\mathbf{curl}^s \mathbf{\Psi}}(\xi) = (2\pi)^s i|\xi|^{s-1}\xi \wedge \widehat{\mathbf{\Psi}}(\xi).$$

Remark 1.1.11 (other non-local calculus identities)**.** *The identities $\operatorname{div}^s \mathbf{\Psi} = I_{1-s} * \operatorname{div} \mathbf{\Psi}$ and $\mathbf{curl}^s \mathbf{\Psi} = I_{1-s} * \mathbf{curl}\, \mathbf{\Psi}$ can be interpreted in the following sense:*

$$\operatorname{div}^s \mathbf{\Psi} = \partial_1^s \mathbf{\Psi}_1 + \cdots + \partial_d^s \mathbf{\Psi}_d.$$

For $d = 2$, we have

$$\mathbf{curl}^s \mathbf{\Psi} = \begin{vmatrix} \partial_1^s & \partial_2^s \\ \mathbf{\Psi}_1 & \mathbf{\Psi}_2 \end{vmatrix},$$

and for $d = 3$, we have

$$\mathbf{curl}^s \mathbf{\Psi} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_1^s & \partial_2^s & \partial_3^s \\ \mathbf{\Psi}_1 & \mathbf{\Psi}_2 & \mathbf{\Psi}_3 \end{vmatrix},$$

which are consistent with the local case.

Remark 1.1.12 (extension to Lipschitz functions)**.** *Corollary 1.1.9 and Proposition 1.1.10 can be extended to Lipschitz functions. This was proven in [22, Proposition 2.2].*

Using the Fourier representation of ∇^s and the vectorial Riesz transform, defined as

$$\widehat{\mathcal{R}w}(\xi) = i\frac{\xi}{|\xi|}\widehat{w}(\xi),$$

we deduce that

$$\nabla^s w = \mathcal{R}(-\Delta)^{\frac{s}{2}} w.$$

Recall that we extended $(-\Delta)^s$ to arbitrary values of s (cf. Definition (1.1.6)). Thus, we extend ∇^s to any $s > 0$ via

$$\nabla^s w = \mathcal{R}(-\Delta)^{\frac{s}{2}} w, \quad \forall w \in C_c^\infty(\mathbb{R}^d).$$

Coordinate-wise we deduce that $\nabla^s = (\partial_1^s, \ldots, \partial_d^s)$, where ∂_k^s denotes the operator defined via

$$\partial_k^s w = \mathcal{R}_k(-\Delta)^{\frac{s}{2}} w, \quad \forall w \in C_c^\infty(\mathbb{R}^d),\ s > 0,\ k = 1, \ldots, d.$$

The notation is justified by the following lemma, which generalizes Theorem 1.1.8 and shows that these fractional derivatives extend those of Definition 1.1.7 to arbitrary values of s.

Lemma 1.1.13. *Let $s = m + s' > 0$ with $m \in \mathbb{N}$ and $s' \in (0, 1)$. Then, for all $w \in C_c^\infty(\mathbb{R}^d)$,*

$$\partial_k^s w = I_{1-s'} * \partial_k^{m+1} w,$$

for all $k = 1, \ldots, d$. Therefore, for all $w \in C_c^\infty(\mathbb{R}^d)$,

$$\widehat{\partial_k^s w}(\xi) = (2\pi)^s i|\xi|^{s-1}\xi_k \widehat{w}(\xi).$$

Note that we also deduce

$$\nabla^s w = \nabla(-\Delta)^{\frac{s-1}{2}} w.$$

Therefore, in the spirit of Definition (1.1.6), an integral representation of this extension reads

$$\nabla^s w(x) := \begin{cases} \mu(d,s)\ \text{p.v.} \displaystyle\int_{\mathbb{R}^d} h \frac{w(x+h) - \sum_{|\alpha| \le 2\lfloor \frac{s-1}{2} \rfloor + 1} \partial_\alpha w(x) \frac{h^\alpha}{\alpha!}}{|h|^{d+s+1}} dh & \text{if } s \text{ is not odd,} \\ \nabla(-\Delta)^{\frac{s-1}{2}} w(x) & \text{otherwise.} \end{cases} \tag{1.1.11}$$

Again, this representation can be derived, for example, as a consequence of [53, Theorem 4.3].

Remark 1.1.14 (physical relevance of the fractional operators)**.** *Actually, [53] provides a significantly stronger characterization of all the fractional operators defined in this section, from which the Fourier representation of Definitions* (1.1.6) *and* (1.1.11) *follow as a corollary. The aim of that work is to show that the fractional operators ∇^s, div^s and $(-\Delta)^s$ are uniquely determined, up to multiplicative constants, by the properties of rotational invariance* [1]*, translational invariance, and s-homogeneity* [2]

We also extend the operators div^s and $\mathbf{curl}^s$ to arbitrary values of $s > 0$ via its Fourier representation; however, the integral forms of these extensions are not needed for our purposes.

[1]Namely, for every $Q \in O(d)$ it holds that $[(-\Delta)^s w(Q^T\cdot)](x) = (-\Delta)^s w(Q^T x)$ (resp. $\operatorname{div}^s[Q\mathbf{\Psi}(Q^T\cdot)](x) = \operatorname{div}^s\mathbf{\Psi}(Q^T x)$ and $\nabla^s[w(Q^T\cdot)](x) = Q\,\nabla^s w(Q^T x)$).

[2]Namely, for all $\lambda > 0$ it holds that $[(-\Delta)^s w(\lambda\cdot)](x) = \lambda^{2s}(-\Delta)^s w(\lambda x)$ (resp. $\operatorname{div}^s[\mathbf{\Psi}(\lambda\cdot)](x) = \lambda^s \operatorname{div}^s\mathbf{\Psi}(\lambda x)$ and $\nabla^s[w(\lambda\cdot)](x) = \lambda^s\,\nabla^s w(\lambda x)$).

1.1.3 Fractional calculus identities

In view of Corollary 1.1.9, we will denote the Riesz gradient by ∇^s. The main goal of this section is to extend to the fractional setting some vector calculus identities that hold in the local case. They will be useful for the analysis carried out in chapters 2 and 3. Throughout this section, if we write $\mathbf{curl}^s\mathbf{\Psi}$ it is implied that $\mathbf{\Psi}$ is a field in $\mathbb{R}^2$ or $\mathbb{R}^3$.

At the start of this section, we commented that the fractional Laplacian can be regarded as the composition of certain non-local operators. As in the local case, such operators are div^s and ∇^s. The identity follows by calculating the symbol of the composition $\operatorname{div}^s \circ \nabla^s$.

Lemma 1.1.15. *Let $0 \leq \alpha, \beta$. It holds,*

$$\operatorname{div}^{\alpha} \nabla^{\beta} w = (-\Delta)^{\frac{\alpha+\beta}{2}} w,$$

for all $w \in C_c^\infty(\mathbb{R}^d)$.

Remark 1.1.16. *Let $s \in (0,1)$ and $w \in C_c^\infty(\mathbb{R}^d)$. Then, we deduce*

$$\operatorname{div}^s \nabla^s w = (-\Delta)^s w.$$

This formula can be interpreted in the following way:

$$(-\Delta)^s w = \sum_{k=1}^{d} \partial_k^s(\partial_k^s w).$$

Another classical identities that we can recover are $\mathbf{curl} \circ \nabla = 0$ and $\operatorname{div} \circ \mathbf{curl} = 0$.

Proposition 1.1.17. *Let α, $\beta > 0$, $w \in C_c^\infty(\mathbb{R}^d)$ and $\mathbf{\Psi} \in [C_c^\infty(\mathbb{R}^d)]^d$. Then,*

$$\mathbf{curl}^{\alpha} \nabla^{\beta} w = 0,$$
$$\operatorname{div}^{\alpha} \mathbf{curl}^{\beta} \mathbf{\Psi} = 0.$$

Proof. First, we have $\xi \wedge \xi = 0$ for all $\xi \in \mathbb{R}^d$. Therefore,

$$\widehat{\mathbf{curl}^{\alpha}\nabla^{\beta}} w(\xi) = (2\pi)^\alpha i|\xi|^{\alpha-1}\xi \wedge \widehat{\nabla^{\beta} w}(\xi) = (2\pi)^\alpha i|\xi|^{\alpha-1}\xi \wedge ((2\pi)^\beta i|\xi|^{\beta-1}\xi \widehat{w}(\xi)) = 0.$$

Similarly, $\xi \cdot (x \wedge \xi) = 0$ for all $x, \xi \in \mathbb{R}^d$. Then,

$$\operatorname{div}\widehat{^{\alpha}\mathbf{curl}}^{\beta} w(\xi) = (2\pi)^\alpha i|\xi|^{\alpha-1}\xi \cdot \widehat{\mathbf{curl}^{\beta}} w(\xi) = (2\pi)^\alpha i|\xi|^{\alpha-1}\xi \cdot [(2\pi)^\beta i|\xi|^{\beta-1}\xi \wedge \widehat{\mathbf{\Psi}}(\xi)] = 0.$$

□

Given any skew-symmetric matrix $A \in \mathbb{R}^{3\times 3}$ there exists a unique vector $a \in \mathbb{R}^3$, such that

$$Ax = a \wedge x,$$

for all $x \in \mathbb{R}^d$, and $|A| = 2|a|$. One can define the classical $\mathbf{curl}\, w$ as the skew vector associated to the skew part of the differential of w. That means, $\mathbf{curl}\, w$ is the unique vector such that

$$[\nabla w - (\nabla w)^t]x = \mathbf{curl}\, w \wedge x,$$

and $|\nabla w - (\nabla w)^t| = 2|\mathbf{curl}\, w|$. The same holds in the fractional setting.

Lemma 1.1.18. *Let $s \in (0,1)$, $d = 2$ or $d = 3$, and $w \in [C_c^\infty(\mathbb{R}^d)]^d$. Then,*

$$|\nabla^s w|^2 = \operatorname{tr}([\nabla^s w]^2) + |\mathbf{curl}^s w|^2.$$

Proof. For a matrix $A \in \mathbb{R}^{d\times d}$ we have $|A|^2 = \operatorname{tr}(AA^T) = \operatorname{tr}(A^2) + \frac{1}{2}|A - A^T|^2$. Therefore, we need to prove that

$$|\mathbf{curl}^s w|^2 = \frac{1}{2}|\nabla^s w - [\nabla^s w]^t|^2.$$

If $w = (w_1, \dots, w_d)$ then by definition $(\nabla^s w)_{ij} = \partial_j^s w_i$. Now assume $d = 3$. By Proposition 1.1.10, we have

$$|\mathbf{curl}^s w|^2 = \left| \begin{pmatrix} \partial_2^s w_3 - \partial_3^s w_2 \\ \partial_3^s w_1 - \partial_1^s w_3 \\ \partial_1^s w_2 - \partial_2^s w_1 \end{pmatrix} \right|^2 = \frac{1}{2}|\nabla^s w - [\nabla^s w]^t|^2.$$

The case $d = 2$ follows immediately upon observing that for $w \in [C_c^\infty(\mathbb{R}^d)]^2$ we have $\mathbf{curl}^s(w,0) = (0,0,\mathbf{curl}^s w)$ ☐

The classical integration by parts formula implies that the Laplacian is a self-adjoint operator, and that the gradient and divergence are adjoint operators. This is crucial for the analysis of mixed formulations regarding the Laplacian. Again, the same can be proven for their fractional counterparts.

Theorem 1.1.19. *Let $\alpha, \beta, s \geq 0$, $v, w \in C_c^\infty(\mathbb{R}^d)$ and $\mathbf{\Psi} \in [C_c^\infty(\mathbb{R}^d)]^d$. Then,*

$$\int_{\mathbb{R}^d} w\,(-\Delta)^{\beta+\alpha} v\,dx = \int_{\mathbb{R}^d} (-\Delta)^\alpha w\,(-\Delta)^\beta v\,dx = \int_{\mathbb{R}^d} (-\Delta)^{\beta+\alpha} w\,v\,dx$$

and

$$\int_{\mathbb{R}^d} \nabla^s w \cdot \mathbf{\Psi}\,dx = -\int_{\mathbb{R}^d} w\,\operatorname{div}^s \mathbf{\Psi}\,dx.$$

Proof. The two identities are a consequence of the Fourier representation and the Plancherel's formula. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^d} w\,(-\Delta)^{\beta+\alpha} v\,dx &= \int_{\mathbb{R}^d} \widehat{w}(\xi)\,\widehat{(-\Delta)^{\beta+\alpha} v}(\xi)\,d\xi \\
&= \int_{\mathbb{R}^d} (2\pi)^{2\alpha}|\xi|^{2\alpha}\widehat{w}(\xi)\,(2\pi)^{2\beta}|\xi|^{2\beta}\widehat{v}(\xi)\,d\xi \\
&= \int_{\mathbb{R}^d} (-\Delta)^\alpha w\,(-\Delta)^\beta v\,dx.
\end{aligned}$$

For the second identity, we must be a bit more careful, since the symbols of div^s and ∇^s are complex-valued. Therefore, after Plancherel's formula, complex conjugation of the second term in the L^2-inner product can not be omitted. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^d} \nabla^s w \cdot \mathbf{\Psi}\,dx &= \int_{\mathbb{R}^d} \widehat{\nabla^s w}(\xi) \cdot \overline{\widehat{\mathbf{\Psi}}(\xi)}\,d\xi \\
&= \int_{\mathbb{R}^d} \left[(2\pi)^s i|\xi|^{s-1}\xi\,\widehat{w}(\xi)\right] \cdot \overline{\widehat{\mathbf{\Psi}}(\xi)}\,d\xi \\
&= -\int_{\mathbb{R}^d} \widehat{w}(\xi)\,\overline{(2\pi)^s i|\xi|^{s-1}\xi \cdot \widehat{\mathbf{\Psi}}(\xi)}\,d\xi \\
&= -\int_{\mathbb{R}^d} w\,\operatorname{div}^s \mathbf{\Psi}\,dx.
\end{aligned}$$

☐

Remark 1.1.20. *By a density argument, all the results proved in this section extend to the spaces defined in the following sections, we refer to Theorem 1.2.5.*

1.2 Fractional differentiability spaces

The notion of fractional differentiability plays a central role in the modern analysis of partial differential equations, especially in the study of nonlocal operators, anomalous diffusion processes, and integro-differential models.

Several functional frameworks have been developed to describe non-integer orders of smoothness, each arising from different analytical motivations: variational formulations, trace theory, interpolation theory, and pseudo-differential calculus. Among these, fractional Sobolev spaces (also known as Sobolev–Slobodeckij spaces) and Bessel potential spaces play a fundamental role.

This section is devoted to the systematic introduction of the main spaces we use to describe fractional regularity. We first focus on Sobolev–Slobodeckij spaces, which are defined through Gagliardo seminorms and naturally arise in variational formulations and trace theory. We then discuss the Hilbert case and its connection with Bessel potential spaces, which provide the natural functional setting for fractional operators defined via Fourier multipliers and pseudo-differential symbols.

1.2.1 Sobolev–Slobodeckij spaces

Among all the possible ways of defining non-integer orders of differentiability, Sobolev spaces $W^{s,p}$ for $s \in (0,1)$ and $p \in [1,\infty)$, have gained great importance in the analysis of PDEs, both in the local and non-local settings. In this section we explore the properties of fractional Sobolev spaces, which are the main tool of this work. We closely follow [26].

Definition 1.2.1 (Fractional Sobolev spaces)**.** *Let $\Omega \subset \mathbb{R}^d$, $s \in (0,1)$ and $p \in [1,\infty]$. We define the fractional Sobolev space $W^{s,p}(\Omega)$ as*

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\, dx\, dy < \infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\, dx\, dy \right)^{\frac{1}{p}}.$$

Moreover, we denote

$$|u|_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\, dx\, dy \right)^{\frac{1}{p}},$$

the Gagliardo seminorm of u.

In the case $p = \infty$, we define $W^{s,\infty}(\Omega) := C^s(\Omega)$, the space of s-Hölder continuous functions on Ω.

As in the classical case, when $s \geq t$ the space $W^{s,p}(\Omega)$ is continuously embedded into $W^{t,p}(\Omega)$. Moreover, when Ω has sufficient regularity, we also have the embedding in the asymptotic case, namely $W^{1,p}(\Omega) \subset W^{s,p}(\Omega)$.

Proposition 1.2.2 (See [26, Proposition 2.1 and Proposition 2.2]). *Let $\Omega \subset \mathbb{R}^d$ be open, $p \in [1, \infty)$ and $0 < t \leq s < 1$. Then for every measurable function $u : \Omega \to \mathbb{R}$ we have*

$$\|u\|_{W^{s,p}(\Omega)} \leq C(d, p, t, s) \|u\|_{W^{t,p}(\Omega)}.$$

In particular, $W^{s,p}(\Omega) \subset W^{t,p}(\Omega)$. Moreover, if Ω is a Lipschitz domain, then for every measurable function $u : \Omega \to \mathbb{R}$ we have

$$\|u\|_{W^{1,p}(\Omega)} \leq C(d, p, s) \|u\|_{W^{s,p}(\Omega)}.$$

The regularity of the domain in the previous proposition is very close to being optimal; see [26, Example 9.1].

A natural question is whether Definition 1.2.1 can be extended to $s \geq 1$. The answer is no. If $u : \Omega \to \mathbb{R}$ is measurable and satisfies

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \, dx \, dy < \infty$$

for some $s \geq 1$, then u must be constant.

Therefore, for $k + s \in \mathbb{R}$ with $s \in (0, 1)$ and $k \in \mathbb{N}$, we define $W^{k+s,\,p}(\Omega)$ as the space of functions in L^p whose weak derivatives of order k belong to $W^{s,\,p}(\Omega)$, that is,

$$W^{k+s,\,p}(\Omega) := \{u \in W^{k,\,p}(\Omega) : \partial_\alpha u \in W^{s,\,p}(\Omega) \text{ for every multi-index } \alpha \text{ with } |\alpha| = k\},$$

endowed with the norm

$$\|u\|_{W^{k+s,\,p}(\Omega)} := \left(\|u\|^p_{W^{k,\,p}(\Omega)} + \sum_{|\alpha|=k} \|\partial_\alpha u\|^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}}.$$

Proposition 1.2.2 extends naturally to this case. Moreover, these spaces are complete normed spaces.

Proposition 1.2.3 (See, for example, [3, Lemma 7.44]). *Let $s > 0$ and $p \in [1, \infty]$. Then $(W^{s,p}(\Omega), \|\cdot\|_{W^{s,p}(\Omega)})$ is a Banach space.*

There are several ways to define the notion of vanishing on the boundary for these spaces.

Definition 1.2.4. *Let $s > 0$. We define $W^{s,\,p}_0(\Omega)$ as the completion of $C^\infty_c(\Omega)$ with respect to the $W^{s,\,p}(\Omega)$ norm.*

We also define $\widetilde{W}^{s,p}(\Omega)$ as the set of functions in $W^{s,p}(\mathbb{R}^d)$ supported in Ω, endowed with the norm

$$\|u\|_{\widetilde{W}^{s,p}(\Omega)} := \|u\|_{W^{s,p}(\mathbb{R}^d)}.$$

Note that if $s \notin \mathbb{N}$, the norms in $W^{s,p}(\Omega)$ and in $\widetilde{W}^{s,p}(\Omega)$ do not coincide, since the former involves integration over $\mathbb{R}^d$. Nevertheless, we have the following density result.

Theorem 1.2.5 (See [37, Theorem 1.4.2.2]). *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz open set. Then $C^\infty_c(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ for all $s > 0$.*

Remark 1.2.6 (Schwartz class density). *By definition $\widetilde{W}^{s,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$. Consequently,*

$$C^\infty_c(\mathbb{R}^d) \subset \mathcal{S} \cap W^{s,p}(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d) \cap W^{s,p}(\mathbb{R}^d),$$

are all dense subsets of $W^{s,p}(\mathbb{R}^d)$.

Remark 1.2.7 (Counterexample for $sp > 1$)**.** *Let $\Omega = B(0,1) \subset \mathbb{R}^d$. Then, the spaces $W^{s,p}(\Omega)$ and $\widetilde{W}^{s,p}(\Omega)$ are different when $sp \geq 1$. Indeed, let $\chi_\Omega : \mathbb{R}^d \to \mathbb{R}$ be the characteristic function of Ω. It is clear that $\chi_\Omega \in W^{s,p}(\Omega)$. However,*

$$|\chi_\Omega|^p_{W^{s,p}(\mathbb{R}^d)} = 2 \iint_{\Omega \times \Omega^c} \frac{1}{|x-y|^{d+sp}} \, dx \, dy.$$

Now, let $\varepsilon > 0$ and define $A_\varepsilon = \{x \in \Omega : \, d(x, \partial\Omega) < \varepsilon\}$ and $B_\varepsilon = \{y \in \Omega^c : \, d(y, \partial\Omega) < \varepsilon\}$. Then,

$$\iint_{\Omega \times \Omega^c} \frac{1}{|x-y|^{d+sp}} \, dx \, dy \geq \iint_{A_\varepsilon \times B_\varepsilon} \frac{1}{|x-y|^{d+sp}} \, dx \, dy.$$

A change of variables to polar coordinates near the boundary yields

$$\iint_{A_\varepsilon \times B_\varepsilon} \frac{1}{|x-y|^{d+sp}} \, dx \, dy \simeq \int_0^\varepsilon \int_0^\varepsilon \frac{1}{(t+\sigma)^{1+sp}} \, dt \, d\sigma,$$

and this last integral diverges for $sp > 1$.

Finally, for $s < 0$ Sobolev spaces are defined by duality.

Definition 1.2.8. *Let $s < 0$, $p \in [1, \infty)$ and $q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We define $W^{s,p}(\Omega)$ as the dual of $\widetilde{W}^{-s,q}(\Omega)$, and $\widetilde{W}^{s,p}(\Omega)$ as the dual of $W^{-s,q}(\Omega)$.*

Having defined the fractional Sobolev spaces, we proceed to state classical results and properties of these spaces, many of them analogous to their local counterparts. First, whenever the domain Ω is regular enough but trace operators are unavailable, smooth functions are dense in $W^{s,p}(\Omega)$; see [37, Theorem 1.4.2.4].

Theorem 1.2.9. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then, for all $s \in (0, \frac{1}{p}]$, $W^{s,p}_0(\Omega) = W^{s,p}(\Omega)$.*

We also recover some classical inequalities regarding fractional norms.

Proposition 1.2.10 (Poincaré inequality for zero average)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $p \in (1, \infty)$ and $s \in (0,1)$. Given $u \in W^{s,p}(\Omega)$, we denote*

$$u_\Omega := \frac{1}{|\Omega|} \int_\Omega u(y) \, dy.$$

Then

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq C h_\Omega^s |u|_{W^{s,p}(\Omega)}, \tag{1.2.1}$$

where h_Ω is the diameter of Ω and C depends only on d, Ω and p.

Proof. Applying Hölder's inequality if $p > 1$, we write

$$\int_\Omega |u - u_\Omega|^p \, dx = \frac{1}{|\Omega|^p} \int_\Omega \left| \int_\Omega (u(x) - u(y)) \, dy \right|^p dx \leq \frac{1}{|\Omega|} \int_\Omega \int_\Omega |u(x) - u(y)|^p \, dy \, dx.$$

Therefore,

$$\int_\Omega |u - u_\Omega|^p \, dx \leq \frac{h_\Omega^{d+sp}}{|\Omega|} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dy \, dx.$$

Thus, the claimed identity (1.2.1) follows with $C = \left(\frac{h_\Omega^d}{|\Omega|} \right)^{\frac{1}{p}}$.

□

Proposition 1.2.11 (Poincaré inequality for zero trace). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $p \in [1, \infty)$ and $s \in (0,1)$. Given $u \in \widetilde{W}^{s,p}(\Omega)$, we have*

$$\|u\|_{L^p(\Omega)} \le C |u|_{W^{s,p}(\mathbb{R}^d)}, \tag{1.2.2}$$

where C depends only on d, Ω, s and p.

Proof. By [26, Lemma 6.1], there exists some constant $c(n, s, p) > 0$ such that for all $x \in \Omega$,

$$c(d, s, p)|\Omega|^{-\frac{sp}{d}} \le \int_{\Omega^c} \frac{1}{|x-y|^{d+sp}}\, dy.$$

On the other hand, since $u = 0$ on Ω^c we know that $|u(x)|^p = |u(x) - u(y)|^p$ for all $x \in \Omega$, $y \in \Omega^c$. So, we obtain

$$c(d, s, p)|\Omega|^{-\frac{sp}{d}} \int_\Omega |u(x)|^p\, dx \le \int_\Omega \int_{\Omega^c} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\, dx\, dy,$$

and the Poincaré inequality follows straightforwardly. □

The second inequality implies that the seminorm $|\cdot|_{\widetilde{W}^{s,p}(\Omega)}$ is in fact a norm on $\widetilde{W}^{s,p}(\Omega)$. Another well-known inequality is the following; see [37, Theorem 1.4.4.4].

Proposition 1.2.12 (Hardy inequality). *Let Ω be a bounded Lipschitz domain and $\delta(x) = d(x, \partial\Omega)$. Then, if $s - \frac{1}{p}$ is not an integer, there exists $c = c(\Omega, d, s, p) > 0$ such that*

$$\int_\Omega \frac{|\partial_\alpha v(x)|^p}{\delta(x)^{(s-|\alpha|)p}}\, dx \le c\, \|v\|^p_{W^{s,p}(\Omega)} \quad \forall\, v \in W^{s,p}_0(\Omega),\ \alpha \in \mathbb{N}^n,\ |\alpha| \le s. \tag{1.2.3}$$

As a consequence of the previous proposition, we deduce

Corollary 1.2.13. *If $s \in (0,1)$, $s \neq \frac{1}{p}$, there exist a constant $C = C(\Omega, d, p, s) > 0$ such that*

$$|v|_{W^{s,p}(\mathbb{R}^d)} \le C\|v\|_{W^{s,p}(\Omega)},$$

for all $v \in W^{s,p}_0(\Omega)$.

Proof. It is sufficient to show it for $v \in C^\infty_c(\Omega)$. We have,

$$\begin{aligned}
|v|^p_{W^{s,p}(\mathbb{R}^d)} &= \iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{d+sp}}\, dx\, dy + 2 \iint_{\Omega\times\Omega^c} \frac{|v(x)|^p}{|x-y|^{d+sp}}\, dx\, dy \\
&\le |v|_{W^{s,p}(\Omega)} + \int_\Omega |v(x)|^p \int_{B(x,\delta(x)/2)^c} \frac{1}{|x-y|^{d+sp}}\, dy\, dx \\
&\le |v|_{W^{s,p}(\Omega)} + C(d, s, p) \int_\Omega \frac{|v(x)|^p}{\delta(x)^{sp}}\, dx,
\end{aligned}$$

and the result follows by the Hardy inequality (1.2.3). □

Since $C^\infty_c(\Omega)$ is dense both in $W^{s,p}_0(\Omega)$ and $\widetilde{W}^{s,p}(\Omega)$, we deduce that $W^{s,p}_0(\Omega) = \widetilde{W}^{s,p}(\Omega)$ if $s \neq \frac{1}{p}$. This also implies that, if $0 < s < \frac{1}{p}$, then $W^{s,p}(\Omega)$, $W^{s,p}_0(\Omega)$ and $\widetilde{W}^{s,p}(\Omega)$ coincide. We summarize this discussion in the following corollary.

Corollary 1.2.14. *Let Ω be a bounded Lipschitz domain. If $s - \frac{1}{p} \notin \mathbb{N}$, we have*

$$W_0^{s,p}(\Omega) = \widetilde{W}^{s,p}(\Omega).$$

Moreover, if $0 < s < \frac{1}{p}$, then

$$W^{s,p}(\Omega) = W_0^{s,p}(\Omega) = \widetilde{W}^{s,p}(\Omega).$$

We will need compact embeddings to prove weak continuity results for the fractional operators. The next result is a fractional version of the classical Rellich-Kondrachov Theorem.

Theorem 1.2.15 (See [26, Theorem 7.1])**.** *Let $s \in (0,1)$, $p \in [1,\infty)$, $q \in [1,p]$, $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\{v_k\}_{k\in\mathbb{N}} \subset W^{s,p}(\Omega)$ be a bounded sequence. Then $\{v_k\}$ has a convergent subsequence in $L^q(\Omega)$.*

We conclude this part by addressing the consistency between the definition of fractional Sobolev spaces and the classical integer-order Sobolev spaces. That is, if the Gagliardo seminorm recovers the classical case when $s \to 0^+$ and $s \to 1^-$. Up to normalizing constants, the answer is yes. The asymptotic $s \to 1^-$ was proved in [14], while the case $s \to 0^+$ was proved in [46]. Recall the constant $\nu(d,s)$ given by (1.1.2), which satisfies

$$\nu(d,s) \sim s(s-1) \quad \text{when } s \to 0^+ \text{ or } s \to 1^-.$$

Proposition 1.2.16. *Let $v \in L^p(\Omega)$, $p \in (1,\infty)$. We have,*

$$\lim_{s\to 1^-} (1-s)\|v\|^p_{W^{s,p}(\Omega)} = C(d,p)\|v\|^p_{W^{1,p}(\Omega)},$$

with the convention that $\|v\|_{W^{1,p}(\Omega)} = \infty$ if $v \notin W^{1,p}(\Omega)$. Furthermore, if there exist $s_0 >$ such that $v \in W^{s_0,p}(\mathbb{R}^d)$, then

$$\lim_{s\to 0^+} s\|v\|^p_{W^{s,p}(\Omega)} = C(d,p)\|v\|^p_{L^p(\mathbb{R}^d)}.$$

1.2.2 The Hilbert case and Bessel potential spaces

We dedicate this section to study the space $W^{s,2}(\Omega)$. In this case, we obtain a Hilbert space with the inner product

$$(u,v) := \langle u,v\rangle_{L^2(\Omega)} + \langle u,v\rangle_{W^{s,2}(\Omega)},$$

where

$$\langle u,v\rangle_{W^{s,2}(\Omega)} := \iint_{\Omega\times\Omega} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,dy\,dx.$$

For $s = k + \sigma$, with $k \in \mathbb{N}$ and $\sigma \in (0,1)$, we define

$$(u,v) := \sum_{|\alpha|\le k-1} \langle \partial_\alpha u, \partial_\alpha v\rangle_{L^2(\Omega)} + \sum_{|\alpha|=k} \langle \partial_\alpha u, \partial_\alpha v\rangle_{W^{\sigma,2}(\Omega)}.$$

In the Hilbert setting $p = 2$, the variational structures associated with the fractional operators admit a natural functional framework given by the fractional Sobolev spaces. This situation changes substantially when considering integrability exponents $p \neq 2$. In that case, the natural functional spaces associated with fractional pseudo-differential operators are no longer Sobolev spaces, but rather the so-called *Bessel potential spaces*. These spaces provide the correct L^p-framework for fractional operators defined via Fourier multipliers, and coincide with Sobolev spaces only in the Hilbert case $p = 2$.

We now introduce these spaces and summarize their main properties.

Definition 1.2.17. *Let $s \geq 0$ and $p \in (1, \infty)$. We define*

$$H^{s,p}(\mathbb{R}^d) = \{u \in L^p(\mathbb{R}^d) : \mathscr{F}^{-1}((1 + 4\pi^2|\xi|^2)^{\frac{s}{2}} \mathscr{F}u) \in L^p(\mathbb{R}^d)\},$$

endowed with the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^d)} = \|\mathscr{F}^{-1}((1 + 4\pi^2|\cdot|^2)^{\frac{s}{2}} \mathscr{F}u)\|_{L^p(\mathbb{R}^d)}.$$

The next theorem gives a summary of the properties of this spaces.

Theorem 1.2.18 (See [3, Theorem 7.63, p.221])**.** *Let $s \geq 0$ and $p \in (1, \infty)$. It holds,*

1. *$C^\infty(\mathbb{R}^d)$ is dense in $H^{s,p}(\mathbb{R}^d)$.*
2. *If $t < s$, then $H^{s,p}(\mathbb{R}^d) \hookrightarrow H^{t,p}(\mathbb{R}^d)$, for all $p > 1$.*
3. *If $t \leq s$ and $1 < p \leq q \leq \dfrac{dp}{d - (s-t)p}$, then*
$$H^{s,p}(\mathbb{R}^d) \hookrightarrow H^{t,q}(\mathbb{R}^d).$$
4. *If $0 \leq \mu \leq s - \dfrac{d}{p} < 1$, then*
$$H^{s,p}(\mathbb{R}^d) \hookrightarrow C^{0,\mu}(\mathbb{R}^d).$$
5. *If $1 < p < \infty$ and $\varepsilon > 0$, then for every s we have*
$$H^{s+\varepsilon,p}(\mathbb{R}^d) \hookrightarrow W^{s,p}(\mathbb{R}^d) \hookrightarrow H^{s-\varepsilon,p}(\mathbb{R}^d).$$

The fractional Laplacian links Sobolev spaces with integrability index $p = 2$, and Bessel potential spaces.

Proposition 1.2.19 (See [26, Proposition 3.3])**.** *Let $s \in (0, 1)$, then*

$$|v|^2_{W^{s,2}(\mathbb{R}^d)} = \frac{2(2\pi)^{2s}}{\nu(d,s)} \||\cdot|^s \hat{v}\|^2_{L^2(\mathbb{R}^d)} = \frac{2}{\nu(d,s)} \|(-\Delta)^{\frac{s}{2}} v\|^2_{L^2(\mathbb{R}^d)} \tag{1.2.4}$$

for all $v \in W^{s,2}(\mathbb{R}^d)$. In particular,

$$W^{s,2}(\mathbb{R}^d) = H^{s,2}(\mathbb{R}^d).$$

Proof. We begin by showing the first equality in (1.2.4). Plancherel's formula combined with the change of variables $z = x - y$ yields

$$\begin{aligned}
|v|^2_{W^{s,2}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(x) - v(x-z)|^2}{|z|^{d+2s}}\,dxdz \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|v(x) - (\tau_z v)(x)|^2}{|z|^{d+2s}}\,dxdz \\
&= \int_{\mathbb{R}^d} \left\| \frac{v - \tau_z v}{|z|^{\frac{d}{2}+s}} \right\|^2_{L^2(\mathbb{R}^d)} dz \\
&= \int_{\mathbb{R}^d} \left\| \widehat{\left(\frac{v - \tau_z v}{|z|^{\frac{d}{2}+s}} \right)} \right\|^2_{L^2(\mathbb{R}^d)} dz.
\end{aligned}$$

Recall the identity $1 - \cos(\theta) = \frac{1}{2}|1 - e^{-i\theta}|^2$ for all $\theta \in \mathbb{R}$. By properties of the Fourier transform (c.f. Proposition 1.1.3), we have

$$\begin{aligned}
|v|^2_{W^{s,2}(\mathbb{R}^d)} &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|1 - e^{-2\pi i\xi\cdot z}|^2}{|z|^{d+2s}} |\hat{v}(\xi)|^2 d\xi dz \\
&= 2\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1 - \cos(2\pi\xi\cdot z)}{|z|^{d+2s}} |\hat{v}(\xi)|^2 d\xi dz.
\end{aligned}$$

Therefore, if we show

$$\int_{\mathbb{R}^d} \frac{1 - \cos(2\pi\xi\cdot z)}{|z|^{d+2s}}\,dz = \frac{|2\pi\xi|^{2s}}{\nu(d,s)},$$

the first identity follows. Define $f(\xi) = \int_{\mathbb{R}^d} \frac{1-\cos(2\pi\xi\cdot z)}{|z|^{d+2s}}\,dz$. Suitable changes of variable show

$$f(Q\xi) = f(\xi) \quad \forall\, \xi \in \mathbb{R}^d,\, Q \in O(d), \qquad f(\alpha\xi) = \alpha^{2s} f(\xi) \quad \forall\, \alpha > 0,\, \xi \in \mathbb{R}^d.$$

These properties, combined with (1.1.5), imply

$$f(\xi) = |\xi|^{2s} f(e_1) = |\xi|^{2s} \int_{\mathbb{R}^d} \frac{1 - \cos(2\pi z_1)}{|z|^{d+2s}}\,dz = \frac{|2\pi\xi|^{2s}}{\nu(d,s)}.$$

The second equality in (1.2.4) follows from the Fourier symbol of the fractional Laplacian; c.f. formula (1.1.4).

Lastly, the first equality and the relation $(1 + |\xi|^2)^s \simeq 1 + |\xi|^{2s}$, imply

$$\|u\|^2_{W^{s,2}(\mathbb{R}^d)} \simeq \int_{\mathbb{R}^d} (1 + |\xi|^{2s})\hat{u}(\xi)\,d\xi \simeq \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \hat{u}(\xi)\,d\xi \simeq \|u\|_{H^{s,2}(\mathbb{R}^d)},$$

and therefore the equivalence of spaces follows. □

The previous equivalence relies heavily on Plancherel's formula. It is a known fact that the Fourier transform is not an isomorphism between $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ unless $p = q = 2$. So, in general, the Bessel potential spaces do not coincide with the Sobolev spaces.

By Theorem 1.1.19, we obtain that

$$\int_{\mathbb{R}^d} w\,(-\Delta)^s v\,dx = \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}} w\,(-\Delta)^{\frac{s}{2}} v\,dx = \int_{\mathbb{R}^d} (-\Delta)^s w\,v\,dx.$$

Thus, as an immediate consequence of Proposition 1.2.19, we deduce.

Corollary 1.2.20. *Let Ω be an open set and let $u, v \in C_c^\infty(\Omega)$. Then,*

$$\frac{\nu(d,s)}{2}\langle u, v\rangle_{W^{s,2}(\mathbb{R}^d)} = \langle(-\Delta)^s u, v\rangle_{L^2(\Omega)}. \tag{1.2.5}$$

Remark 1.2.21 (Extension to $\widetilde{W}^{s,2}(\Omega)$)**.** *Via a density argument we can extend the previous corollary to the space $\widetilde{W}^{s,2}(\Omega)$, c.f. Theorem 1.3.5.*

Remark 1.2.22 (A more general integration by parts formula)**.** *The previous proposition is a particular case of [27, Lemma 3.3]. The general result states that, given sufficiently smooth functions $u, v : \mathbb{R}^d \to \mathbb{R}$, one has*

$$\frac{\nu(d,s)}{2}\iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,dy\,dx = \int_\Omega v(x)(-\Delta)^s u(x)\,dx + \int_{\mathbb{R}^d\setminus\Omega} v(x)\mathcal{N}_s u(x)\,dx,$$

where $Q = (\mathbb{R}^d\times\mathbb{R}^d)\setminus(\Omega^c\times\Omega^c)$ and $\mathcal{N}_s$ is the so-called nonlocal Neumann derivative *operator introduced in [27]:*

$$\mathcal{N}_s u(x) := \nu(d,s)\int_\Omega \frac{u(x)-u(y)}{|x-y|^{d+2s}}\,dy, \quad x \in \mathbb{R}\setminus\overline{\Omega}.$$

For the purposes of this thesis, Corollary 1.2.20 is sufficient.

The previous Corollary shows how to write weak formulations for problems involving the fractional Laplacian. For this purpose, given $\Omega \subset \mathbb{R}^d$ an open set, we define the spaces

$$H^s(\Omega) := W^{s,2}(\Omega), \quad \widetilde{H}^s(\Omega) := \widetilde{W}^{s,2}(\Omega), \quad H_0^s(\Omega) := W_0^{s,2}(\Omega),$$

and the norm

$$\|u\|_{H^s(\Omega)} := \|u\|_{L^2(\Omega)} + |u|_{H^s(\Omega)},$$

with

$$|u|_{H^s(\Omega)} = \sqrt{\frac{\nu(d,s)}{2}}|u|_{W^{s,2}(\Omega)}.$$

Due to the Poincaré inequality (1.2.2) and Corollary 1.2.20, the $H^s(\mathbb{R}^d)$ seminorm is a norm in $\widetilde{H}^s(\Omega)$. With these definitions at hand, we consider the fractional Poisson problem in Ω, namely

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c := \mathbb{R}^d\setminus\Omega. \end{cases} \tag{P}$$

Weak solutions of (P) are defined by multiplying the equation by a test function and using the integration by parts formula (1.2.5). That is, the weak formulation of problem (P) is:

$$\text{find } u \in \widetilde{H}^s(\Omega) \text{ such that } \langle u, v\rangle_{H^s(\mathbb{R}^d)} = \int_\Omega fv\,dx \quad \forall v \in \widetilde{H}^s(\Omega).$$

A direct application of the Lax-Milgram theorem yields the well-posedness of the fractional Poisson problem.

Proposition 1.2.23. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $f \in H^{-s}(\Omega)$. Then, Problem* (P) *is well posed: there exists a unique solution $u \in \widetilde{H}^s(\Omega)$, and the following stability estimate holds*

$$|u|_{H^s(\mathbb{R}^d)} \le \|f\|_{H^{-s}(\Omega)}.$$

Proof. As noted above, the $H^s(\mathbb{R}^d)$ seminorm defines a norm on $\widetilde{H}^s(\Omega)$; hence coercivity follows. Continuity follows from the Cauchy–Schwarz inequality. □

1.3 Connections between Fractional vector calculus operators, Sobolev spaces and Bessel potential spaces

Having defined the fractional calculus operators and related variational spaces, we finish the chapter by establishing several connections that will be useful throughout this work, and by extending some of the definitions and results proved in 1.1 from smooth functions to functions in H^s.

1.3.1 Characterization of Bessel potential spaces via fractional gradients

Let $w \in C_c^\infty(\mathbb{R}^d)$. By formula (1.2.4) and Theorem 1.1.8, we deduce

$$|w|_{H^s(\mathbb{R}^d)} = \|\nabla^s w\|_{L^2(\mathbb{R}^d)}.$$

Therefore, Proposition 1.2.19 can be restated as follows.

Corollary 1.3.1. *Let $s \in (0,1)$. Then, the operator ∇^s extends to a continuous operator in $H^s(\mathbb{R}^d)$ such that*

$$\widehat{\nabla^s w}(\xi) = (2\pi)^s i|\xi|^{s-1}\xi\widehat{w}(\xi),$$

for all $w \in H^s(\mathbb{R}^d)$. Moreover, it holds

$$H^s(\mathbb{R}^d) = \{w \in L^2(\mathbb{R}^d) : \|\nabla^s w\|_{L^2(\mathbb{R}^d)} < \infty\},$$

with equivalence of inner products given by

$$\langle w, v\rangle_{H^s(\mathbb{R}^d)} = \langle \nabla^s w, \nabla^s v\rangle_{L^2(\mathbb{R}^d)} = \langle (-\Delta)^{\frac{s}{2}} w, (-\Delta)^{\frac{s}{2}} v\rangle_{L^2(\mathbb{R}^d)}.$$

Remark 1.3.2. *By linearity and Proposition 1.1.10, the operators* div^s *and* $\mathbf{curl}^s$ *also extend to continuous operators in $[H^s(\mathbb{R}^d)]^d$.*

Again, we would like to use Plancherel's formula to deduce a relation like

$$\|\nabla^s w\|_{L^p(\mathbb{R}^d)} \simeq |w|_{W^{s,p}(\mathbb{R}^d)} \quad \text{for } p \neq 2,$$

but we have the same problem as before. In fact, the previous equivalence of semi-norms is false in general, and is a consequence of the final result of this section: the finiteness of $\|\nabla^s w\|_{L^p(\mathbb{R}^d)}$ characterizes the Bessel potential spaces, which, do not coincide with the Sobolev-Slobodeckij spaces unless $p = 2$.

Definition 1.3.3. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Define the norm*

$$\|w\|_{X^{s,p}(\mathbb{R}^d)} := \|w\|_{L^p(\mathbb{R}^d)} + \|\nabla^s w\|_{L^p(\mathbb{R}^d)}.$$

We define

$$X^{s,p}(\mathbb{R}^d) := \overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_{X^{s,p}(\mathbb{R}^d)}}.$$

To define ∇^s on H^s-functions, we are extending it by density. Therefore,

$$X^{s,2}(\mathbb{R}^d) = \{w \in L^2(\mathbb{R}^d) : \|\nabla^s w\|_{L^2(\mathbb{R}^d)} < \infty\},$$

thus, by Corollary 1.3.1,

$$H^{s,2}(\mathbb{R}^d) = X^{s,2}(\mathbb{R}^d).$$

In fact, this result holds true for all $p \in (1,\infty)$; see [52, Theorem 1.7].

Theorem 1.3.4. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Then,*

$$H^{s,p}(\mathbb{R}^d) = X^{s,p}(\mathbb{R}^d).$$

1.3.2 Mapping properties and some continuity results

We proved that the operators defined in Section 1.1 are pseudo-differential operators and we found their symbols. Therefore, from Corollary 1.3.1, we obtain the following mapping properties.

Theorem 1.3.5. *Let $s, t \geq 0$. Then:*

(i) The operator $\nabla^s : H^{t+s}(\mathbb{R}^d) \to [H^t(\mathbb{R}^d)]^d$ is continuous.

(ii) The operator $\operatorname{div}^s : [H^{t+s}(\mathbb{R}^d)]^d \to H^t(\mathbb{R}^d)$ is continuous.

(iii) For $d = 2$ or $d = 3$, the operator $\mathbf{curl}^s : [H^{t+s}(\mathbb{R}^d)]^d \to [H^t(\mathbb{R}^d)]^{d(d-1)/2}$ is continuous.

(iv) The fractional Laplacian $(-\Delta)^s : H^{t+2s}(\mathbb{R}^d) \to H^t(\mathbb{R}^d)$ is continuous.

Proof. First, we prove (i). Given $u \in H^{t+s}(\mathbb{R}^d)$ we use Corollary 1.3.1 to deduce

$$\begin{aligned} |\nabla^s u|_{H^r(\mathbb{R}^d)} &= \|\nabla^r(\nabla^s u)\|_{L^2(\mathbb{R}^d)} \\ &= \|(2\pi)^r |\cdot|^r \widehat{\nabla^s u}\|_{L^2(\mathbb{R}^d)} \\ &= \|(2\pi)^{r+s} |\cdot|^{r+s} \widehat{u}\|_{L^2(\mathbb{R}^d)} \\ &= |u|_{H^{s+r}(\mathbb{R}^d)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\nabla^s u\|^2_{L^2(\mathbb{R}^d)} &= 2\pi \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{u}(\xi)|^2 \, d\xi \\ &\leq 2\pi \left(\int_{B_1} |\widehat{u}(\xi)|^2 \, d\xi + \int_{B_1^c} |\xi|^{2(s+r)} |\widehat{u}(\xi)|^2 \, d\xi \right) \\ &\leq 2\pi \left(\|u\|^2_{L^2(\mathbb{R}^d)} + |u|^2_{H^{s+r}(\mathbb{R}^d)} \right). \end{aligned}$$

The proof of (ii), (iii) and (iv) follow analogously from the Fourier symbol of div^s, $\mathbf{curl}^s$ and $(-\Delta)^s$ respectively. □

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. It is a classical result that a sequence $\{v_k\} \subset W^{1,p}(\Omega)$ converges weakly to $v \in W^{1,p}(\mathbb{R}^d)$ if and only if $v_k \to v$ in $L^p(\Omega)$ and $\nabla v_k \rightharpoonup \nabla v$ in $L^p(\Omega)$. We want to extend this result to the fractional setting. Recall that, thanks to the Poincaré inequality (1.2.2), the seminorm $|\cdot|_{W^{s,p}(\mathbb{R}^d)}$ is a norm in $\widetilde{W}^{s,p}(\Omega)$ equivalent to $\|\cdot\|_{W^{s,p}(\Omega)}$. First, as a consequence of Theorem 1.2.15, we deduce the following.

Corollary 1.3.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$ and $p \in [1,\infty)$. Then, $\widetilde{W}^{s,p}(\Omega)$ is compactly embedded in $L^p(\mathbb{R}^d)$.*

Proof. The result follows immediately from the bound

$$\|w\|_{W^{s,p}(\Omega)} \leq C |w|_{W^{s,p}(\mathbb{R}^d)}$$

for all $w \in \widetilde{W}^{s,p}(\Omega)$ (c.f. the Poincaré inequality (1.2.2)). Let $\{w_k\} \subset \widetilde{W}^{s,p}(\Omega)$ be a bounded sequence. By the previous estimate and Theorem 1.2.15, the sequence $\{w_k|_\Omega\}$ has a convergent subsequence in $L^p(\Omega)$. The result then follows by extending the limit by zero to $L^p(\mathbb{R}^d)$ and recalling that functions in $\widetilde{W}^{s,p}(\Omega)$ are supported in Ω. $\square$

Proposition 1.3.7. *Let $\{v_k\} \subset \widetilde{H}^s(\Omega)$. Then $v_k \rightharpoonup v$ in $\widetilde{H}^s(\Omega)$ if and only if*

$$v_k \to v \quad \text{in } L^2(\mathbb{R}^d) \qquad \text{and} \qquad \nabla^s v_k \rightharpoonup \nabla^s v \quad \text{in } L^2(\mathbb{R}^d).$$

Proof. Notice that, using the identity

$$\int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} w \, dx = \int_{\mathbb{R}^d} \nabla^s u \cdot \nabla^s w \, dx,$$

it is clear that $v_k \rightharpoonup v$ in $H^s(\mathbb{R}^d)$ if and only if

$$v_k \rightharpoonup v \quad \text{in } L^2(\mathbb{R}^d) \qquad \text{and} \qquad \nabla^s v_k \rightharpoonup \nabla^s v \quad \text{in } L^2(\mathbb{R}^d).$$

First, suppose that $v_k \to v$ in $L^2(\mathbb{R}^d)$ and $\nabla^s v_k \rightharpoonup \nabla^s v$ in $L^2(\mathbb{R}^d)$. Therefore, $v_k \rightharpoonup v$ in $H^s(\mathbb{R}^d)$ and the strong convergence in L^2 implies $\operatorname{supp} v \subset \Omega$, thus $v_k \rightharpoonup v$ in $\widetilde{H}^s(\Omega)$.

Now, assume that $v_k \rightharpoonup v$ in $\widetilde{H}^s(\Omega)$. Again, we know that $\nabla^s v_k \rightharpoonup \nabla^s v$ and $v_k \rightharpoonup v$ in $L^2(\mathbb{R}^d)$. If $\{v_k\}$ does not converge to v in $L^2(\mathbb{R}^d)$, there exists a subsequence $\{v_{k_j}\}$ and $\delta > 0$ such that

$$\|v_{k_j} - v\|_{L^2(\mathbb{R}^d)} > \delta \quad \forall j \in \mathbb{N}.$$

By Corollary 1.3.6, there exists a further subsequence that converges in $L^2(\mathbb{R}^d)$, necessarily to a different limit. This yields a contradiction, since $v_{k_j} \rightharpoonup v$ in $L^2(\mathbb{R}^d)$. $\square$

The aim of Chapter 3 is to study a fractional version of the Oseen–Frank model, which in the local case consists in minimizing the Oseen–Frank energy

$$E(v) = \frac{1}{2} \int_\Omega k_1 (\operatorname{div} v)^2 + k_2 (v \cdot \mathbf{curl}\, v)^2 + k_3 |v \times \mathbf{curl}\, v|^2 + (k_2 + k_4)(\operatorname{tr}([\nabla v]^2) - (\operatorname{div} v)^2) \, dx,$$

subject to the unit-length constraint $|v| = 1$. The four constants k_i are known as Frank's constants, and correspond to splay (k_1), twist (k_2), bend (k_3), and saddle-splay ($k_2 + k_4$); more details in Chapter 3. The unit-length constraint is preserved under weak limits thanks to the previous proposition. To establish the existence of minimizers in the fractional setting, we require weak continuity (or weak lower continuity) of all terms appearing in the energy. The splay term and the saddle-splay term can be dealt with Theorem 1.3.5. We shall now treat the twist term and the bend term.

Lemma 1.3.8. *Let $\{u_k\}, \{v_k\} \subset L^2(\mathbb{R}^d)$ such that $|u_k| \le C < \infty$ for all $k \in \mathbb{N}$, $v_k \rightharpoonup v$ in $L^2(\mathbb{R}^d)$ and $u_k \to u$ in $L^2(\mathbb{R}^d)$. Then, $v_k u_k$ converges weakly to vu in $L^2(\mathbb{R}^d)$.*

Proof. Let $w \in L^\infty(\mathbb{R}^d)$. Then, we have

$$\int_{\mathbb{R}^d} w u_k v_k \, dx = \int_{\mathbb{R}^d} w u v_k \, dx + \int_{\mathbb{R}^d} w (u_k - u) v_k \, dx \to \int_{\mathbb{R}^d} w u v \, dx,$$

as $k \to \infty$. Indeed, the convergence follows by observing that

$$\left| \int_{\mathbb{R}^d} w (u_k - u) v_k \, dx \right| \le \|w\|_{L^\infty(\mathbb{R}^d)} \|u - u_k\|_{L^2(\mathbb{R}^d)} \|v_k\|_{L^2(\mathbb{R}^d)},$$

and that $\{v_k\}$ is bounded in $L^2(\mathbb{R}^d)$. This shows that $v_k u_k$ converges weakly to vu in $L^1(\mathbb{R}^d)$. Combining this with the fact that $v_k u_k$ is bounded in L^2 (since $\{u_k\}$ is bounded in $L^\infty(\mathbb{R}^d)$), we obtain the weak convergence in $L^2(\mathbb{R}^d)$. □

Proposition 1.3.7 and Lemma 1.3.8 allow us to deduce the weak continuity of the splay and saddle-splay terms.

Lemma 1.3.9. *Let $\{v_k\} \subset [\widetilde{H}^s(\Omega)]^d$ be such that $|v_k| \leq C < \infty$ for all $n \in \mathbb{N}$ and $v_k \rightharpoonup v$ in $[H^s(\mathbb{R}^d)]^d$. Then we have,*

$$\begin{aligned} \mathrm{div}^s v_k &\rightharpoonup \mathrm{div}^s v, \\ \mathbf{curl}^s v_k &\rightharpoonup \mathbf{curl}^s v, \\ v_k \cdot \mathbf{curl}^s v_k &\rightharpoonup v \cdot \mathbf{curl}^s v, \\ v_k \times \mathbf{curl}^s v_k &\rightharpoonup v \times \mathbf{curl}^s v, \end{aligned}$$

as $k \to \infty$, all in $L^2(\mathbb{R}^d)$.

Proof. The first two assertions are a direct consequence of Theorem 1.3.5. The remaining ones follow from Lemma 1.3.8 together with the fact that, by Proposition 1.3.7, $v_k \to v$ in $L^2(\mathbb{R}^d)$. □

Chapter 2

Mixed Problems and the Fractional Darcy Problem

We call a problem a *mixed problem* if it consists of a system of PDEs in which two function spaces are used to obtain two different solution variables. In many cases, the second variable is introduced to obtain additional information about the physical system under study and is usually related to derivatives of the original variable. For instance, in the classical Poisson problem, it is possible to introduce the gradient of the solution as a new variable and thus obtain a first–order system of PDEs. From the point of view of flow problems in porous media (or Darcy problems), this corresponds to introducing the superficial velocity of the fluid as an additional variable to the pressure.

In Section 2.1 we develop a general framework for the analysis of mixed formulations and saddle–point problems. We begin by studying the finite–dimensional setting, where the main ideas can be presented in a simple algebraic context and where the structure arising from constrained optimization becomes transparent. We then extend the analysis to Hilbert spaces, introducing the abstract mixed formulation and the associated operator framework. Finally, we establish necessary and sufficient conditions for well–posedness in the sense of Hadamard, leading to the classical inf–sup (Babuška–Brezzi) condition and the corresponding stability estimates that will be used throughout the analysis of the fractional Darcy problem defined in Section 2.2.

In Section 2.2 we develop a mixed formulation of the fractional Poisson problem (P) that allows for the simultaneous approximation of the solution and its fractional gradient. After introducing the underlying nonlocal vector calculus framework, we prove well-posedness of the formulation and construct a stabilized variational version amenable to standard finite element discretizations. We conclude by establishing Sobolev regularity of the solutions, which are consequence of the Sobolev regularity of solutions to the fractional Poisson problem and the mapping properties of the fractional gradient.

Sections 2.4 and 2.5 are devoted to the finite element discretization of the fractional Darcy problem. In Section 2.4 we describe the employed finite element framework, some technical results on interpolation are derived and a priori error bounds are shown. We finish this chapter with Section 2.5, where several several numerical experiments that illustrate our theoretical findings are displayed.

The material presented in Sections 2.2, 2.3, 2.4 and 2.5 is largely based on [9].

2.1 Mixed problems in Hilbert spaces

In this section, we provide a sufficiently general framework for the mixed problems studied in this thesis that guarantees well-posedness. For our purposes, Hilbert space theory is sufficient, although we remark that the ideas can be easily adapted to the Banach space setting. The first analysis of this type of problems in a more general setting was developed in [15], where the inf–sup condition or the Babuška–Brezzi condition was formulated for the first time (see Theorem 2.1.8). Also notable are the works [5] and [24], where many of the main ideas of the field are introduced in the context of specific problems.

2.1.1 Saddle-Point problems in finite dimension

Mixed formulations in partial differential equations and constrained variational problems typically lead to linear systems with a *saddle-point structure*

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{2.1.1}$$

where A and B are linear operators between function spaces. In this section we study the finite-dimensional case, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$ are real matrices, and $f, x \in \mathbb{R}^n$ and $g, y \in \mathbb{R}^m$ are vectors. In the next section we analyze the case in which A and B are operators between Hilbert spaces, aiming to generalize the results presented here. Apart from the benefit of viewing the results in a more elementary context than that of Hilbert spaces, discretizations of mixed problems usually require solving a problem of the form (2.1.1) in finite dimension.

We seek conditions that guarantee the well-posedness of problem (2.1.1), namely existence, uniqueness of solutions, and stability with respect to the parameters. The goal of this section is therefore to find necessary and sufficient conditions on the matrices A and B such that:

1. The matrix
$$M = \begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$$
is invertible.

2. There exists a constant $c > 0$ such that if x, y, f, g satisfy (2.1.1), then
$$|x| + |y| \le c(|f| + |g|).$$
We call this property *stability*.

Note that stability implies the invertibility of M. Indeed, if $f = 0$, $g = 0$, and stability holds, then necessarily $x = 0$ and $y = 0$. We postpone the discussion of stability for problem (2.1.1) to the general case, i.e., when A and B are operators between Hilbert spaces of arbitrary dimension. We refer to Theorem 2.1.8.

Remark 2.1.1 (Why saddle point?). *Problem* (2.1.1) *arises in the context of constrained optimization. In this case, the variable y is the Lagrange multiplier associated with the constraint $Bx = g$. More precisely, consider the following constrained minimization problem:*

$$\min_{x\in\mathbb{R}^n,\, Bx=g} J(x), \tag{2.1.2}$$

for $J(x) = \frac{1}{2}x \cdot Ax - f \cdot x$*. Define the Lagrangian* $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *by*

$$\mathcal{L}(x, y) = J(x) + y \cdot (Bx - g) = \frac{1}{2}x \cdot Ax - f \cdot x + y \cdot (Bx - g).$$

It is a classical result that $x^* \in \mathbb{R}^n$ *is a solution of the minimization problem* (2.1.2) *if and only if there exists* $y^* \in \mathbb{R}^m$ *such that* (x^*, y^*) *is a saddle point of the Lagrangian* $\mathcal{L}$*.*

The stationary points of $\mathcal{L}$ *satisfy*

$$\nabla \mathcal{L}(x, y) = \begin{pmatrix} Ax - f + B^t y \\ Bx - g \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix} = 0.$$

Therefore, $x^* \in \mathbb{R}^n$ *is a solution of the minimization problem* (2.1.2) *if and only if there exists* $y^* \in \mathbb{R}^m$ *such that* (x^*, y^*) *is a solution of* (2.1.1)*.*

The matrix

$$M = \begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix}$$

is invertible if and only if it is injective, i.e., if when $f = 0$ and $g = 0$ the problem (2.1.1) has the unique solution $x = 0$ and $y = 0$. Let us then assume that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ satisfy

$$\begin{aligned} &Ax + B^t y = 0, \\ &Bx = 0. \end{aligned}$$

Projecting onto $\mathrm{Ker}B$ in the first equation and recalling that $\mathrm{Ker}B = [\mathrm{Im}B^t]^\perp$, we obtain

$$\pi_{\mathrm{Ker}B} Ax = 0.$$

Moreover, the second equation implies that $x \in \mathrm{Ker}B$. Thus, we deduce that if $\pi_{\mathrm{Ker}B} A|_{\mathrm{Ker}B} : \mathrm{Ker}B \to \mathrm{Ker}B$ is injective, then $x = 0$.

On the other hand, if $x = 0$, then $B^t y = 0$. Hence, if B^t is injective (equivalently, if B is surjective), we deduce that $y = 0$. In fact, these two conditions are equivalent to the existence and uniqueness of solutions of the system (2.1.1).

Theorem 2.1.2. *Let* $A \in \mathbb{R}^{n \times n}$ *and* $B \in \mathbb{R}^{m \times n}$ *be matrices. The matrix*

$$M = \begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix}$$

is invertible if and only if the following conditions hold:

$$\begin{aligned} &\pi_{\mathrm{Ker}B} A|_{\mathrm{Ker}B} : \mathrm{Ker}B \to \mathrm{Ker}B \text{ is an isomorphism}, \\ &B : \mathbb{R}^n \to \mathbb{R}^m \text{ is surjective.} \end{aligned}$$

Proof. The converse was shown in the previous discussion. Suppose that M is invertible. Let $g \in \mathbb{R}^m$. Then there exist unique $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix},$$

which implies the surjectivity of B.

To prove the injectivity of $\pi_{\mathrm{Ker}B}A|_{\mathrm{Ker}B}$, observe that if $\pi_{\mathrm{Ker}B}Ax = 0$, then $Ax \in [\mathrm{Ker}B]^{\perp} = \mathrm{Im}B^t$, so there exists $y \in \mathbb{R}^m$ such that $Ax = -B^t y$. Since $x \in \mathrm{Ker}B$, the pair (x, y) satisfies

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which implies $x = 0$. □

Usually, verifying that the map $\pi_{\mathrm{Ker}B}A|_{\mathrm{Ker}B}$ is invertible simplifies in specific contexts.

Corollary 2.1.3. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$ be matrices. If B is surjective and $x \cdot Ax > 0$ for all $x \in \mathrm{Ker}B$, then M is invertible.*

Proof. The corollary follows directly from Theorem 2.1.2, noting that $x \cdot Ax > 0$ for all $x \in \mathrm{Ker}B$ if and only if $\pi_{\mathrm{Ker}B}A|_{\mathrm{Ker}B}$ is positive definite. Indeed, for $x \in \mathrm{Ker}B$ we have

$$x \cdot Ax = \pi_{\mathrm{Ker}B}x \cdot \pi_{\mathrm{Ker}B}Ax + \pi_{\mathrm{Ker}B}x \cdot (Ax - \pi_{\mathrm{Ker}B}Ax) = \pi_{\mathrm{Ker}B}x \cdot \pi_{\mathrm{Ker}B}Ax.$$

□

2.1.2 Mixed formulations in Hilbert spaces

In this section we fix notation and define the *mixed formulation* on a Hilbert space of arbitrary dimension.

Let H and K be Hilbert spaces, and let $b : H \times K \to \mathbb{R}$ be a bilinear form. We say that b is *continuous* if there exists $\mu > 0$ such that

$$b(p, \Phi) \leq \mu \|p\|_H \|\Phi\|_K,$$

for all $p \in H$ and $\Phi \in K$. In this case, we define its operator norm $\|b\| = \|b\|_{H'\times K'}$ by

$$\|b\| := \sup_{p\in H,\, \Phi\in K} \frac{b(p, \Phi)}{\|p\|_H \|\Phi\|_K}.$$

Recall that b induces a linear map $B : K \to H'$, defined by

$$b(p, \Phi) = \langle p, B\Phi\rangle_{H\times H'}. \tag{2.1.3}$$

Conversely, any linear operator $B : K \to H'$ defines a bilinear form via (2.1.3). Analogously, one defines $B^t : H \to K'$, and we have

$$b(p, \Phi) = \langle p, B\Phi\rangle_{H'\times H} = \langle B^t p, \Phi\rangle_{K'\times K}.$$

The continuity of b and B are equivalent, with $\|b\| = \|B\|$.

Recall that given a subspace W of a Hilbert space H, we define W^0, the *polar space* of W, as

$$W^0 := \{u \in H' : \langle u, v\rangle = 0 \ \forall v \in W\}.$$

This definition is meaningful in Banach spaces, since in Hilbert spaces one deduces

$$W^0 = R_H(W^{\perp}), \tag{2.1.4}$$

where $R_H : H \to H'$ is the isomorphism given by the Riesz representation theorem, from which one deduces

$$(W^0)^0 = \overline{W}. \tag{2.1.5}$$

Moreover, one has the identities

$$(\mathrm{Im}B)^0 = \mathrm{Ker}B^t, \quad (\mathrm{Im}B^t)^0 = \mathrm{Ker}B.$$

Before defining the mixed formulation, we recall the following classical functional analysis result. The latter is a corollary of the Banach's closed range theorem, whose proof can be found, for example, in [7, Theorem 4.1.5].

Theorem 2.1.4 (Corollary of Banach's closed range theorem)**.** *Let H and K be Hilbert spaces, and let $B : K \to H'$ be a continuous linear operator. The following are equivalent:*

1. *B is surjective,*
2. *$\mathrm{Im}(B^t)$ is closed in K' and B^t is injective,*
3. *There exists $\beta > 0$ such that $\|B^t p\|_{K'} \geq \beta \|p\|_H$ for all $p \in H$.*

Proof. The Banach's closed range theorem states that

$$\mathrm{Im}B \text{ is closed } \textit{if and only if } \mathrm{Im}B^t \text{ is closed.}$$

Combined with the identities $(\mathrm{Im}B)^0 = \mathrm{Ker}B^t$, $(\mathrm{Im}B^t)^0 = \mathrm{Ker}B$ and (2.1.5), we immediately obtain the equivalence of (1) and (2).

Now, assume (3). Then B^t is injective. Let $\{B^t q_n\} \subset \mathrm{Im}(B^t)$ be a Cauchy sequence. Then

$$\|q_n - q_m\|_H \leq \beta^{-1} \|B^t(q_n - q_m)\|_{K'}, \quad \forall n, m \in \mathbb{N},$$

so $\{q_n\}$ is Cauchy and converges to some $q \in H$. By continuity, $B^t q_n \to B^t q$, proving that $\mathrm{Im}(B^t)$ is closed.

Finally, assume (2). Then $B^t : K \to \mathrm{Im}(B^t)$ is an isomorphism of Hilbert spaces, whose inverse is continuous, yielding (3). □

Now we are ready to define the *mixed formulation* and to derive its well-posedness.

Let H and K be two real Hilbert spaces. Assume that we are given continuous bilinear forms $a : K \times K \to \mathbb{R}$ and $b : H \times K \to \mathbb{R}$, with associated operators $A : K \to K'$ and $B : K \to H'$ defined by (2.1.3). Let $f \in H'$ and $g \in K'$. We define the mixed formulation as: find $(p, \mathbf{\Phi}) \in H \times K$ such that

$$\begin{cases} a(\mathbf{\Phi}, \mathbf{\Psi}) + b(p, \mathbf{\Psi}) = \langle g, \mathbf{\Psi} \rangle_{K' \times K}, & \forall \mathbf{\Psi} \in K, \\ \qquad\qquad b(q, \mathbf{\Phi}) = \langle f, q \rangle_{H' \times H}, & \forall q \in H. \end{cases} \tag{2.1.6}$$

Equivalently, the problem can be formulated in terms of the associated operators as: find $(p, \mathbf{\Phi}) \in H \times K$ such that

$$\begin{cases} A\mathbf{\Phi} + B^t p = g, \\ \qquad B\mathbf{\Phi} = f, \end{cases} \tag{2.1.7}$$

which corresponds to a direct generalization of problem (2.1.1).

If we define $\mathcal{L} : (H \times K) \times (H \times K) \to \mathbb{R}$ by

$$\mathcal{L}((p, \boldsymbol{\Phi}), (q, \boldsymbol{\Psi})) := a(\boldsymbol{\Phi}, \boldsymbol{\Psi}) + b(p, \boldsymbol{\Psi}) - b(q, \boldsymbol{\Phi}),$$

then problem (2.1.6) can be written in variational form as: find $(p, \boldsymbol{\Phi}) \in H \times K$ such that

$$\mathcal{L}((p, \boldsymbol{\Phi}), (q, \boldsymbol{\Psi})) = \langle f, q\rangle_{H' \times H} - \langle g, \Psi\rangle_{K' \times K}, \quad \forall (q, \boldsymbol{\Psi}) \in H \times K. \tag{2.1.8}$$

Remark 2.1.5. *When the bilinear form a is symmetric, the mixed formulation can be interpreted as a constrained minimization problem, in the spirit of Remark 2.1.1. Indeed, $(p, \boldsymbol{\Phi}) \in H \times K$ satisfies* (2.1.8) *if and only if it is a saddle point of the Lagrangian associated with the minimization problem*

$$\min_{\substack{\boldsymbol{\Psi} \in K, \\ B\boldsymbol{\Psi} = f}} J(\boldsymbol{\Psi}),$$

where $J(\boldsymbol{\Psi}) = \frac{1}{2} a(\boldsymbol{\Psi}, \boldsymbol{\Psi}) - \langle g, \boldsymbol{\Psi}\rangle_{K' \times K}$.

2.1.3 Well-posedness of the problem and the inf–sup condition

When we speak of well-posedness, we mean it in the sense of Hadamard. That is, we seek necessary and sufficient conditions ensuring existence, uniqueness, and continuous dependence of the solutions on the data for problem (2.1.6). Recall formulas (2.1.4) and

$$R_{H'}([\mathrm{Im}B]^{\perp}) = (\mathrm{Im}B)^0 = \mathrm{Ker}B^t, \quad R_{K'}([\mathrm{Im}B^t]^{\perp}) = (\mathrm{Im}B^t)^0 = \mathrm{Ker}B.$$

Therefore,

$$[\mathrm{Im}B]^{\perp} = R_H(\mathrm{Ker}B^t), \quad [\mathrm{Im}B^t]^{\perp} = R_K(\mathrm{Ker}B).$$

Moreover, by the Hahn–Banach theorem, we can deduce that $W' \simeq R_H(W)$ for every closed subspace W of a Hilbert space H. For this reason, with an abuse of notation, **we will denote** $R_H(W)$ by W' until the end of this section.

It is clear that, in order for problem (2.1.7) to admit a solution, the operator B must be surjective. Moreover, projecting the first equation of (2.1.7) onto $(\mathrm{Ker}B)'$ yields

$$\pi_{[KerB]'} A\boldsymbol{\Phi} = \pi_{[KerB]'} g.$$

Therefore, requiring that $\pi_{[KerB]'} A|_{\mathrm{Ker}B}$ be invertible and that B be surjective appears to be sufficient. In fact, as in the finite-dimensional case, this is also a necessary condition.

Theorem 2.1.6. *Let $a : K \times K \to \mathbb{R}$ and $b : H \times K \to \mathbb{R}$ be continuous bilinear forms, with associated operators $A : K \to K'$ and $B : K \to H'$. Then problem* (2.1.6) *admits a unique solution for all $f \in H'$ and $g \in K'$* ***if and only if*** *the following conditions hold:*

$$\begin{aligned} &\pi_{[KerB]'} A|_{KerB} : \mathrm{Ker}B \to [KerB]' \text{ is an isomorphism}, \\ &\mathrm{Im}(B) = H'. \end{aligned} \tag{2.1.9}$$

Proof. Assume that for every $(g, f) \in H' \times K'$ there exist unique $(p, \boldsymbol{\Phi}) \in H \times K$ solving (2.1.6). It is clear that this implies $Im(B) = H'$. Now, let $g \in [KerB]' \subset H'$. Then there exist unique $(p, \boldsymbol{\Phi}) \in H \times K$ such that

$$\begin{cases} A\boldsymbol{\Phi} + B^t p = g, \\ \qquad B\boldsymbol{\Phi} = 0. \end{cases}$$

Recalling that $[KerB]' = Im(B^t)^\perp$, we deduce that $\pi_{[KerB]'}A\mathbf{\Phi} = g$. Since $\mathbf{\Phi} \in KerB$ and it is unique, we conclude that $\pi_{[KerB]'}A|_{KerB}$ is an isomorphism.

Now suppose that A and B satisfy (2.1.9), and let $g \in K'$ and $f \in H'$. Since B is surjective, there exists $\mathbf{\Phi}_f \in K$ such that $B\mathbf{\Phi}_f = f$. Since $\pi_{[KerB]'}A|_{KerB}$ is an isomorphism, there exists a unique $\mathbf{\Phi}_0 \in KerB$ such that $\pi_{[KerB]'}(g - A\mathbf{\Phi}_f - A\mathbf{\Phi}_0) = 0$. Theorem 2.1.4 implies that $g - A\mathbf{\Phi}_0 \in [KerB]'^\perp = Im(B^t)$, and from the injectivity of B^t we deduce that there exists a unique $p \in H$ such that

$$B^t p = g - A\mathbf{\Phi}_f - A\mathbf{\Phi}_0.$$

Setting $\mathbf{\Phi} = \mathbf{\Phi}_0 + \mathbf{\Phi}_f$, we conclude that

$$\begin{cases} B^t p = g - A(\mathbf{\Phi}_f - \mathbf{\Phi}_0) = g - A\mathbf{\Phi}, \\ B\mathbf{\Phi} = B\mathbf{\Phi}_f + B\mathbf{\Phi}_0 = f. \end{cases}$$

□

In the spirit of Corollary 2.1.3, the following theorem is stated in the most common framework encountered in the problems studied in this thesis. Observe that the operator $\pi_{[KerB]'}A|_{KerB}$ is the operator associated with the restriction $a|_{KerB\times KerB} : KerB \times KerB \to \mathbb{R}$, that is,

$$a(\mathbf{\Phi}, \mathbf{\Psi}) = \langle \pi_{[KerB]'}A\mathbf{\Phi}, \mathbf{\Psi}\rangle = \langle \mathbf{\Phi}, [\pi_{[KerB]'}A]^t\mathbf{\Psi}\rangle, \quad \forall \mathbf{\Phi}, \mathbf{\Psi} \in KerB.$$

Theorem 2.1.7 (Sufficient condition for problem (2.1.6))**.** *Let $a : K \times K \to \mathbb{R}$ and $b : H \times K \to \mathbb{R}$ be continuous bilinear forms, with associated operators $A : K \to K'$ and $B : K \to H'$. Suppose that a is coercive on the kernel of B, and that B is surjective. Then problem* (2.1.6) *admits a unique solution for all $f \in H'$ and $g \in K'$.*

Proof. If a is coercive in Ker$B$ then there exists a constant $\alpha > 0$ such that $\|\pi_{[KerB]'}A\mathbf{\Phi}\|_{K'} \geq \alpha\|\mathbf{\Phi}\|_K$ and $\|[\pi_{[KerB]'}A]^t\mathbf{\Phi}\|_{K'} \geq \alpha\|\mathbf{\Phi}\|_K$ for all $\mathbf{\Phi} \in$ Ker$B$. Hence, by Theorem 2.1.4, the operator $\pi_{[KerB]'}A|_{KerB}$ is an isomorphism. The result then follows immediately from Theorem 2.1.6. □

We conclude this section by addressing the issue of stability, which naturally leads to the inf–sup condition, also known as the Babuška–Brezzi condition. This condition can be understood as a reformulation of Theorem 2.1.6 in terms of the bilinear forms a and b.

From Theorem 2.1.4, we deduce that B is surjective if and only if there exists a constant $\beta > 0$ such that

$$\|B^t p\|_{K'} \geq \beta\|p\|_H, \quad \forall p \in H,$$

which is equivalent to

$$\inf_{p\in H} \frac{\|B^t p\|_{K'}}{\|p\|_H} = \inf_{p\in H} \sup_{\mathbf{\Phi}\in K} \frac{\langle B^t p, \mathbf{\Phi}\rangle_{K'\times K}}{\|\mathbf{\Phi}\|_K\|p\|_H} \geq \beta.$$

This can be written as

$$\inf_{p\in H} \sup_{\mathbf{\Phi}\in K} \frac{b(p, \mathbf{\Phi})}{\|\mathbf{\Phi}\|_K\|p\|_H} \geq \beta,$$

which we call an *inf–sup condition*. Similarly, one can show that $\pi_{[KerB]'}A|_{KerB}$ is an isomorphism if and only if there exists a constant $\alpha > 0$ such that

$$\inf_{\mathbf{\Phi}\in KerB} \sup_{\mathbf{\Psi}\in KerB} \frac{a(\mathbf{\Phi}, \mathbf{\Psi})}{\|\mathbf{\Phi}\|_K\|\mathbf{\Psi}\|_K} \geq \alpha,$$

$$\inf_{\mathbf{\Psi}\in KerB} \sup_{\mathbf{\Phi}\in KerB} \frac{a(\mathbf{\Phi}, \mathbf{\Psi})}{\|\mathbf{\Phi}\|_K\|\mathbf{\Psi}\|_K} \geq \alpha.$$

We conclude this part with what may be regarded as the two main results of the analysis carried out so far in this chapter.

Theorem 2.1.8 (Inf–sup condition)**.** *Let $a : K \times K \to \mathbb{R}$ and $b : H \times K \to \mathbb{R}$ be continuous bilinear forms, with associated operators $A : K \to K'$ and $B : K \to H'$. Then problem* (2.1.6) *admits a unique solution for all $f \in H'$ and $g \in K'$ if and only if there exist constants $\beta > 0$ and $\alpha > 0$ such that b satisfies the inf–sup condition*

$$\inf_{p \in H} \sup_{\mathbf{\Phi} \in K} \frac{b(p, \mathbf{\Phi})}{\|\mathbf{\Phi}\|_K \|p\|_H} \geq \beta,$$

and a satisfies the double inf–sup condition

$$\begin{aligned} \inf_{\mathbf{\Phi} \in KerB} \sup_{\mathbf{\Psi} \in KerB} \frac{a(\mathbf{\Phi}, \mathbf{\Psi})}{\|\mathbf{\Phi}\|_K \|\mathbf{\Psi}\|_K} &\geq \alpha, \\ \inf_{\mathbf{\Psi} \in KerB} \sup_{\mathbf{\Phi} \in KerB} \frac{a(\mathbf{\Phi}, \mathbf{\Psi})}{\|\mathbf{\Phi}\|_K \|\mathbf{\Psi}\|_K} &\geq \alpha. \end{aligned} \tag{2.1.10}$$

In this case, if $(p, \mathbf{\Phi}) \in H \times K$ is the solution of (2.1.6)*, then*

$$\begin{aligned} \|\mathbf{\Phi}\|_K &\leq \frac{1}{\alpha} \|g\|_{K'} + \frac{2\|a\|}{\alpha\beta} \|f\|_{H'}, \\ \|p\|_H &\leq \frac{2\|a\|}{\alpha\beta} \|g\|_{K'} + \frac{\|a\|^2}{\alpha\beta^2} \|f\|_{H'}. \end{aligned}$$

Proof. Given $f \in H'$ and $g \in K'$, the existence and uniqueness of a solution $(p, \mathbf{\Phi}) \in H \times K$ to problem (2.1.6) follows directly from Theorem 2.1.6 and the previous discussion. We now derive the estimates. Let $\mathbf{\Phi}_f \in [KerB]^\perp$ be such that $B\mathbf{\Phi}_f = f$, and let $(p_0, \mathbf{\Phi}_0) \in H \times K$ satisfy

$$\begin{cases} a(\mathbf{\Phi}_0, \mathbf{\Psi}) + b(p_0, \mathbf{\Psi}) = \langle g - A\Phi_f, \mathbf{\Psi} \rangle_{K' \times K}, & \forall \mathbf{\Psi} \in K, \\ b(q, \mathbf{\Phi}_0) = 0, & \forall q \in H. \end{cases} \tag{2.1.11}$$

By uniqueness, we have

$$\mathbf{\Phi} = \mathbf{\Phi}_f + \mathbf{\Phi}_0, \quad p = p_0. \tag{2.1.12}$$

Since $\mathbf{\Phi}_f \in [KerB]^\perp$ and $B|_{[KerB]^\perp} : [KerB]^\perp \to H'$ is an isomorphism, it follows that

$$\beta \|\mathbf{\Phi}_f\|_K \leq \|B\mathbf{\Phi}_f\|_{H'} = \|f\|_{H'}. \tag{2.1.13}$$

Since $[\mathrm{Ker}B]' = [ImB^t]^\perp$, we deduce $\pi_{[\mathrm{Ker}B]'} B^t p_0 = 0$, thus using the first equation in (2.1.11), we obtain

$$\pi_{[\mathrm{Ker}B]'} A\mathbf{\Phi}_0 = \pi_{[\mathrm{Ker}B]'} [g - A\mathbf{\Phi}_f].$$

The first inequality (2.1.10) then yields

$$\alpha \|\mathbf{\Phi}_0\|_K \leq \|\pi_{[\mathrm{Ker}B]'} A\mathbf{\Phi}_0\|_{K'} \leq \|g\|_{K'} + \|A\mathbf{\Phi}_f\|_{K'} \leq \|g\|_{K'} + \|a\| \|\mathbf{\Phi}_f\|_K.$$

Using (2.1.13), we conclude that

$$\|\mathbf{\Phi}_0\|_K \leq \frac{1}{\alpha} \|g\|_{K'} + \frac{\|a\|}{\alpha\beta} \|f\|_{H'}.$$

Finally, from (2.1.12) we obtain

$$\|\mathbf{\Phi}\|_K \leq \frac{1}{\alpha} \|g\|_{K'} + \frac{2\|a\|}{\alpha\beta} \|f\|_{H'}.$$

Analogously, using that $\alpha \leq \|a\|$, we conclude that

$$\beta\|p\|_H \leq \|B^t p\|_{K'} \leq \|g\|_{K'} + \|A\mathbf{\Phi}\|_{K'} \leq \frac{2\|a\|}{\alpha}\|g\|_{K'} + \frac{2\|a\|^2}{\alpha\beta}\|f\|_{H'}.$$

□

Finally, we state Theorem 2.1.8 in a form suitable for the setting of the following section, where we study a mixed formulation of the fractional Laplacian. Again, this is the case when a is coercive.

Corollary 2.1.9. *Let $a : K \times K \to \mathbb{R}$ and $b : H \times K \to \mathbb{R}$ be continuous bilinear forms, with associated operators $A : K \to K'$ and $B : K \to H'$. Assume that the restriction of a to $Ker B$ is coercive with coercivity constant $\alpha > 0$, and that b satisfies the inf–sup condition*

$$\inf_{p \in H} \sup_{\mathbf{\Phi} \in K} \frac{b(p, \mathbf{\Phi})}{\|\mathbf{\Phi}\|_K \|p\|_H} \geq \beta.$$

Then, problem (2.1.6) *admits a unique solution for all $f \in K'$ and $g \in H'$. Moreover, if $(p, \mathbf{\Phi}) \in H \times K$ is the solution of* (2.1.6), *then*

$$\|\mathbf{\Phi}\|_K \leq \frac{1}{\alpha}\|g\|_{K'} + \frac{2\|a\|}{\alpha\beta}\|f\|_{H'},$$
$$\|p\|_H \leq \frac{2\|a\|}{\alpha\beta}\|g\|_{K'} + \frac{\|a\|^2}{\alpha\beta^2}\|f\|_{H'}.$$

Remark 2.1.10 (Discretization of a mixed problems)**.** *Note that, contrary to coercivity, the inf-sup condition* (2.2.5) *is not guaranteed under restriction to arbitrary subspaces. Therefore, when designing finite element discretizations of mixed problems, choosing any conforming discrete space is not enough. To illustrate this issue, let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $f \in L^2(\Omega)$. Consider the Darcy problem (or the Laplacian in mixed form)*

$$\begin{cases} \mathbf{\Phi} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{\Phi} = f & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases}$$

or, in weak form: find $(p, \mathbf{\Phi}) \in L^2(\Omega) \times H(\operatorname{div}; \Omega)$ such that

$$\int_\Omega \mathbf{\Phi} \cdot \mathbf{\Psi} - \int_\Omega p \operatorname{div} \mathbf{\Psi} + \int_\Omega q \operatorname{div} \mathbf{\Phi} = \int_\Omega f q \quad \forall (q, \mathbf{\Psi}) \in L^2(\Omega) \times H(\operatorname{div}; \Omega),$$

where

$$H(\operatorname{div}; \Omega) = \{\mathbf{\Psi} \in [L^2(\Omega)]^d : \operatorname{div} \mathbf{\Psi} \in L^2(\Omega)\}.$$

This problem is well-posed; see [32, Chapter 51]. However, the stability of piece-wise polynomial discretizations is not guaranteed. One customary cure is to consider the $H(\operatorname{div})$ conforming Raviart-Thomas elements. First, we define

$$\mathbb{RT}_{d,k} := [\mathcal{P}_k]^d \oplus x\mathcal{P}_k^{(H)}$$

where $\mathcal{P}_k^{(H)}$ denotes the space of homogeneous polynomials of degree k. For example, for $k = 1$ and $d = 2$, a basis of such space is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 x_2 \\ x_2^2 \end{pmatrix}, \begin{pmatrix} x_1^2 \\ x_1 x_2 \end{pmatrix} \right\}.$$

Thus, given a simplicial mesh $\mathcal{T}_h(\Omega)$, the Raviart-Thomas element space is defined as

$$\mathcal{P}_k(\mathcal{T}_h(\Omega)) \times \mathbb{RT}_{d,k}(\mathcal{T}_h(\Omega)) := \{(q_h, \mathbf{\Psi}_h) : q_h|_T \in \mathcal{P}_k,\ \mathbf{\Psi}_h|_T \in \mathbb{RT}_{d,k}\}.$$

This discretization is stable and well-posed; we refer to [31, Chapter 19] for properties of the Raviart-Thomas spaces and to [32, Chapter 51] for the well-posedness of the discrete problem.

2.2 Fractional Darcy problem

The mixed formulation of the classical Poisson problem (to which we will refer as the *fractional Darcy problem*) introduces the flux as an additional variable, leading to a system of coupled equations. Using fractional calculus identities, in this section we explore a mixed formulation of the fractional Poisson problem (P), establish its well-posedness and study its Sobolev regularity with the aim of obtaining covergence rates for the finite element approximations. Since a direct discretization of this problem appears to be out of reach, we adapt the stabilized approach of [45], which yields a coercive and well-posed formulation.

2.2.1 Problem formulation

Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. In this section, we consider the fractional Poisson problem in Ω defined in the first chapter, namely

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c := \mathbb{R}^d \setminus \Omega. \end{cases} \tag{P}$$

We propose to study the mixed formulation of this problem, cf. (D) below, to approximate both the solution u and its so-called fractional gradient in primal form. Above, $s \in (0,1)$, $f \in L^2(\Omega)$, and $(-\Delta)^s$ denotes the fractional Laplacian (cf. Definition (1.1.1)).

Recall that the fractional Laplacian can be regarded as the composition (cf. Lemma 1.1.15 and Remark 1.1.20)

$$(-\Delta)^s w = \operatorname{div}^s \nabla^s w, \quad \forall w \in \widetilde{H}^s(\Omega),$$

where div^s and ∇^s are given by Definition 1.1.5; we refer to Chapter 1 for a summary of the properties of this operators. As in the local case, the idea is, given $p \in \widetilde{H}^s(\Omega)$ the solution of (P), to introduce $\mathbf{\Phi} := -\nabla^s p$ as a new variable in problem (P). Therefore,

$$\operatorname{div}^s \mathbf{\Phi}|_\Omega = f \in L^2(\Omega),$$

this motivates the definition of

$$H(\operatorname{div}^s; \Omega) := \left\{ \mathbf{\Psi} \in \left(L^2(\mathbb{R}^d)\right)^d : (\operatorname{div}^s \mathbf{\Psi})\big|_\Omega \in L^2(\Omega) \right\},$$

endowed with the norm

$$\|\mathbf{\Psi}\|_{H(\operatorname{div}^s;\Omega)} = \left(\|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)} + \|(\operatorname{div}^s \mathbf{\Psi})\big|_\Omega\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Furthermore, we will denote by $\widetilde{L}^2(\Omega)$ the space of functions in $L^2(\Omega)$ that are extended by zero to Ω^c. Finally, recall the integration by parts formula (cf. Theorem 1.1.19 and Remark 1.1.20): given $w \in \widetilde{H}^s(\Omega)$, $\mathbf{\Psi} \in H(\operatorname{div}^s;\Omega)$, we have

$$\int_{\mathbb{R}^d} \nabla^s w \cdot \mathbf{\Psi} = -\int_{\mathbb{R}^d} w \operatorname{div}^s \mathbf{\Psi}. \tag{2.2.1}$$

With the notation established above, the *fractional Darcy problem* reads: find $(p, \mathbf{\Phi}) \in \widetilde{L}^2(\Omega) \times H(\operatorname{div}^s;\Omega)$ such that

$$\begin{cases} \mathbf{\Phi} + \nabla^s p = 0 & \text{in } \mathbb{R}^d, \\ \operatorname{div}^s \mathbf{\Phi} = f & \text{in } \Omega, \\ p = 0 & \text{in } \Omega^c. \end{cases} \tag{D}$$

Following the standard terminology for the Darcy problem, occasionally we will refer to p as *pressure* and to $\mathbf{\Phi}$ as *flux*. Clearly, if $(p, \mathbf{\Phi})$ solves the problem above, then $u = p$ solves (P). The difference between (P) and (D) is that, obviously, we have introduced the flux variable $\mathbf{\Phi} := -\nabla^s p$; due to the nonlocal nature of the problem, this definition needs to be imposed in the whole space $\mathbb{R}^d$ and not just in the domain Ω.

From a computational perspective, in comparison to the classical (local) divergence, the operator div^s brings two apparent difficulties. On the one hand, it does not map piecewise polynomial functions into piecewise polynomials of one degree less. On the other hand, the nonlocal nature of the operator implies that $\operatorname{div}^s \mathbf{\Psi}$ can have an unbounded support even when $\mathbf{\Psi}$ does not. To illustrate these statements, we let $h > 0$ and $\mathbf{\Psi} : \mathbb{R} \to \mathbb{R}$ given by

$$\mathbf{\Psi}(x) = \begin{cases} 1 - \frac{|x|}{h} & \text{if } x \in [-h, h], \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.2}$$

Note that in the one-dimensional case we have $\partial^s = \nabla^s = \operatorname{div}^s$. Thus, we have $\partial^s \mathbf{\Psi}(x) = \mathcal{I}_{1-s} \mathbf{\Psi}'(x)$, where $\mathcal{I}_{1-s}$ denotes the Riesz transform (cf. Theorem 1.1.8 and Remark 1.1.12), therefore

$$\partial^s \mathbf{\Psi}(x) = c(1, 1-s) \int_{-\infty}^{+\infty} \frac{\mathbf{\Psi}'(y)}{|x-y|^s} dy = \frac{c(1, 1-s)}{h} \left(\int_{-h}^{0} \frac{dy}{|x-y|^s} - \int_0^h \frac{dy}{|x-y|^s} \right).$$

Consequently, a straightforward calculation gives

$$\partial^s \mathbf{\Psi}(x) = \frac{c(1, 1-s)}{h(1-s)} \left((x+h)^{1-s} - 2x^{1-s} + (x-h)^{1-s} \right), \qquad x > h.$$

Thus, the construction and analysis of $H(\operatorname{div}^s)$-conforming finite elements a-la Raviart-Thomas does not seem straightforward in this setting. Indeed, an important property of such spaces is that $\operatorname{div}(\mathbb{RT}_{d,k}) \subset \mathcal{P}_k$; see Remark 2.1.10. For this reason, in this thesis we pursue an alternative route by adapting the approach of Masud and Hughes [45] and employing a stabilized formulation that can be discretized with continuous Lagrange elements.

2.2.2 Well-posedness of the fractional Darcy problem

This section analyzes the well-posedness of the fractional Darcy problem (D). For simplicity, we assume the right-hand side $f \in L^2(\Omega)$. Using the integration by parts formula (2.2.1), its weak formulation reads: find $(p,\mathbf{\Phi}) \in \widetilde{L}^2(\Omega) \times H(\operatorname{div}^s;\Omega)$ such that, for all $(q,\mathbf{\Psi}) \in \widetilde{L}^2(\Omega) \times H(\operatorname{div}^s;\Omega)$,

$$\int_{\mathbb{R}^d} \mathbf{\Phi}\cdot\mathbf{\Psi} - \int_{\mathbb{R}^d} p \operatorname{div}^s \mathbf{\Psi} + \int_{\mathbb{R}^d} q \operatorname{div}^s \mathbf{\Phi} = \int_{\mathbb{R}^d} fq. \tag{2.2.3}$$

We point out that all but the first of the integrals above can be actually computed in Ω.

Let us introduce some additional notation. First, we define the forms

$$\begin{aligned} a\colon H(\operatorname{div}^s;\Omega) \times H(\operatorname{div}^s;\Omega) \to \mathbb{R}, &\quad a(\mathbf{\Phi},\mathbf{\Psi}) = \int_{\mathbb{R}^d} \mathbf{\Phi}\cdot\mathbf{\Psi}, \\ b\colon \widetilde{L}^2(\Omega) \times H(\operatorname{div}^s;\Omega) \to \mathbb{R}, &\quad b(q,\mathbf{\Psi}) = \int_{\Omega} q \operatorname{div}^s \mathbf{\Psi}, \\ F\colon \widetilde{L}^2(\Omega) \to \mathbb{R}, &\quad F(q) = \int_\Omega fq. \end{aligned} \tag{2.2.4}$$

We also define $B\colon H(\operatorname{div}^s;\Omega) \to L^2(\Omega)$ such that

$$(B\mathbf{\Psi}, q)_{L^2(\Omega)} := b(q,\mathbf{\Psi}), \quad \forall \mathbf{\Psi} \in H(\operatorname{div}^s;\Omega),\ q \in \widetilde{L}^2(\Omega).$$

Notice that a is symmetric and that the problem above has a clear saddle-point structure. Therefore, by the analysis carried out in the previous section, to prove the well-posedness of (2.2.3) it suffices to show that

- the form a is coercive in $\ker B$;
- the form b satisfies an inf-sup condition,

see Corollary 2.1.9. The fact that a is coercive in $\ker B$ follows straightforwardly upon observing that $\ker B = \{\mathbf{\Psi} \in H(\operatorname{div}^s;\Omega)\colon \operatorname{div}^s \mathbf{\Psi} = 0 \text{ in } \Omega\}$. This yields that, for every $\mathbf{\Psi} \in \ker B$,

$$a(\mathbf{\Psi},\mathbf{\Psi}) = \|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)} = \|\mathbf{\Psi}\|^2_{H(\operatorname{div}^s;\Omega)}.$$

In view of Theorem 2.1.7, to show the existence and uniqueness of solutions it is sufficient to show that $B\mathbf{\Psi} = \operatorname{div}^s \mathbf{\Psi}|_\Omega$ is surjective. Furthermore, as a consequence of the Poincaré inequality (1.2.2), we obtain an inf-sup condition for b.

Lemma 2.2.1 (Surjectivity of div^s)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. The operator $\operatorname{div}^s|_\Omega$ such that $\operatorname{div}^s|_\Omega \mathbf{\Psi} := (\operatorname{div}^s \mathbf{\Psi})|_\Omega$ maps $H(\operatorname{div}^s;\Omega)$ onto $\widetilde{L}^2(\Omega)$. Consequently, b satisfies an inf-sup condition: there exists $\beta > 0$ such that*

$$\inf_{p \in \widetilde{L}^2(\Omega)} \sup_{\mathbf{\Phi} \in H(\operatorname{div}^s;\Omega)} \frac{b(p,\mathbf{\Phi})}{\|p\|_{L^2(\Omega)} \|\mathbf{\Phi}\|_{H(\operatorname{div}^s;\Omega)}} \geq \beta. \tag{2.2.5}$$

Proof. The fact that $\operatorname{div}^s|_\Omega \mathbf{\Psi} \in L^2(\Omega)$ for all $\mathbf{\Psi} \in H(\operatorname{div}^s;\Omega)$ is evident from the definition. Next, given $p \in \widetilde{L}^2(\Omega)$, we consider $w \in \widetilde{H}^s(\Omega)$ such that

$$(\nabla^s w, \nabla^s v)_{L^2(\mathbb{R}^d)} = (p, v)_{L^2(\mathbb{R}^d)}, \quad \forall v \in \widetilde{H}^s(\Omega).$$

This is equivalent to stating that $w \in \widetilde{H}^s(\Omega)$ solves the fractional Poisson problem (P) with right-hand side p. By Proposition 1.2.23, such w is well-defined. We let $\mathbf{\Psi} := -\nabla^s w$, that obviously satisfies $\operatorname{div}^s \mathbf{\Psi} = (-\Delta)^s w = p$ in Ω. This shows that $\operatorname{div}^s|_\Omega$ is surjective. Additionally, we have

$$\|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)} = \|\nabla^s w\|^2_{L^2(\mathbb{R}^d)} = (p,w)_{L^2(\mathbb{R}^d)} \leq \|p\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)},$$

and the Poincaré inequality (1.2.2) gives

$$\|w\|_{L^2(\Omega)} \le C_P \|\nabla^s w\|_{L^2(\mathbb{R}^d)} = C_P \|\boldsymbol{\Psi}\|_{L^2(\mathbb{R}^d)},$$

so that $\|\boldsymbol{\Psi}\|_{L^2(\mathbb{R}^d)} \le C_P \|p\|_{L^2(\Omega)}$. We therefore deduce that

$$\|\boldsymbol{\Psi}\|^2_{H(\mathrm{div}^s;\Omega)} = \|\boldsymbol{\Psi}\|^2_{L^2(\mathbb{R}^d)} + \|\mathrm{div}^s \boldsymbol{\Psi}|_\Omega\|^2_{L^2(\Omega)} \le (1 + C_P^2)\|p\|^2_{L^2(\Omega)}.$$

Finally, we conclude

$$\sup_{\boldsymbol{\Phi} \in H(\mathrm{div}^s;\Omega)} \frac{b(p, \boldsymbol{\Phi})}{\|\boldsymbol{\Phi}\|_{H(\mathrm{div}^s;\Omega)}} \ge \frac{b(p, \boldsymbol{\Psi})}{\|\boldsymbol{\Psi}\|_{H(\mathrm{div}^s;\Omega)}} \ge \frac{\|p\|_{L^2(\Omega)}}{\sqrt{1 + C_P^2}}.$$

This shows that b satisfies the inf-sup condition (2.2.5) with constant $\beta := \sqrt{1 + C_P^2}^{\,-1}$. □

As a corollary, we deduce the well-posedness of the fractional Darcy problem (cf. Corollary 2.1.9).

Proposition 2.2.2 (Well-posedness)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain. Then, problem* (2.2.3) *has a unique solution $(p, \boldsymbol{\Phi}) \in \widetilde{L}^2(\Omega) \times H(\mathrm{div}^s;\Omega)$, and there hold*

$$\|p\|_{L^2(\Omega)} \le 2\sqrt{1 + C_P^2}\, \|f\|_{L^2(\Omega)},$$
$$\|\boldsymbol{\Phi}\|_{H(\mathrm{div}^s;\Omega)} \le 2(1 + C_P^2)\, \|f\|_{L^2(\Omega)},$$

with C_P being the constant in the Poincaré inequality (1.2.2)*.*

2.2.3 Stabilization

Although the formulation developed in the previous section is well-posed, its saddle-point structure makes H^s finite element approximation challenging. In this section, we build on the ideas of [45] to develop a stabilized variational formulation of the fractional Darcy problem, which is amenable to be treated with continuous Lagrange elements. We also present numerical experiments that illustrate the importance of the stabilization.

Remark 2.2.3 (Use of stabilization)**.** *In the classical (local) setting, stabilized mixed formulations introduce additional differential operators, which are often associated with increased matrix fill-in and conditioning issues at the discrete level. In the present nonlocal setting, however, these considerations are less decisive. The fractional operators defined in the first chapter already lead to dense discrete operators, independently of whether stabilization is employed (cf. Appendix A). Moreover, alternative mixed discretizations based on $H(\mathrm{div}^s)$-conforming elements are not readily available in this context. For these reasons, stabilization provides a natural and practical way to obtain a well-posed discrete formulation using standard continuous Lagrange elements.*

We now describe the stabilized formulation for the fractional Darcy problem. As mentioned above, a $\mathcal{P}^1$–$\mathcal{P}^1$ discretization is generally a poor choice, both in the local and nonlocal settings, since the discrete inf–sup condition is not guaranteed. To tackle this, we shrink the domain of the pressure and pursue a coercive formulation. To shorten the notation, we write

$$\begin{aligned} &\mathcal{L}\colon \left(\widetilde{L}^2(\Omega) \times H(\mathrm{div}^s;\Omega)\right) \times \left(\widetilde{L}^2(\Omega) \times H(\mathrm{div}^s;\Omega)\right) \to \mathbb{R}, \\ &\mathcal{L}((p, \boldsymbol{\Phi}), (q, \boldsymbol{\Psi})) := a(\boldsymbol{\Phi}, \boldsymbol{\Psi}) - b(p, \boldsymbol{\Psi}) + b(q, \boldsymbol{\Phi}), \end{aligned} \tag{2.2.6}$$

so that we can rewrite (2.2.3) as

$$\mathcal{L}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) = F(q). \tag{2.2.7}$$

We introduce the stabilized form in $\mathbb{V} := \widetilde{H}^s(\Omega) \times H(\mathrm{div}^s;\Omega)$, $\mathcal{L}_{\mathrm{stab}} \colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$,

$$\mathcal{L}_{\mathrm{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) := \mathcal{L}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) + \frac{1}{2}\int_{\mathbb{R}^d} (\mathbf{\Phi} + \nabla^s p)\cdot(-\mathbf{\Psi} + \nabla^s q)\,. \tag{2.2.8}$$

By (2.2.6), we note that this can be rewritten as

$$\begin{aligned}\mathcal{L}_{\mathrm{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) =& \frac{1}{2}\int_{\mathbb{R}^d} \mathbf{\Phi}\cdot\mathbf{\Psi} + \frac{1}{2}\int_{\mathbb{R}^d} \nabla^s p\cdot\mathbf{\Psi} - \frac{1}{2}\int_{\mathbb{R}^d} \nabla^s q\cdot\mathbf{\Phi} \\ &+ \frac{1}{2}\int_{\mathbb{R}^d} \nabla^s p\cdot\nabla^s q.\end{aligned} \tag{2.2.9}$$

With this, we consider the stabilized problem: find $(p,\mathbf{\Phi}) \in \mathbb{V}$ such that

$$\mathcal{L}_{\mathrm{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) = F(q) \quad \forall (q,\mathbf{\Psi}) \in \mathbb{V}. \tag{2.2.10}$$

We make three important remarks concerning the definition of $\mathcal{L}_{\mathrm{stab}}$. First, the non-local nature of the problem implies that the stabilization term above needs to be computed in the whole $\mathbb{R}^d$. Second, we have shrunk the domain of $\mathcal{L}$ by replacing $\widetilde{L}^2(\Omega)$ by $\widetilde{H}^s(\Omega)$ so that the stabilization term is well-defined, in exchange of shrinking the domain of the pressure we obtained a coercive formulation. Third, it is consistent: if $(p,\mathbf{\Phi}) \in \mathbb{V}$ in the solution of our problem we have $\nabla^s p + \mathbf{\Phi} = 0$, and so the term that we are adding is zero (see also Lemma 2.2.5). Let us consider the following norm in $\mathbb{V}$,

$$|||(q,\mathbf{\Psi})||| := \left[\frac{1}{2}\left(\|\nabla^s q\|^2_{L^2(\mathbb{R}^d)} + \|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)}\right)\right]^{1/2}. \tag{2.2.11}$$

The fact that $|||\cdot|||$ is a seminorm is evident. We further note that if $|||(q,\mathbf{\Psi})||| = 0$ then obviously $\mathbf{\Psi} = 0$ in $\mathbb{R}^d$ and the Poincaré inequality (1.2.2) implies that $q = 0$ in Ω (and thus in $\mathbb{R}^d$), and it follows that $|||\cdot|||$ is actually a norm.

Remark 2.2.4 (Strong form of the stabilized problem)**.** *The strong form of the stabilized problem* (2.2.10) *reads*

$$\begin{cases} \nabla^s p + \mathbf{\Phi} = 0 & \text{in } \mathbb{R}^d, \\ \frac{1}{2}\mathrm{div}^s\mathbf{\Phi} + \frac{1}{2}(-\Delta)^s p = f & \text{in } \Omega, \\ p = 0 & \text{in } \Omega^c. \end{cases}$$

More generally, for $\theta \in (0,1)$, one may consider

$$\begin{cases} \nabla^s p + \mathbf{\Phi} = 0 & \text{in } \mathbb{R}^d, \\ (1-\theta)\mathrm{div}^s\mathbf{\Phi} + \theta(-\Delta)^s p = f & \text{in } \Omega, \\ p = 0 & \text{in } \Omega^c. \end{cases}$$

This family of equivalent problems corresponds to replacing $\mathcal{L}$ with the stabilized form $\mathcal{L}_\theta : \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ given by

$$\mathcal{L}_\theta((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) := \mathcal{L}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) + \theta\int_{\mathbb{R}^d} (\mathbf{\Phi} + \nabla^s p)\cdot(-\mathbf{\Psi} + \nabla^s q)\,.$$

It is straightforward to verify that all such choices of θ lead to a formulation that is stable and coercive in $\mathbb{V}$:

$$\mathcal{L}_\theta((p,\mathbf{\Phi}),(p,\mathbf{\Phi})) \geq 2\max\{\theta, 1-\theta\}|||(p,\mathbf{\Phi})|||^2 \qquad \forall (p,\mathbf{\Phi}) \in \mathbb{V},$$
$$|\mathcal{L}_\theta((p,\mathbf{\Phi}),(q,\mathbf{\Psi}))| \leq 2\max\{\theta, 1-\theta\}|||(p,\mathbf{\Phi})|||\,|||(q,\mathbf{\Psi})||| \qquad \forall (p,\mathbf{\Phi}),(q,\mathbf{\Psi}) \in \mathbb{V}.$$

We choose $\theta = \frac{1}{2}$, as this value minimizes the continuity constant.

Now, let us show that the stabilized form defines an equivalent problem.

Lemma 2.2.5 (Equivalence)**.** *A pair $(p,\mathbf{\Phi}) \in \mathbb{V} = \widetilde{H}^s(\Omega) \times H(\mathrm{div}^s;\Omega)$ solves the stabilized problem* (2.2.10) *if and only if it solves* (2.2.7)*.*

Proof. If $(p,\mathbf{\Phi}) \in \mathbb{V}$ solves problem (2.2.7) then using $q = 0$ in that formulation and integrating by parts (cf. formula (2.2.1)), we deduce $\mathbf{\Phi} + \nabla^s p = 0$ in $\mathbb{R}^d$. Therefore, the stabilization term vanishes,

$$\frac{1}{2}\int_{\mathbb{R}^d} (\mathbf{\Phi} + \nabla^s p)\cdot(-\mathbf{\Psi} + \nabla^s q) = 0 \quad \forall (q,\mathbf{\Psi}) \in \mathbb{V},$$

and it follows that $(p,\mathbf{\Phi})$ solves the stabilized problem (2.2.10).

Conversely, if $(p,\mathbf{\Phi}) \in \mathbb{V}$ solves the stabilized problem (2.2.10), then for every $\mathbf{\Psi} \in H(\mathrm{div}^s;\Omega)$ we have

$$\begin{aligned} 0 = \mathcal{L}_{\mathrm{stab}}((p,\mathbf{\Phi}),(0,\mathbf{\Psi})) &= \frac{1}{2}\int_{\mathbb{R}^d} \mathbf{\Phi}\cdot\mathbf{\Psi} - \frac{1}{2}\int_{\mathbb{R}^d} p\,\mathrm{div}^s\mathbf{\Psi} \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \mathbf{\Phi}\cdot\mathbf{\Psi} + \frac{1}{2}\int_{\mathbb{R}^d} \nabla^s p\cdot\mathbf{\Psi}, \end{aligned}$$

which again implies $\nabla^s p + \mathbf{\Phi} = 0$ in $\mathbb{R}^d$. Therefore, the stabilization term is zero and $(p,\mathbf{\Phi})$ solves the fractional Darcy problem (2.2.7). □

We have obtained an equivalent formulation to the fractional Darcy problem, but the stabilized form has the key advantage of being coercive.

Lemma 2.2.6 (Coercivity)**.** *We have*

$$\mathcal{L}_{\mathrm{stab}}((p,\mathbf{\Phi}),(p,\mathbf{\Phi})) = |||(p,\mathbf{\Phi})|||^2 \quad \forall (p,\mathbf{\Phi}) \in \mathbb{V}.$$

Proof. The result follows by a direct computation using (2.2.9). Indeed, if $(p,\mathbf{\Phi}) \in \mathbb{V}$, then

$$\mathcal{L}_{\mathrm{stab}}((p,\mathbf{\Phi}),(p,\mathbf{\Phi})) = \frac{1}{2}\|\mathbf{\Phi}\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\nabla^s p\|^2_{L^2(\mathbb{R}^d)}.$$

□

In addition to being coercive, it is straightforward to verify that the form $\mathcal{L}_{\mathrm{stab}}$ is continuous.

Lemma 2.2.7 (Continuity)**.** *We have*

$$\mathcal{L}_{\mathrm{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) \leq |||(p,\mathbf{\Phi})|||\,|||(q,\mathbf{\Psi})||| \quad \forall (p,\mathbf{\Phi}),(q,\mathbf{\Psi}) \in \mathbb{V}.$$

Proof. Let $(p,\mathbf{\Phi}),(q,\mathbf{\Psi}) \in \mathbb{V}$. Using (2.2.9), we have

$$\begin{aligned} |\mathcal{L}_{\mathrm{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi}))| = \Big| &\frac{1}{2}\int_{\mathbb{R}^d} \mathbf{\Phi}\cdot\mathbf{\Psi} + \frac{1}{2}\int_{\mathbb{R}^d} \nabla^s p\cdot\mathbf{\Psi} - \frac{1}{2}\int_{\mathbb{R}^d} \nabla^s q\cdot\mathbf{\Phi} \\ &+ \frac{1}{2}\int_{\mathbb{R}^d} \nabla^s p\cdot\nabla^s q\Big|. \end{aligned}$$

Therefore, by Young's inequality, we deduce

$$
\begin{aligned}
|\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi}))| &\leq \frac{1}{2}\left(\|\mathbf{\Phi}\|_{L^2(\mathbb{R}^d)} + \|\nabla^s p\|_{L^2(\mathbb{R}^d)}\right)\left(\|\mathbf{\Psi}\|_{L^2(\mathbb{R}^d)} + \|\nabla^s q\|_{L^2(\mathbb{R}^d)}\right) \\
&\leq |||(p,\mathbf{\Phi})|||\,|||(q,\mathbf{\Psi})|||.
\end{aligned}
$$

□

Finally, the combination of the two lemmas above with the Lax-Milgram theorem gives rise to the well-posedness of our problem.

Proposition 2.2.8 (Well-posedness of stabilized formulation)**.** *Given $f \in L^2(\Omega)$, problem* (2.2.10) *has a unique solution $(p,\mathbf{\Phi}) \in \mathbb{V}$. Moreover, we have the stability estimate*

$$
|||(p,\mathbf{\Phi})||| \leq \sqrt{2}\,C_P\|f\|_{L^2(\Omega)},
$$

with C_P being the constant in the Poincaré inequality (1.2.2).

Proof. The Lax-Milgram theorem implies the existence and uniqueness of a solution $(p,\mathbf{\Phi}) \in \mathbb{V}$. Additionally, by the coercivity of the stabilized form and identity (2.2.10), we deduce

$$
|||(p,\mathbf{\Phi})|||^2 = \mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(p,\mathbf{\Phi})) = F(p) \leq \|p\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}.
$$

The desired estimate follows now by the Poincaré inequality (1.2.2) and noticing that $\|p\|_{L^2(\mathbb{R}^d)} \leq \sqrt{2}|||(p,\mathbf{\Phi})|||$. □

2.2.4 The need of stabilization

Our aim in introducing the stabilized formulation is to be able to use standard $\mathcal{P}_1$-$\mathcal{P}_1^d$ discretization for problem (D). In the local case, such equal-order pairs are generically unstable on arbitrary meshes. In the nonlocal setting, and especially for small values of s, one might expect to have better stability properties. However, our numerical experiments indicate that, as in the local case, the stability of the standard mixed formulation of the fractional Laplacian is not guaranteed when continuous Lagrange elements are employed for both the pressure and the flux.

We illustrate this point with a computational experiment. Using the finite element approximations described in Section 2.4, and implementing the matrices as outlined in Appendix A, we consider problem (D) with constant right-hand side $f = 1$ on the square domain $\Omega = (-1,1)^2$.

Figure 2.1 shows some pressures computed with continuous, piecewise linear finite elements on structured meshes with $s = 0.25$ and $s = 0.75$ for the finite element counterparts of (2.2.3) and (2.2.10). The results clearly show that the non-stabilized formulation produces spurious oscillations.

To further illustrate this observation, we compute the pressure H^s-errors on quasi-uniform meshes for problem (D) with $\Omega = B(0,1)$, $f \equiv 1$ and different values of s, for both the stabilized and the non-stabilized formulations. In this case, the pressure is given by (2.3.1) (see also Theorem 2.5.1). The results are summarized in Table 2.1 and show that the errors for the standard mixed formulation are significantly larger than those for its stabilized counterpart.

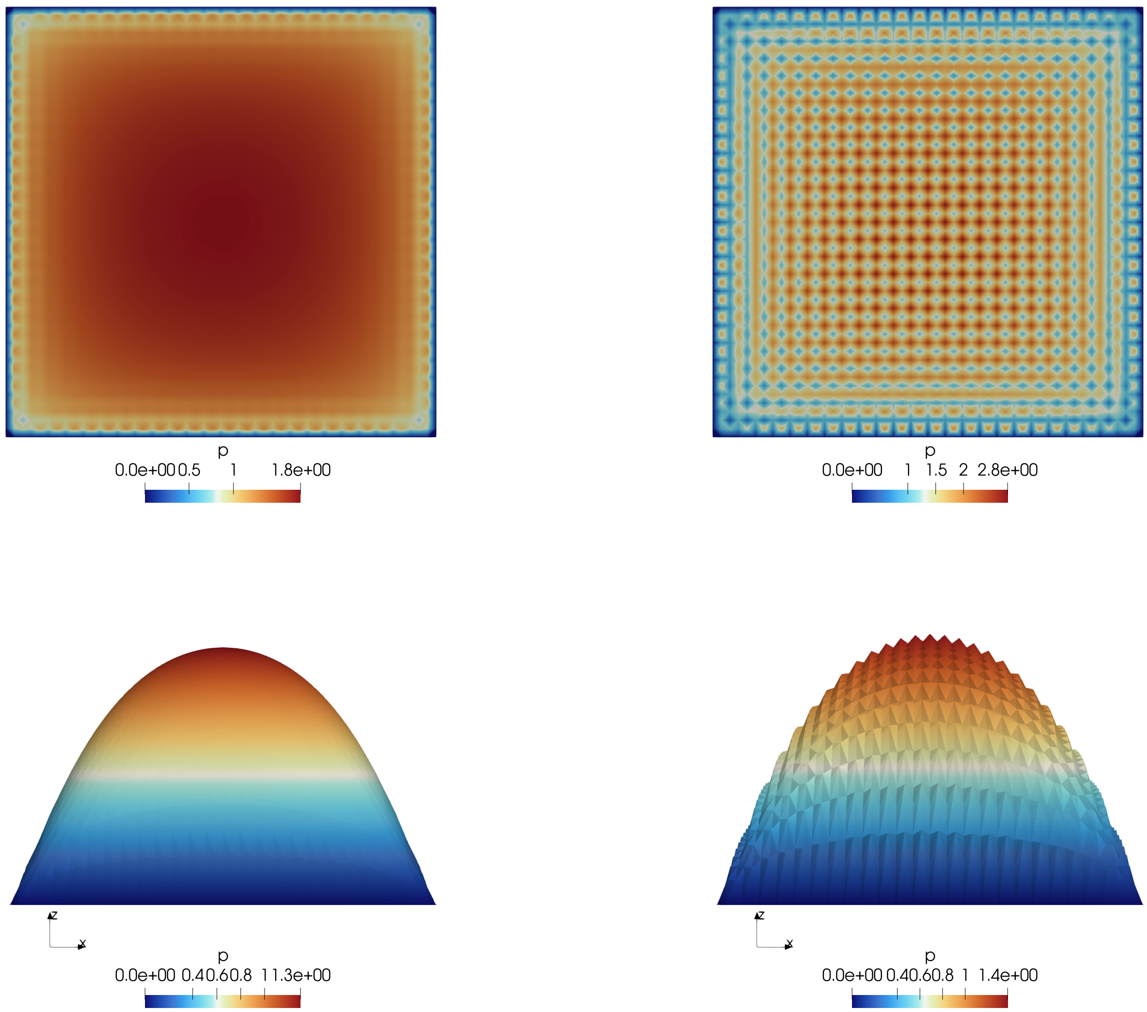


Figure 2.1: Finite element approximations of the pressure p in (D) for $s = 0.25$ (top row) and $s = 0.75$ (bottom) with the stabilization factor (left column) and without it (right). Here, $f \equiv 1$, $\Omega = (-1,1)^2$, the mesh size is $h = 0.04$ and the computational domain $B_H = B(0,2)$. See Section 2.4 for details on the finite element framework.

2.3 Sobolev regularity

Coercivity ensures that any conforming discretization satisfies a best-approximation property. To derive convergence rates, however, interpolation estimates and solution regularity are required. We address the latter in this section.

Sobolev regularity up to $\partial\Omega$ of u, the solution of (P), was established in [11]; for f in the Besov space $B^{-s+1/2}_{2,1}(\Omega)$, the solution u lies in the space $\cap_{\varepsilon>0}\widetilde{H}^{s+1/2-\varepsilon}(\Omega)$. This is the maximal expected regularity not only for arbitrary Lipschitz domains, but also for smooth domains as well. Indeed, a remarkable example arises when $\Omega = B(0,1)$ and $f \equiv 1$. The solution in this case corresponds to the first exit time of a $2s$-stable Lévy process in Ω and is given by

$$u(x) = k_{s,d}(1-|x|^2)^s \chi_{B(0,1)}(x), \tag{2.3.1}$$

$s = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.02$
Non-stabilized	0.8293	0.6234	0.4714	0.4452
Stabilized	0.1405	0.0985	0.0690	0.0617
$s = 0.5$	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.02$
Non-stabilized	0.7908	0.5833	0.4338	0.4124
Stabilized	0.1056	0.0705	0.0488	0.0443
$s = 0.8$	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.02$
Non-stabilized	0.7140	0.5177	0.3961	0.3786
Stabilized	0.0731	0.0503	0.0351	0.0314

Table 2.1: H^s-error for the pressure p on problem (2.5.1) for different values of s, using the standard (without stabilization) and stabilized mixed formulations, respectively.

with $k_{s,d} > 0$ an explicit constant (cf. Theorem 2.5.1). Despite both the domain and the right hand side being smooth, the solution satisfies

$$u \in \cap_{\varepsilon>0}\widetilde{H}^{s+1/2-\varepsilon}(\Omega), \quad u \notin \widetilde{H}^{s+1/2}(\Omega).$$

We point out that the hypothesis $f \in B^{-s+1/2}_{2,1}(\Omega)$ is weaker than $f \in L^2(\Omega)$ when $s > 1/2$; if $s \leq 1/2$, one can perform a simple interpolation argument to show that $u \in \cap_{\varepsilon>0}\widetilde{H}^{2s-\varepsilon}(\Omega)$ provided $f \in L^2(\Omega)$.

Moreover, having at hand the regularity of u, one can deduce a regularity estimate for the flux by means of standard mapping properties of ∇^s (cf. Theorem 1.3.5). We summarize the preceding discussion about regularity of solutions in the following proposition.

Proposition 2.3.1 (Regularity of solutions)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain and $f \in L^2(\Omega)$. Consider $(p, \mathbf{\Phi})$ the weak solution of* (D)*. We have*

$$\begin{aligned}
\|p\|_{\widetilde{H}^{s+\frac{1}{2}-\varepsilon}(\Omega)} + |\mathbf{\Phi}|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{R}^d)} &\leq \frac{C}{\sqrt{\varepsilon|1-2s|}}\|f\|_{L^2(\Omega)}, && \text{for } s > \frac{1}{2}, \\
\|p\|_{\widetilde{H}^{2s-\varepsilon}(\Omega)} + |\mathbf{\Phi}|_{H^{s-\varepsilon}(\mathbb{R}^d)} &\leq \frac{C}{\sqrt{\varepsilon|1-2s|}}\|f\|_{L^2(\Omega)}, && \text{for } s < \frac{1}{2}, \\
\|p\|_{\widetilde{H}^{1-\varepsilon}(\Omega)} + |\mathbf{\Phi}|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{R}^d)} &\leq \frac{C}{\varepsilon}\|f\|_{L^2(\Omega)}, && \text{for } s = \frac{1}{2},
\end{aligned} \tag{2.3.2}$$

for any $\varepsilon < \min\{2s, \frac{1}{2} + s\}$, with constants $C = C(d, \Omega)$.

Remark 2.3.2 (Regularity in case $f \in B^{-s+1/2}_{2,1}(\Omega)$)**.** *We stated the previous proposition under the simplifying assumption $f \in L^2(\Omega)$. However, as we already commented, the technique from [11] is better suited for f belonging to a certain Besov space. It turns out that, if $s > 1/2$, the assumption $f \in L^2(\Omega)$ in Proposition 2.3.1 can be relaxed to $f \in B^{-s+1/2}_{2,1}(\Omega)$. Additionally, if $s \leq 1/2$, then assuming $f \in B^{-s+1/2}_{2,1}(\Omega)$ is stronger than assuming $f \in L^2(\Omega)$, but one also has the stronger and uniform in s estimate*

$$\|p\|_{\widetilde{H}^{s+\frac{1}{2}-\varepsilon}(\Omega)} + |\mathbf{\Phi}|_{H^{\frac{1}{2}-\varepsilon}(\mathbb{R}^d)} \leq \frac{C}{\sqrt{\varepsilon}}\|f\|_{B^{-s+1/2}_{2,1}(\Omega)}. \tag{2.3.3}$$

See [10, Corollary 2.1]; we also refer to that work for a short discussion on Besov spaces.

The inequality (2.3.3) highlights that Besov regularity of the right-hand side translates into improved Sobolev-type estimates for the solution. In what follows, we pursue an analogous result involving weighted Bessel-type norms, where the regularity is expressed in terms of the Hölder continuity of the data. Indeed, Hölder regularity of solutions to (P) is well-known for Lipschitz domains and bounded right-hand sides, and was established in [47]. Furthermore, when f has some additional Hölder regularity one can deduce estimates for u in certain weighted Hölder norms.

For $x, y \in \Omega$, let $\delta(x) = d(x, \partial\Omega)$ and $\delta(x, y) = \min\{\delta(x), \delta(y)\}$. For $\beta > 0$ with $\beta = k + \beta'$, $k \in \mathbb{N}$ and $\beta' \in [0, 1)$, and $\theta \geq -\beta$ we define

$$|u|_{C^{k,\beta'}(\Omega)} := \sup_{x,y\in\Omega} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\beta'}},$$

i.e, the semi-norm of the Hölder space $C^{k,\beta'}(\Omega)$. We also define

$$|u|_{\beta}^{(\theta)} := \sup_{x,y\in\Omega} \delta(x,y)^{\beta+\theta} \frac{|D^k u(x) - D^k u(y)|}{|x-y|^{\beta'}},$$

together with the norm:

- for $\theta \geq 0$,

$$\|u\|_{\beta}^{(-\theta)} := \sum_{j=0}^{k} \|\delta^{j+\theta} D^j u\|_{L^\infty(\Omega)} + |u|_{\beta}^{(\theta)},$$

- for $0 > \theta \geq -\beta$,

$$\|u\|_{\beta}^{(-\theta)} := \|u\|_{C^{\lfloor -\theta \rfloor, -\theta + \lfloor \theta \rfloor}(\Omega)} + \sum_{j=1}^{k} \|\delta^{j+\theta} D^j u\|_{L^\infty(\Omega)} + |u|_{\beta}^{(\theta)}.$$

With these definitions in place, we recall the following results on the Hölder regularity of solutions to (P); see [47, Propositions 1.1 and 1.4].

Proposition 2.3.3 (Weighted Hölder regularity). *Let $\Omega \subset \mathbb{R}^d$ a Lipschitz domain satisfying the exterior ball condition and $f \in L^\infty(\Omega)$. Then the solution u of* (P) *belongs to $C^{0,s}(\mathbb{R}^d)$ and*

$$\|u\|_{C^{0,s}(\overline{\Omega})} \leq C(\Omega, s)\|f\|_{L^\infty(\Omega)}. \tag{2.3.4}$$

Moreover, let $\beta > 0$ such that neither β nor $\beta + 2s$ is an integer. Then, if $f \in C^\beta(\Omega)$ with $\|f\|_{\beta}^{(-s)} < \infty$, $u \in C^{\beta+2s}(\Omega)$ and

$$\|u\|_{\beta+2s}^{(-s)} \leq C(\Omega, s, \beta)\left(\|u\|_{C^{0,s}(\Omega)} + \|f\|_{\beta}^{(-s)}\right). \tag{2.3.5}$$

Let β be such that

$$\beta \in \begin{cases} (1, 2-2s) & \text{if } s \in (0, \frac{1}{2}), \\ (0, 2-2s) & \text{if } s \in [\frac{1}{2}, 1). \end{cases}$$

Estimate (2.3.5) yields

$$|u|_{\beta+2s}^{(-s)} = \sup_{x,y\in\Omega} \delta(x,y)^{\beta+s} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\beta+2s-1}} \leq C(\Omega, s, \beta, \|f\|_{\beta}^{(s)}) \tag{2.3.6}$$

and

$$\sup_{x\in\Omega} |\delta(x)^{1-s} \nabla u(x)| \leq C(\Omega, s, \beta, \|f\|_{\beta}^{(s)}). \tag{2.3.7}$$

These two estimates allow us to bound the weighted quantity $|\delta(x)^\alpha \nabla^\gamma u(x)|$ for $x \in \Omega$.

Theorem 2.3.4 (Weighted pressure estimates)**.** *Let $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain satisfying the exterior ball condition, the function $f \in C^\beta(\Omega)$ with $\|f\|_\beta^{(-s)} < \infty$ for some $\beta \in [1-s, 2-2s) \setminus \{1\}$, and $(p, \mathbf{\Phi})$ be the weak solution of* (D)*. Then, p satisfies*

$$\|\delta^{\frac{1}{2}-\varepsilon} \nabla^{1+s-2\varepsilon} p\|_{L^2(\Omega)} \leq \frac{C(\Omega, s, f)}{(\beta - 1 + s + 2\varepsilon)(s - 2\varepsilon)\sqrt{\varepsilon}} < \infty \quad \textit{for all } \varepsilon \in (0, s/2).$$

Proof. Let $\gamma \in (1,2)$, $\alpha > 0$ and $\beta > 0$ be such that $f \in C^\beta(\Omega)$ with $\|f\|_\beta^{(-s)} < \infty$. Specific requirements on these parameters are derived in the following. According to (1.1.7), for $x \in \Omega$ we have

$$|\delta(x)^\alpha \nabla^\gamma p(x)| \leq C \int_{\mathbb{R}^d} \delta(x)^\alpha \frac{|p(x+h) - p(x) - \nabla p(x) \cdot h|}{|h|^{d+\gamma}} dh.$$

We split the integration domain as $\mathbb{R}^d = B(0, \delta(x)/2) \cup B(0, \delta(x)/2)^c$ and denote the corresponding integrals as (I) and (II), respectively.

For $h \in B(0, \delta(x)/2)$, by the mean value theorem we can write $p(x+h) - p(x) = \nabla p(x + \lambda(h)\, h) \cdot h$ for some $\lambda \in (0,1)$. Moreover, we have $\delta(x) \simeq \delta(x, x + \lambda(h)\, h)$. Therefore, if $\beta + 2s - \gamma > 0$, estimate (2.3.6) gives

$$\begin{aligned} (I) &\leq C\delta(x)^\alpha \int_{B(0,\delta(x)/2)} \frac{|\nabla p(x + \lambda(h)\, h) - \nabla p(x)|}{|h|^{d+\gamma-1}} dh \\ &\leq C(\Omega, s, \beta, f)\, \delta(x)^{\alpha-\beta-s} \int_{B(0,\delta(x)/2)} \frac{dh}{|h|^{d+\gamma-\beta-2s}} \\ &\leq \frac{C(\Omega, s, \beta, f)}{\beta + 2s - \gamma} \delta(x)^{\alpha-\gamma+s}. \end{aligned}$$

In order to bound (II) we employ the Hölder regularity (2.3.4) and the estimate (2.3.7) to deduce

$$\begin{aligned} (II) &\leq C \left[\delta^\alpha(x) \int_{B(0,\delta(x)/2)^c} \frac{1}{|h|^{d+\gamma-s}} dh + \delta(x)^{\alpha-1+s} \int_{B(0,\delta(x)/2)^c} \frac{1}{|h|^{d+\gamma-1}} dh \right] \\ &\leq \frac{C(\Omega, s, \beta, f)}{\gamma - 1} \delta(x)^{\alpha-\gamma+s}, \end{aligned}$$

because we are assuming $\gamma > 1$. Therefore, we conclude

$$|\delta(x)^\alpha \nabla^\gamma p(x)| \leq \frac{C(\Omega, s, \beta, f)}{(\beta + 2s - \gamma)(\gamma - 1)} \delta(x)^{\alpha-\gamma+s}$$

for $\beta + 2s - \gamma > 0$. Taking squares and integrating over Ω, we obtain

$$\begin{aligned} \|\delta^\alpha \nabla^\gamma p\|_{L^2(\Omega)}^2 &\leq \frac{C(\Omega, s, \beta, f)}{(\beta + 2s - \gamma)^2(\gamma - 1)^2} \int_\Omega \delta(x)^{2(\alpha-\gamma+s)} dx \\ &\leq \frac{C(\Omega, s, \beta, f)}{(\beta + 2s - \gamma)^2(\gamma - 1)^2(1 + 2(\alpha + s - \gamma))}, \end{aligned}$$

where we have used that for $\theta \in [0,1]$ we have (cf. [18, Lemma 2.14])

$$\int_\Omega \delta(x)^\theta \, dx = \mathcal{O}\left(\frac{1}{1-\theta}\right),$$

and assumed

$$\beta > \gamma - 2s, \quad \alpha > -\frac{1}{2} - s + \gamma. \tag{2.3.8}$$

As long as these two conditions are met, we can guarantee $\delta^\alpha \nabla^\gamma p \in L^2(\Omega)$. We choose, for $\varepsilon > 0$, $\alpha = \frac{1}{2} - \varepsilon$ and $\gamma = 1 + s - 2\varepsilon$ to conclude

$$\|\delta^{\frac{1}{2}-\varepsilon}\nabla^{1+s-2\varepsilon}p\|_{L^2(\Omega)} \leq \frac{C(\Omega, s, f)}{(\beta - 1 + s + 2\varepsilon)(s - 2\varepsilon)\sqrt{\varepsilon}}.$$

□

Remark 2.3.5 (Global estimates). *The same technique allows one to show that, for Ω_λ being a neighborhood of Ω, one has $\delta^{\frac{1}{2}-\varepsilon}\nabla^{1+s-2\varepsilon}p \in L^2(\Omega_\lambda)$. Indeed, it suffices to observe that, for $x \in \Omega_\lambda \setminus \Omega$, it holds that*

$$|\nabla^{1+s-2\varepsilon}p(x)| \leq C \int_{B(0,\delta(x))^c} \frac{|p(x+h)|}{|h|^{d+1+s-2\varepsilon}}\, dh,$$

and therefore the same argument as for the term (II) in the previous proof can be applied. On regions uniformly away from $\partial\Omega$, this bound also shows that $\nabla^{1+s-2\varepsilon}$ is smooth; see Lemma 2.4.5 below.

The regularity provided by Theorem 2.3.4 is analogous to the fractional weighted Sobolev regularity established in [2, Proposition 3.12]. This suggests that graded meshes could be employed to exploit such regularity and achieve higher convergence rates in the numerical scheme. However, proving that these improved rates are indeed attained would require a weighted (or enhanced) Poincaré inequality in the spirit of [2, Proposition 4.7], which remains a subject of ongoing research. Nevertheless, our numerical experiments in Section 2.5.3 exhibit improved convergence orders when graded meshes are used.

Remark 2.3.6 (Choice of parameters). *The choice of parameters in Theorem 2.3.4 is motivated by its application for finite element approximations in $d = 2$ dimensions. Indeed, one can show $\delta^\alpha \nabla^\gamma p \in L^2(\Omega)$ as long as* (2.3.8) *is satisfied, but the application of weighted regularity estimates requires the use of adapted meshes. For the discretization of the Poisson problem* (P) *in $d = 2$ dimensions, reference [8] shows that the choice of parameters we made in Theorem 2.3.4 gives rise to optimal interpolation error bounds in $\widetilde{H}^s(\Omega)$, with respect to the number of degrees of freedom, over shape-regular meshes.*

2.4 Finite element discretization of the Darcy problem

By replacing $\mathcal{L}$ with $\mathcal{L}_{\text{stab}}$, we obtain a coercive formulation. Consequently, any conforming finite element space yields a stable discretization. In this work, we focus on linear Lagrange elements for both the pressure p and the flux $\mathbf{\Phi}$.

Let us begin by describing the discrete framework that we will use. We observe that we are approximating $\mathbf{\Phi}$, which is not compactly supported, and both the form a in (2.2.4) and the stabilization term in (2.2.8) involve integration in $\mathbb{R}^d$. To tackle this issue, we consider $H = H(h) \geq 1$ and our computational domains will be a family of sets B_H containing Ω and such that $H \simeq d(\overline{\Omega}, B_H^c)$ and $|B_H| \simeq H^d$. To fix ideas, given H, one can regard B_H to be an approximation of a ball of radius H. We emphasize that, in theory, in order to approximate the long-range interactions in the fractional Laplacian, one must have $H(h) \to \infty$ as $h \to 0$.

Let $\{\mathcal{T}_h(B_H)\}_{h>0}$ be a family of simplicial meshes of $\overline{B_H}$, whose elements $\{T\}_{T\in\mathcal{T}_h(B_H)}$ are assumed to be closed, that satisfy:

- Shape-regularity, i.e., there exist a constant $\sigma > 0$ such that

$$\sup_{h>0} \sup_{T \in \mathcal{T}_h} \frac{h_T}{\rho_T} = \sigma, \tag{2.4.1}$$

where $h_T = diam(T)$ and ρ_T is the diameter of the largest ball contained in T.

- For every $h > 0$, the set

$$\mathcal{T}_h(\Omega) = \{T \in \mathcal{T}_h(B_H) : T \cap \Omega \neq \emptyset\}$$

is a simplicial triangulation of $\overline{\Omega}$.

We denote by $\mathcal{N}_h = \{z_i : i = 1, \ldots, N_h\}$ the set of nodes of $\mathcal{T}_h(B_H)$. We set $n_h = \#(\mathcal{N}_h \cap \Omega)$ and assume that the nodes are labeled so that those belonging to Ω come first, i.e., $\mathcal{N}_h \cap \Omega = \{z_1, \ldots, z_{n_h}\}$. Let $\{\varphi_i\}_{i=1}^{N_h}$ be the standard piecewise linear Lagrange nodal basis associated with $\mathcal{N}_h$ and B_i be the largest ball centered at z_i and contained in $supp(\varphi_i)$. The finite element space is defined as

$$\mathbb{V}_h = \{(q_h, \mathbf{\Psi}_h) \in \mathcal{P}_1(\mathcal{T}_h(B_H)) \times [\mathcal{P}_1(\mathcal{T}_h(B_H))]^d \subset \mathbb{V} : q_h|_{\Omega^c} = 0\}, \tag{2.4.2}$$

where we assume that discrete functions are extended by zero outside of the computational domain B_H. This introduces a consistency error that must be controlled by choosing H accordingly (cf. Lemma 2.4.5).

The discretization of problem (2.2.10) reads: find $(p_h, \mathbf{\Phi}_h) \in \mathbb{V}_h$ such that

$$\mathcal{L}_{\text{stab}}((p_h, \mathbf{\Phi}_h), (q_h, \mathbf{\Psi}_h)) = F(q_h) \quad \forall (q_h, \mathbf{\Psi}_h) \in \mathbb{V}_h. \tag{2.4.3}$$

The coercive formulation and the fact that $\mathbb{V}_h \subset \mathbb{V}$ immediately imply the existence and uniqueness of solutions to (2.4.3), and the best approximation property stated below.

Proposition 2.4.1 (Best approximation)**.** *Let* $(p, \mathbf{\Phi}) \in \mathbb{V}$ *and* $(p_h, \mathbf{\Phi}_h) \in \mathbb{V}_h$ *be the solutions to* (2.2.10) *and* (2.4.3)*, respectively. We have the following Galerkin orthogonality:*

$$\mathcal{L}_{\text{stab}}((p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h), (q_h, \mathbf{\Psi}_h)) = 0 \quad \forall (q_h, \mathbf{\Psi}_h) \in \mathbb{V}_h. \tag{2.4.4}$$

Consequently, we obtain

$$|||(p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h)||| = \min_{(q_h, \mathbf{\Psi}_h) \in \mathbb{V}_h} |||(p - q_h, \mathbf{\Phi} - \mathbf{\Psi}_h)|||. \tag{2.4.5}$$

Proof. Let $(q_h, \mathbf{\Psi}_h) \in \mathbb{V}_h \subset \mathbb{V}$. First, we have

$$\begin{aligned}
\mathcal{L}_{\text{stab}}((p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h), (q_h, \mathbf{\Psi}_h)) &= \mathcal{L}_{\text{stab}}((p, \mathbf{\Phi}), (q_h, \mathbf{\Psi}_h)) - \mathcal{L}_{\text{stab}}((p_h, \mathbf{\Phi}_h), (q_h, \mathbf{\Psi}_h)) \\
&= F(q_h) - F(q_h) \\
&= 0.
\end{aligned}$$

In second place, by the coercivity and continuity of $\mathcal{L}_{\text{stab}}$ and the Galerkin orthogonality, we deduce that, for all $(q_h, \mathbf{\Psi}_h) \in \mathbb{V}_h$,

$$\begin{aligned}
|||(p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h)|||^2 &= \mathcal{L}_{\text{stab}}((p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h), (p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h)) \\
&= \mathcal{L}_{\text{stab}}((p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h), (p - q_h, \mathbf{\Phi} - \mathbf{\Psi}_h)) \\
&\leq |||(p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h)||| \, |||(p - q_h, \mathbf{\Phi} - \mathbf{\Psi}_h)|||,
\end{aligned}$$

and (2.4.5) follows. $\square$

2.4.1 Quasi-interpolation

Our next task to derive convergence rates is to obtain interpolation estimates. The standard Lagrange interpolation is not a feasible option in our setting because of the low regularity of solutions and its lack of stability in the corresponding low-order fractional Sobolev spaces. Therefore, we will use the quasi-interpolation operator introduced in [20]. Other suitable choices of quasi-interpolation (e.g. Clément [21], Scott-Zhang [51]) would also be adequate for our purposes.

Definition 2.4.2 (Quasi-interpolation operators)**.** *We define* $\Pi_h : L^1(\Omega) \to \mathcal{P}_1(\mathcal{T}_h(B_H))$ *and* $\mathbf{\Pi}_h : (L^1(B_H))^d \to \mathcal{P}_1^d(\mathcal{T}_h(B_H))$ *as*

$$\Pi_h q := \sum_{z_i \in \Omega} \left(\frac{1}{|B_i|} \int_{B_i} q(x)\, dx \right) \varphi_i,$$
$$\mathbf{\Pi}_h \mathbf{\Psi} := \sum_{z_i \in B_H} \left(\frac{1}{|B_i|} \int_{B_i} \mathbf{\Psi}(x)\, dx \right) \varphi_i.$$

We refer to [20, 13] for basic properties of this operator. We are concerned with its stability and approximation properties with respect to fractional-order seminorms. To this end, given $T \in \mathcal{T}_h(B_H)$, we define the sets (see Figure 2.2)

$$S_T^1 = \bigcup_{T' \in \mathcal{T}_h(B_H) \colon T' \cap T \neq \emptyset} T', \qquad S_T^2 = \bigcup_{T' \in \mathcal{T}_h(B_H) \colon T' \cap S_T^1 \neq \emptyset} T'.$$

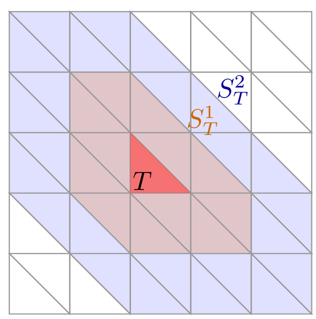


Figure 2.2: Illustration of the sets S_T^1 and S_T^2.

It is well known that fractional semi-norms are, in general, not additive with respect to domain decompositions. However, by allowing some overlapping, localization becomes possible. In [33], the following localization of the Gagliardo semi-norm is provided:

$$|q|^2_{H^s(\Omega)} \leq C \sum_{T \in \mathcal{T}_h(\Omega)} \left(\iint_{T \times (S_T^1 \cap \Omega)} \frac{|q(x) - q(y)|^2}{|x-y|^{d+2s}} dy\, dx + \frac{\|q\|^2_{L^2(T)}}{h_T^{2s}} \right). \tag{2.4.6}$$

The constant above depends on d, s, and the shape-regularity parameter σ from (2.4.1). We are interested in an estimate of this kind for the $\widetilde{H}^s(\Omega)$-norm, i.e, for $|q|_{H^s(\mathbb{R}^d)}$ when $q \in \widetilde{H}^s(\Omega)$. This is the goal of the next lemma (see also [12, Lemma 4.1]).

Lemma 2.4.3 (Localization of the $\widetilde{H}^s(\Omega)$-norm)**.** *Let Ω be a bounded Lipschitz domain and $\mathcal{T}_h(B_H)$ as above. Then it holds,*

$$\|q\|^2_{\widetilde{H}^s(\Omega)} = |q|^2_{H^s(\mathbb{R}^d)} \leq C \sum_{T\in\mathcal{T}_h(\Omega)} \left(\iint_{T\times S_T^1} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx + \frac{\|q\|^2_{L^2(T)}}{h_T^{2s}} \right),$$

for all $q \in \widetilde{H}^s(\Omega)$.

Proof. Let $\Omega_h = \bigcup_{T\in\mathcal{T}_h(\Omega)} S_T^1$ and $q \in \widetilde{H}^s(\Omega)$. Observing that the integral over $\Omega_h^c \times \Omega_h^c$ in the definition of $|q|_{H^s(\mathbb{R}^d)}$ is zero, we obtain

$$\begin{aligned}\frac{2}{\nu(d,s)}|q|^2_{H^s(\mathbb{R}^d)} &= \iint_{\Omega_h\times\Omega_h} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx + 2\iint_{\Omega_h\times\Omega_h^c} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx \\ &\leq \frac{2}{\nu(d,s)}|q|_{H^s(\Omega_h)} + 2\int_\Omega |q(x)|^2 \int_{\Omega_h^c} |x-y|^{-d-2s} dy\, dx.\end{aligned}$$

Now, for every element $T \subset \Omega$, by the shape-regularity of $\mathcal{T}_h$ we have $d(T,\Omega_h^c) \geq Ch_T$. Thus, integrating in polar coordinates we derive

$$\int_\Omega |q(x)|^2 \int_{\Omega_h^c} |x-y|^{-d-2s} dy\, dx \leq C \sum_{T\in\mathcal{T}_h(\Omega)} \frac{\|q\|^2_{L^2(T)}}{h_T^{2s}},$$

that yields

$$|q|^2_{H^s(\mathbb{R}^d)} \leq C \left(|q|^2_{H^s(\Omega_h)} + \sum_{T\in\mathcal{T}_h(\Omega)} \frac{\|q\|^2_{L^2(T)}}{h_T^{2s}} \right). \tag{2.4.7}$$

Next, using (2.4.6) we deduce

$$|q|^2_{H^s(\Omega_h)} \leq C(d,s) \sum_{T\subset\Omega_h} \left(\iint_{T\times S_T^1} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx + \frac{\|q\|^2_{L^2(T)}}{h_T^{2s}} \right).$$

The sum above contains many terms that are zero, as interactions between elements in Ω^c vanish. Indeed, $q \equiv 0$ in Ω^c, and hence for elements $T, T' \subset \Omega_h \backslash \Omega$ it holds

$$\iint_{T\times T'} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx = \iint_{T'\times T} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx = 0.$$

Thus,

$$\iint_{T\times S_T^1} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx = \iint_{T\times(S_T^1\cap\Omega)} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx,$$

for every $T \subset \Omega_h \setminus \Omega$. Therefore, we deduce

$$|q|_{H^s(\Omega_h)} \leq C(d,s) \sum_{T\in\mathcal{T}_h(\Omega)} \left(\iint_{T\times S_T^1} \frac{|q(x)-q(y)|^2}{|x-y|^{d+2s}} dy\, dx + \frac{\|q\|^2_{L^2(T)}}{h_T^{2s}} \right).$$

The proof follows by combining this estimate with (2.4.7). □

Lemma 2.4.3 allows to obtain global interpolation estimates on $\widetilde{H}^s(\Omega)$ from local considerations. Let $s \in (0,1)$, $t \in (s,2]$ and σ the shape-regularity constant (2.4.1). For $T \in \mathcal{T}_h(B_H)$, we have

$$\iint_{T\times S_T^1} \frac{|(q-\Pi_h q)(x)-(q-\Pi_h q)(y)|^2}{|x-y|^{d+2s}}\,dy\,dx \le \frac{C(d,\sigma,t)}{1-s} h_T^{2(t-s)} |q|^2_{H^t(S_T^2)}, \tag{2.4.8}$$

see [13, Proposition 4.10]. Additionally, for $t \in [0,2]$ we have the L^2-approximation bound

$$\|q-\Pi_h q\|_{L^2(T)} \le C h_T^t |q|_{H^t(S_T^1)}, \tag{2.4.9}$$

with a constant C independent of T and h. For $t = 0$, the inequality follows from [20, Lemma 3.1], while for $t = 2$ the inequality follows from [20, Lemma 3.2]. By interpolation between the cases $t = 0$ and $t = 2$, we obtain (2.4.9). Naturally, estimates (2.4.8) and (2.4.9) also hold for $\mathbf{\Pi}_h$.

Combining the local interpolation estimates (2.4.8) and (2.4.9), and the localization provided by Lemma 2.4.3, we deduce global interpolation estimates. In particular, for quasi-uniform meshes, these read as follows.

Lemma 2.4.4 (Global interpolation estimates)**.** *Let $s \in (0,1)$, $t \in (s,2]$, $r \in (0,2]$, and $\mathcal{T}_h(B_H)$ be a quasi-uniform mesh. Then, we have*

$$\begin{aligned} \|q-\Pi_h q\|_{\widetilde{H}^s(\Omega)} &\le C(d,\sigma,t) h^{t-s} \|q\|_{\widetilde{H}^t(\Omega)}, \\ \|\mathbf{\Psi}-\mathbf{\Pi}_h\mathbf{\Psi}\|_{L^2(B_H)} &\le C(d,\sigma,t) h^r |\mathbf{\Psi}|_{H^r(\mathbb{R}^d)}, \end{aligned} \tag{2.4.10}$$

for all $q \in \widetilde{H}^t(\Omega)$ and $\mathbf{\Psi} \in [H^r(\mathbb{R}^d)]^d$.

The coercivity norm $|||(p,\mathbf{\Phi})|||$ involves the $\widetilde{H}^s(\Omega)$-seminorm of p and the $L^2(\mathbb{R}^d)$ norm of $\mathbf{\Phi}$. Thus, taking into account (2.4.10), to conclude an interpolation estimate we need to address $\|\mathbf{\Phi} - \mathbf{\Pi}_h\mathbf{\Phi}\|_{L^2(B_H^c)}$. Since $\mathbf{\Pi}_h\mathbf{\Phi}$ vanishes outside such a set, this calculation reduces to bounding the decay of $|\mathbf{\Phi}|$.

For the sake of simplicity, from this point on **we assume that** $0 \in \Omega$, so that $d(x,\Omega) \simeq |x|$ for all $x \in B_H^c$.

Lemma 2.4.5 (Flux decay)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded, Lipschitz domain such that $0 \in \Omega$, and $(p,\mathbf{\Phi}) \in \mathbb{V}$ be the solution to (2.2.10). We have, for all $x \in \Omega^c$ and all multi-index $\alpha \in \mathbb{N}^d$,*

$$|\partial_\alpha \mathbf{\Phi}(x)| \le \frac{C(d,s,\alpha)}{d(x,\Omega)^{d+s+|\alpha|}} \|p\|_{L^1(\Omega)}.$$

Consequently,

$$\|\partial_\alpha \mathbf{\Phi}\|_{L^2(B_H^c)} \le C(d,s,\alpha,\Omega) H^{-\frac{d}{2}-s-|\alpha|} \|f\|_{L^2(\Omega)}. \tag{2.4.11}$$

Proof. First, for $z \in \mathbb{R}^d \backslash \{0\}$ it holds

$$\frac{\partial_\alpha}{\partial z}\left(\frac{z}{|z|^{d+s+1}}\right) = \frac{Q_\alpha(z)}{|z|^{d+s+1+2|\alpha|}},$$

with $Q_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ a polynomial vector field that is componentwise homogeneous of degree $|\alpha|+1$. Therefore, for $c \in \mathbb{R}$ depending only on the coefficients of the components of Q_α, we have

$$\left|\frac{\partial_\alpha}{\partial z}\left(\frac{z}{|z|^{d+s+1}}\right)\right| \le \frac{c}{|z|^{d+s+|\alpha|}}.$$

Since $\mathbf{\Phi} = \nabla^s p$ in $\mathbb{R}^d$, for all $x \in \Omega^c$ and $y \in \Omega$, we exploit the fact that $|x-y| \geq d(x,\Omega)$ to deduce

$$\begin{aligned}
|\partial_\alpha \mathbf{\Phi}(x)|^2 &= \left| \mu(d,s) \int_\Omega p(y) \frac{\partial_\alpha}{\partial x} \left(\frac{x-y}{|x-y|^{d+s+1}} \right) dy \right|^2 \\
&\leq C \left(\int_\Omega |p(y)| \frac{1}{|x-y|^{d+s+|\alpha|}} dy \right)^2 \\
&\leq \frac{C}{d(x,\Omega)^{2(d+s+|\alpha|)}} \|p\|^2_{L^1(\Omega)},
\end{aligned}$$

with $C = C(d,s,\alpha)$.

Therefore, noticing that $d(x,\Omega) \geq c|x|$ for some constant c, a change of variables to polar coordinates gives the bound

$$\begin{aligned}
\|\partial_\alpha \mathbf{\Phi}\|^2_{L^2(B_H^c)} &\leq C \, \|p\|^2_{L^1(\Omega)} \int_{B_H^c} \frac{1}{d(x,\Omega)^{2(d+s+|\alpha|)}} \, dx \\
&= C \, \|p\|^2_{L^1(\Omega)} \int_H^\infty \rho^{-d-2s-2|\alpha|-1} \, d\rho \\
&\leq C \, \|p\|^2_{L^1(\Omega)} H^{-d-2s-2|\alpha|},
\end{aligned}$$

with a constant $C = C(d,s,\alpha)$. The inequality (2.4.11) follows by observing $\|p\|_{L^1(\Omega)} \leq C(\Omega,s)\|f\|_{L^2(\Omega)}$.
$\square$

2.4.2 Convergence in the stability norm

Collecting the previous interpolation and decay estimates, together with the regularity of solutions, we obtain convergence rates for our numerical scheme. We start with globally uniform meshes.

Proposition 2.4.6 (Order of convergence). *Let $(p, \mathbf{\Phi}) \in \mathbb{V}$ and $(p_h, \mathbf{\Phi}_h) \in \mathbb{V}_h$ be the solutions to* (2.2.10) *and* (2.4.3), *respectively. Assume that $\mathcal{T}_h(B_H)$ is a quasi uniform mesh and that the auxiliary mesh grows according to $H \geq (h|\log h|)^{\frac{-1}{d+2s}}$.*

Then, we have the following convergence rates with respect to the mesh size $h > 0$,

$$|||(p - p_h, \mathbf{\Phi} - \mathbf{\Phi}_h)||| \leq \begin{cases} Ch^{\frac{1}{2}} |\log h|^{\frac{1}{2}} \|f\|_{L^2(\Omega)}, & \text{for } s > \frac{1}{2}, \\ Ch^{\frac{1}{2}} |\log h| \, \|f\|_{L^2(\Omega)}, & \text{for } s = \frac{1}{2}, \\ Ch^s |\log h|^{\frac{1}{2}} \|f\|_{L^2(\Omega)}, & \text{for } s < \frac{1}{2}, \end{cases} \tag{2.4.12}$$

with a constant $C = C(d,s,\sigma)$.

Remark 2.4.7 (Convergence rates in terms of mesh nodes). *For practical purposes, instead of* (2.4.12) *it is more relevant to consider convergence rates in terms of the number of interior nodes. Since we approximate $\mathbb{R}^d$ by meshing B_H and assume $H \geq (h|\log h|)^{\frac{-1}{d+2s}}$, the use of uniform meshes implies that $N_h \gg n_h$, where we recall that n_h and N_h denote the number of nodes in the meshes $\mathcal{T}_h(\Omega)$ and $\mathcal{T}_h(B_H)$, respectively.*

To write (2.4.12) *in terms of N_h, we observe that $|B_H| \simeq H^d$ and, for an quasi uniform mesh with size h, $|B_H| \simeq N_h h^d$, which implies $N_h \simeq \frac{H^d}{h^d}$, and thus*

$$N_h^{-\frac{d+2s}{d(1+d+2s)}} \geq h.$$

This, combined with the previous result, allows us to rewrite Proposition 2.4.6 in terms of the total degrees of freedom on globally quasi-uniform meshes:

$$\|\!|\!|(p-p_h,\mathbf{\Phi}-\mathbf{\Phi}_h)|\!|\!\| \le \begin{cases} CN_h^{-\frac{d+2s}{2d(1+d+2s)}}|\log N_h|^{\frac{1}{2}}\|f\|_{L^2(\Omega)}, & \textit{for } s>\frac{1}{2},\\ CN_h^{-\frac{d+1}{2d(2+d)}}|\log N_h|\,\|f\|_{L^2(\Omega)}, & \textit{for } s=\frac{1}{2},\\ CN_h^{-\frac{s(d+2s)}{d(1+d+2s)}}|\log N_h|^{\frac{1}{2}}\|f\|_{L^2(\Omega)}, & \textit{for } s<\frac{1}{2}. \end{cases} \tag{2.4.13}$$

We emphasize, however, that these bounds correspond to globally quasi-uniform meshes. Since the pressure vanishes on Ω^c, and Lemma 2.4.5 shows that the flux is smooth and decays rapidly away from $\overline{\Omega}$, the mesh on Ω^c can be safely coarsened as the distance from Ω increases. This allows for significantly fewer degrees of freedom than those predicted by (2.4.13)*, without compromising the convergence rate with respect to h. We further discuss this strategy in Section 2.4.4.*

Proof of Proposition 2.4.6. We provide details for the case $s>\frac{1}{2}$. The other ones follow by the same argument. By combining the best approximation property (cf. Proposition 2.4.1) with the interpolation estimates (2.4.10) and Lemma 2.4.5 with $\alpha=0$, we obtain

$$\|\!|\!|(p-p_h,\mathbf{\Phi}-\mathbf{\Phi}_h)|\!|\!\|^2 \le C\left(h^{2t-2s}\|p\|^2_{\widetilde{H}^t(\Omega)} + h^{2r}|\mathbf{\Phi}|^2_{H^r(\mathbb{R}^d)} + H^{-d-2s}\|f\|^2_{L^2(\Omega)}\right), \tag{2.4.14}$$

for $t,r\in(0,2)$. Now, by choosing $t=s+\frac{1}{2}-\varepsilon$ and $\varepsilon=|\log h|^{-1}$ (so that $h^{-\varepsilon}$ is constant for $h<1$), we can estimate the first term in the right hand side above by means of the regularity estimates (cf. Proposition 2.3.1);

$$h^{2t-2s}\|p\|^2_{\widetilde{H}^t(\Omega)} \le Ch|\log h|\|f\|^2_{L^2(\Omega)}.$$

The same argument with $r=\frac{1}{2}-\varepsilon$ and $\varepsilon=|\log h|^{-1}$ shows

$$h^{2r}|\mathbf{\Phi}|^2_{H^r(\mathbb{R}^d)} \le Ch|\log h|\|f\|^2_{L^2(\Omega)}.$$

Finally, it suffices to observe that the third term in the right-hand side of (2.4.14) is (at least) of the same order with respect to h because of our choice of H. □

Remark 2.4.8 (Orders in terms of Besov regularity)**.** *For $f\in B^{-s+1/2}_{2,1}(\Omega)$, the proof of Proposition 2.4.6 can be repeated with the regularity estimate* (2.3.3) *in place of* (2.3.2)*. In this case, e.g. the bound* (2.4.12) *takes the form*

$$\|\!|\!|(p-p_h,\mathbf{\Phi}-\mathbf{\Phi}_h)|\!|\!\| \le Ch^{\frac{1}{2}}|\log h|^{\frac{1}{2}}\|f\|_{B^{-s+1/2}_{2,1}(\Omega)}, \tag{2.4.15}$$

for all $s\in(0,1)$.

2.4.3 Convergence order for the pressure in the L^2-norm

According to the bound (2.4.15), for f smooth enough and $s\in(0,1)$, the order of convergence for the pressure p in (D) in the $\widetilde{H}^s(\Omega)$-norm is $\frac{1}{2}$ (up to logarithmic factors) with respect to the mesh parameter h. Here, we develop an argument along the lines of the well-known Aubin-Nitsche duality trick to derive the order of convergence of p in the L^2 norm.

Proposition 2.4.9 (Aubin-Nitsche argument)**.** *Let* $(p, \mathbf{\Phi}) \in \mathbb{V}$ *be the solution to* (2.2.10) *and* $(p_h, \mathbf{\Phi}_h) \in \mathbb{V}_h$ *be the solution to* (2.4.3). *Assume that* $\mathcal{T}_h(B_H)$ *is a quasi uniform mesh and that the computational domain grows according to* $H \geq (h|\log h|)^{\frac{-1}{d+2s}}$. *Then, if* $f \in B^{-s+1/2}_{2,1}(\Omega)$, *it holds that*

$$\|p - p_h\|_{L^2(\Omega)} \leq C h^{\frac{1}{2}+\min\{s,\frac{1}{2}\}} |\log h|^{\kappa+\frac{1}{2}} \|f\|_{B^{-s+\frac{1}{2}}_{2,1}(\Omega)}, \tag{2.4.16}$$

with a constant $C = C(d, s, \sigma)$. *Here,* $\kappa = 1$ *if* $s = \frac{1}{2}$ *and* $\kappa = \frac{1}{2}$ *otherwise.*

Proof. Define the errors $e^p := p - p_h$ and $e^{\mathbf{\Phi}} := \mathbf{\Phi} - \mathbf{\Phi}_h$. From the Galerkin orthogonality (2.4.4), we deduce

$$\int_{\mathbb{R}^d} \nabla^s e^p \cdot (\nabla^s q_h + \mathbf{\Psi}_h) = \int_{\mathbb{R}^d} e^{\mathbf{\Phi}} \cdot (\nabla^s q_h - \mathbf{\Psi}_h), \tag{2.4.17}$$

for all $(q_h, \mathbf{\Psi}_h) \in \mathbb{V}_h$. Let $(p^e, \mathbf{\Phi}^e) \in \mathbb{V}$ be the weak solution to the dual problem (D) with $f = e^p$. Then, using $\mathbf{\Phi}^e = -\nabla^s p^e$, we obtain

$$\begin{aligned} \|e^p\|^2_{L^2(\Omega)} &= \int_{\mathbb{R}^d} \nabla^s p^e \cdot \nabla^s e^p \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \nabla^s p^e \cdot \nabla^s e^p - \frac{1}{2} \int_{\mathbb{R}^d} \mathbf{\Phi}^e \cdot \nabla^s e^p. \end{aligned} \tag{2.4.18}$$

Combining equations (2.4.18) and (2.4.17), with $q_h = \Pi_h p^e$ and $\mathbf{\Psi}_h = -\mathbf{\Pi}_h \mathbf{\Phi}^e$, we deduce

$$\begin{aligned} 2\|e^p\|^2_{L^2(\Omega)} = &\int_{\mathbb{R}^d} \nabla^s e^p \cdot \nabla^s [p^e - \Pi_h p^e] - \int_{\mathbb{R}^d} \nabla^s e^p \cdot (\mathbf{\Phi}^e - \mathbf{\Pi}_h \mathbf{\Phi}^e) \\ &+ \int_{\mathbb{R}^d} e^{\mathbf{\Phi}} \cdot (\mathbf{\Pi}_h \mathbf{\Phi}^e + \nabla^s \Pi_h p^e). \end{aligned}$$

Since $\mathbf{\Phi}^e = -\nabla^s p^e$, this simplifies to

$$\begin{aligned} 2\|e^p\|^2_{L^2(\Omega)} = &\int_{\mathbb{R}^d} \nabla^s e^p \cdot \big(\nabla^s [p^e - \Pi_h p^e] - (\mathbf{\Phi}^e - \mathbf{\Pi}_h \mathbf{\Phi}^e)\big) \\ &- \int_{\mathbb{R}^d} e^{\mathbf{\Phi}} \cdot \big(\nabla^s [p^e - \Pi_h p^e] + (\mathbf{\Phi}^e - \mathbf{\Pi}_h \mathbf{\Phi}^e)\big). \end{aligned}$$

Thus, we aim to bound the right hand side in the previous equality. First, observe that (2.4.15) implies

$$\|\nabla^s e^p\|_{L^2(\mathbb{R}^d)} + \|e^{\mathbf{\Phi}}\|_{L^2(\mathbb{R}^d)} \leq C h^{\frac{1}{2}} |\log h|^{\frac{1}{2}} \|f\|_{B^{-s+1/2}_{2,1}(\Omega)}.$$

Lastly, the same argument used in Proposition 2.4.6, combining the interpolation estimates (2.4.10) and Lemma 2.4.5 with $\alpha = 0$, but applied to the dual problem gives the bound

$$\|\nabla^s [p^e - \Pi_h p^e]\|_{L^2(\mathbb{R}^d)} + \|\mathbf{\Phi}^e - \mathbf{\Pi}_h \mathbf{\Phi}^e\|_{L^2(\mathbb{R}^d)} \leq C h^{\min\{s,\frac{1}{2}\}} |\log h|^{\kappa} \|e^p\|_{L^2(\Omega)},$$

and (2.4.16) follows. □

Remark 2.4.10 (Comparison with primal formulation)**.** *The convergence rates we have obtained for the pressure for our mixed formulation, both in* $\widetilde{H}^s(\Omega)$ *and in* $L^2(\Omega)$, *are consistent with those reported for primal discretizations of the fractional Laplacian using linear Lagrange finite elements. This behavior can be explained by the fact that the discrete solution produced by the mixed formulation corresponds to an orthogonal projection with respect to the norm* (2.2.11).

2.4.4 On the meshing of Ω^c

We conclude this section by showing how to apply Lemma 2.4.5 to construct meshes with less degrees of freedom than quasi uniform ones, while preserving the convergence rates (2.4.12) and (2.4.16). Lemma 2.4.5 implies that the flux $\mathbf{\Phi}$ has H^2 regularity in any element T such that $d(S_T^1, \partial\Omega) > 0$:

$$|\mathbf{\Phi}|_{H^2(T)} \leq \frac{C(d,s)}{d(T,\Omega)^{d+s+2}} \|p\|_{L^1(\Omega)} h_T^{\frac{d}{2}}.$$

We combine this with the H^2 interpolation estimate (2.4.9),

$$\|\mathbf{\Phi} - \Pi_h \mathbf{\Phi}\|_{L^2(T)} \leq C(d,\sigma) h_T^2 |\mathbf{\Phi}|_{H^2(S_T^1)},$$

to obtain

$$\|\mathbf{\Phi} - \Pi_h \mathbf{\Phi}\|_{L^2(T)} \leq C(d,s,\sigma) \frac{h_T^{2+\frac{d}{2}}}{d(T,\Omega)^{d+s+2}} \|p\|_{L^1(\Omega)}. \tag{2.4.19}$$

This inequality allows us to increase the diameter of elements sufficiently far from Ω, thereby reducing the total number of degrees of freedom without compromising the global convergence rate. Given $h > 0$, we maintain a uniform mesh in Ω with mesh size $h > 0$; this preserves the convergence rates for the pressure, and for the flux on interior elements. In order to retain the flux convergence rates over elements in Ω^c, the L^2 interpolation estimate (2.4.19) shows that it is sufficient to have

$$\frac{h_T^{2+\frac{d}{2}}}{d(T,\Omega)^{d+s+2}} \leq h^{\frac{1}{2}}.$$

Thus, we consider a mesh $\mathcal{T}_h(B_H)$ satisfying

$$h_T = \min\{h,\, h^{\frac{1}{d+4}} d(T,\Omega)^{\frac{2(d+s+2)}{4+d}}\}, \tag{2.4.20}$$

so that we keep an uniform mesh size h across elements T such that $d(T,\Omega) < h^{\frac{3+d}{2(d+s+2)}}$. Moreover, this construction guarantees that the mesh remains locally quasi-uniform. Indeed, in the transition phase (that is, elements T such that $d(T,\Omega) \simeq h^{\frac{3+d}{2(d+s+2)}}$) we have

$$h_T \simeq h^{\frac{1}{d+4}} h^{\frac{3+d}{2(d+s+2)} \frac{2(d+s+2)}{4+d}} = h,$$

showing that the mesh size varies continuously across the interface between the uniform and graded regions.

Figure 2.3 displays a mesh constructed in this fashion for $\Omega = B(0,1) \subset \mathbb{R}^2$. Table 2.2 compares the total number of nodes required by a globally quasi-uniform mesh against a mesh satisfying (2.4.20). The results clearly demonstrate that the improved mesh strategy drastically reduces the number of degrees of freedom—by nearly one order of magnitude—while retaining the same convergence properties (see Table 2.4 below).

Remark 2.4.11 (Shape regularity)**.** *Although the graded construction* (2.4.20) *may reduce the global shape regularity of the mesh, we point out that the meshes considered remain locally quasi-uniform. More precisely, we maintain a uniform mesh in a neighborhood of* Ω *and the interpolation and stability estimates used in the auxiliary elements depend only on the local shape regularity. Consequently, such shape regularity constant remains uniformly bounded on element patches.*

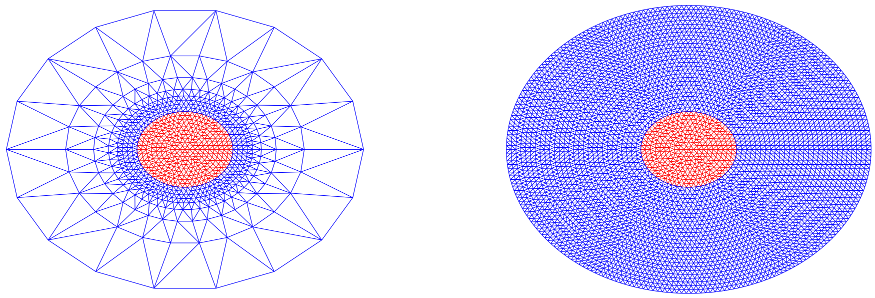


Figure 2.3: Example of a mesh satisfying (2.4.20) (left) and a uniform mesh (right) for $\Omega = B(0,1) \subset \mathbb{R}^2$, $h = \frac{1}{10}$ and $s = \frac{1}{2}$. The mesh in Ω is displayed in red, while the auxiliary elements are displayed in blue.

	$h = 1/30$	$h = 1/50$	$h = 1/75$
Mesh satisfying (2.4.20)	4418	11036	23217
Globally quasi-uniform mesh	29816	123686	478866

Table 2.2: Comparison of the total number of nodes between a globally quasi-uniform mesh and a mesh satisfying (2.4.20) with $s = \frac{1}{2}$. Here, $\Omega = B(0,1)$, and the computational domain diameter grows accordingly to $H = (h|\log h|)^{-\frac{1}{d+2s}}$ for both meshes.

2.5 Numerical experiments

In this section, we present some experiments in dimensions $d = 1$ and $d = 2$. These experiments are intended to validate the theoretical convergence results and to illustrate features that are specific to the proposed mixed formulation.

The main challenges in computing solutions of (D) are the need to evaluate integrals over $\mathbb{R}^d$, the presence of a singular kernel, and the nonlocal nature of the problem, which leads to dense system matrices. A description of the construction of the matrices involved in problem (2.4.3) can be found in Appendix A.

In the following, we test convergence rates of the stabilized method over quasi-uniform and graded meshes, as well as the influence of the computational domain diameter H on the errors. To this end, it is convenient to consider test cases where exact solutions are available. In particular, explicit solutions of (P) are available when Ω is a ball. Consider the polynomials $P_{a,b,n}$ of degree n defined as

$$P_{a,b,n}(t) = \frac{\Gamma(a+n+1)}{n!\,\Gamma(a+b+n+1)} \sum_{j=0}^{n} \binom{n}{j} \frac{\Gamma(a+b+n+j+1)}{\Gamma(a+j+1)} \left(\frac{t-1}{2}\right)^j,$$

and the function $\varphi_s : \mathbb{R}^d \to \mathbb{R}$ such that

$$\varphi_s(x) = (1 - |x|^2)^s \chi_{B(0,1)}.$$

A family of solutions can be built by using this functions.

Theorem 2.5.1 ([29, Theorem 3]). *Let $\Omega = B(0,1) \subset \mathbb{R}^d$. For $s \in (0,1)$ and $n \in \mathbb{N}$, consider*

$$C(n,d,s) = \frac{n!\,\Gamma(\frac{d}{2}+n)}{2^{2s}\Gamma(1+s+n)\,\Gamma(\frac{d}{2}+s+n)}$$

and $p_{n,s} : \mathbb{R}^d \to \mathbb{R}$ such that

$$p_{n,s}(x) = P_{s,\frac{d}{2}-1,n}(2|x|^2 - 1).$$

Then, $u = C(n,d,s)\varphi_s\, p_{n,s}$ is the solution of (P) *with right-hand side $f = p_{n,s}$.*

For $n = 0$, Theorem 2.5.1 gives us an explicit solution for the fractional torsion problem,

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } B(0,1), \\ u = 0 & \text{in } B(0,1)^c, \end{cases} \tag{2.5.1}$$

which, in turn, corresponds to an explicit expression for p in the mixed formulation:

$$\begin{cases} \boldsymbol{\Phi} + \nabla^s p = 0 & \text{in } \mathbb{R}^d, \\ \operatorname{div}^s \boldsymbol{\Phi} = 1 & \text{in } B(0,1), \\ p = 0 & \text{in } B(0,1)^c. \end{cases} \tag{2.5.2}$$

2.5.1 Quasi-uniform meshes

As a first example, we analyze the convergence rates for this problem on quasi-uniform meshes in $d = 1$ and $d = 2$ dimensions. The parameter H is chosen according to Proposition 2.4.6, i.e, $H \geq (h|\log h|)^{\frac{-1}{d+2s}}$. For $d = 1$, we consider a uniform mesh $\mathcal{T}_h([-H,H])$ of mesh size h, and for $d = 2$, we construct $\mathcal{T}_h(B_H)$ following (2.4.20).

For the pressure p, the computed orders of convergence for different values of s in one and two dimensions are respectively presented in Tables 2.3 and 2.4 and Figure 2.4. The outcomes of our computational experiments are consistent with Remark 2.4.8 and Proposition 2.4.9. Figure 2.5 illustrates a representative computed solution on the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1]^2$ with $s = 0.5$ and $f \equiv 1$;we emphasize that no analytic expression for the solution to (P) is available in this case. Away from the reentrant corner, the boundary behavior $p(x) \simeq d(x,\partial\Omega)^s$, which is responsible for the reduced convergence rates of the pressure, is clearly visible. This behavior is also reflected in the flux $\boldsymbol{\Phi}$: its normal components exhibit rapid variations near $\partial\Omega$.

2.5.2 Dependence on H

This section is devoted to exploring numerically a feature specific to the nonlocal setting, namely the interplay between the mesh size and the size of the computational domain, which is required to grow in order to ensure convergence. Our second set of experiments deals with the dependence of discrete solutions to (2.5.2) with respect to the computational domain diameter H. For a fixed h and different values of s, we compute errors for different values of H. We recall that our approach extends the discrete flux $\boldsymbol{\Phi}_h$ by zero outside B_H, so that H effectively acts as a truncation parameter: it introduces an additional source of discretization error that is absent in local problems. Consequently, an improvement of the error is expected as H increases. Naturally, increasing H leads to a larger computational domain, and therefore to an increase in the number of mesh elements and in the associated CPU time. This

Value of s	H^s-order	L^2-order
0.1	0.4691	0.5949
0.2	0.4956	0.6444
0.3	0.5000	0.7968
0.4	0.5004	0.9236
0.5	0.5005	1.0012
0.6	0.5005	0.9966
0.7	0.5009	0.9928
0.8	0.5014	0.9952
0.9	0.5017	1.0045

Table 2.3: Order of convergence for the pressure p in problem (2.5.2) in the one dimensional case.

Value of s	H^s-order in h	L^2-order in h	H^s-order in n_h	L^2-order in n_h
0.1	0.4985	0.5869	-0.2430	-0.2861
0.2	0.4959	0.6817	-0.2417	-0.3323
0.3	0.5170	0.8309	-0.2520	-0.4050
0.4	0.5314	0.9208	-0.2590	-0.4488
0.5	0.5187	0.9989	-0.2528	-0.4869
0.6	0.5189	1.1164	-0.2529	-0.5442
0.7	0.5175	1.2247	-0.2523	-0.5969
0.8	0.5127	1.1923	-0.2499	-0.5811
0.9	0.5131	1.0946	-0.2501	-0.5336

Table 2.4: Order of convergence for the pressure p in problem (2.5.2) the two dimensional case.

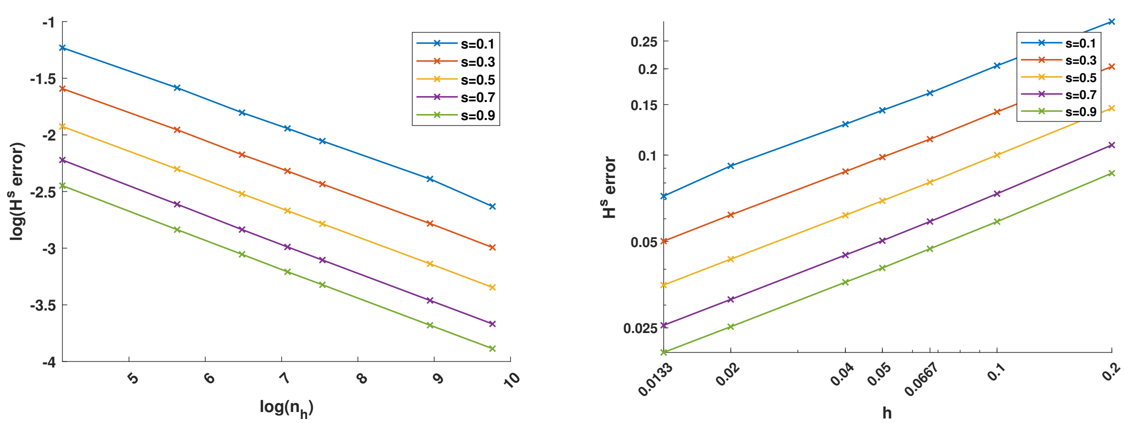


Figure 2.4: Convergence results for $d = 2$ corresponding to Table 2.4 in logarithmic scale.

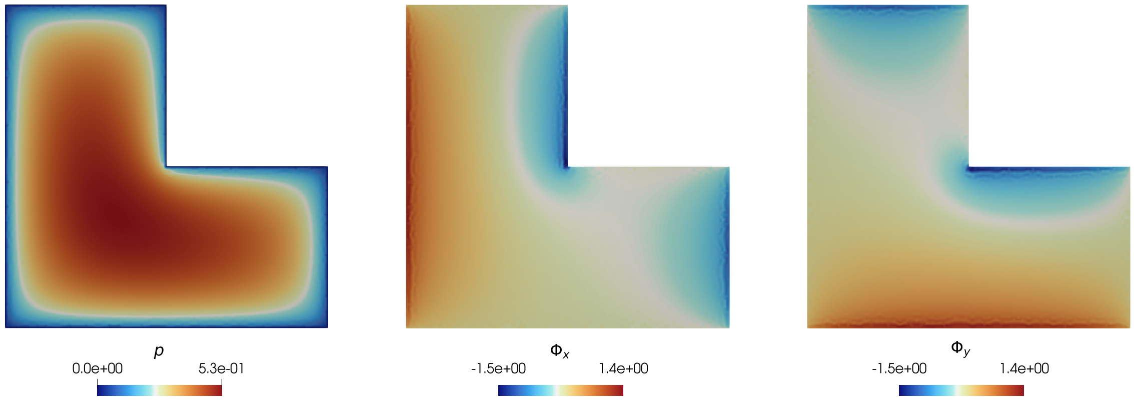


Figure 2.5: Here, $f(x) = 1$, $s = 0.5$, $\Omega = (-1,1)^2 \setminus [0,1]^2$, and $B_H = B(0,2)$ as mentioned before. The left panel shows the computed pressure and the center and right ones display the components of the computed flux inside Ω. We highlight the singular behavior of the normal component of the flux near the boundary of Ω. The mesh contains 38284 elements with 8029 degrees of freedom for the pressure and 2×19334 degrees of freedom for the flux.

highlights the need to balance the truncation error with the discretization error induced by the mesh size h. Proposition 2.4.6 suggests that significant improvements in the error are not to be expected once H is sufficiently large. In particular, beyond a certain threshold, the dominant contribution to the error is governed by the spatial discretization, and further enlarging the computational domain yields only marginal gains. Our results, displayed in Table 2.5, are in good agreement with this heuristic idea. They indicate that, in practice, one may choose H moderately large without compromising accuracy, thereby avoiding unnecessary computational costs.

H^s**-error**	$s = 0.2$	$s = 0.5$	$s = 0.8$
$H = 0.10$	0.2170	0.0865	0.0524
$H = 0.20$	0.1768	0.0843	0.0522
$H = 0.50$	0.1502	0.0814	0.0519
$H = 0.70$	0.1436	0.0808	0.0518
$H = 1.20$	0.1375	0.0803	0.0518
$H = 1.50$	0.1363	0.0802	0.0518
$H = 1.70$	0.1359	0.0803	0.0518
$H = 2.20$	0.1353	0.0803	0.0517

Table 2.5: H^s-error for the pressure in (2.5.2) in $d = 2$ over globally uniform meshes with size $h = 0.08$ and different computational domain diameters.

2.5.3 Graded meshes

As a final example, we explore numerically the behavior of the proposed method on graded meshes that are commonly employed in fractional-order problems. The weighted regularity proven in Theorem 2.3.4 suggests that improved convergence rates could be achieved by using a priori adapted meshes in the spirit of [2]; we refer to that work for a detailed discussion of the connection between weighted regularity and mesh refinement strategies. In what follows, and for the sole purpose of illustration, we consider meshes $\mathcal{T}_h(\Omega)$ constructed so as to reflect this regularity. Concretely, given $H > 0$, the regularity proven in Theorem 2.3.4 can be exploited by requiring that the mesh satisfy the following condition for elements contained in Ω:

$$h_T \approx \begin{cases} h\, d(T, \partial\Omega)^{\frac{1}{2}} & \text{if } T \cap \partial\Omega = \emptyset, \\ h^2 & \text{if } T \cap \partial\Omega \neq \emptyset. \end{cases} \tag{G}$$

Over $\Omega = B(0,1)$ such a mesh can be obtained in the following way. Consider an integer $N > 0$ and the sequence $r_j = (1 - \frac{j}{N})^2$ for $1 \leq j \leq N$. Let $\mathcal{T}_h(B_H)$ be the union of all uniform meshes with mesh size $h_j = r_j - r_{j-1}$ in the domains $\{x \in B(0,1) : r_{j-1} < |x| < r_j\} \subset \Omega$ for $1 \leq j \leq N$. This construction ensures that conditions (G) are satisfied with $h \simeq 1/N$.

The improved orders of approximation for different values of s are displayed in Table 2.6 for the torsion problem (2.5.2). As expected from the weighted regularity in Theorem 2.3.4, first-order convergence for the pressure is attained over these meshes. While we do not provide a theoretical analysis of mesh adaptivity for the mixed formulation, these experiments indicate that mesh grading strategies commonly used for primal discretizations of the fractional Laplacian can also be beneficial in the present mixed setting.

Value of s	H^s order in h
0.1	0.9601
0.2	0.9873
0.3	1.0059
0.4	1.0181
0.5	1.0269
0.6	1.0349
0.7	1.0437
0.8	1.0553
0.9	1.0702

Table 2.6: Order of convergence for the pressure p using graded meshes. Here, $f \equiv 1$ and $\Omega = B(0,1)$.

Chapter 3

Fractional Oseen-Frank model

This chapter is devoted to a relaxation of a classical model of liquid crystals. The nematic phase of a liquid crystal arises in materials composed of elongated molecules which do not exhibit long-range positional order but retain a certain degree of orientational order. In this phase, the molecules tend to align along a preferred direction, described macroscopically by a unit vector field $u : \Omega \to \mathbb{S}^{d-1}$, called the director field or orientation field.

The simplest continuum theory modeling this behavior is the Oseen–Frank model [56]. It is based on the assumptions that molecules preferentially align along a common direction, that they all have the same length, and that long-range electromagnetic interactions are negligible. Under these hypotheses, the equilibrium configuration of a nematic liquid crystal is obtained by minimizing the Oseen–Frank elastic energy subject to the pointwise unit-length constraint $|u| = 1$. In its most general form, the energy consists of the classical *splay*, *twist*, and *bend* contributions, weighted by elastic constants k_1, k_2, k_3, together with a null-Lagrangian term involving k_4. In the so-called one-constant approximation, the model reduces to the Dirichlet energy.

A distinctive feature of nematic materials is the presence of *defects*, which correspond to singularities in the orientation field. The Oseen–Frank model is unable to fully accommodate such configurations with finite energy: defects of co-dimension less than or equal to two necessarily carry infinite energy. This limitation has motivated the development of relaxed models. One important alternative is the Ericksen model [30], which introduces an additional scalar order parameter that allows partial loss of orientational order near singularities, effectively regularizing the energy.

In this chapter, we investigate a different type of relaxation, obtained by replacing the classical local differential operators (gradient, divergence, and curl) appearing in the Oseen–Frank energy with their fractional counterparts. This leads to a nonlocal energy functional defined on H^s-type spaces with $s \in (0, 1)$. The main idea is that lowering the differentiability requirement on the director field from H^1 to H^s permits configurations with weaker regularity, potentially allowing singularities to have finite energy.

Section 3.1 provides basic material on the classical Oseen-Frank energy and applies the direct method of calculus of variations to prove the existence of minimizers.

In section 3.2, we consider fractional analogues of the splay, twist, and bend terms, and we discuss two natural nonlocal extensions of the Oseen–Frank energy. While both share similar structural properties, the modified formulation turns out to be more suitable from the variational point of view, as it ensures coercivity under natural assumptions on the elastic constants. The existence of minimizers again follows from the direct method of the calculus of variations, thanks to the weak lower semicontinuity of the fractional operators. Fractional relaxations were first studied in [4] in the framework of the one-constant approximation.

The numerical approximation of minimizers of a simplified version of the fractional Oseen-Frank energy is addressed in Section 3.3. There, based on [36], we develop a finite element discretization combined with a Marchuk-Yanenko operator-splitting scheme. The method separates the quadratic fractional energy from the nonlinear unit-length constraint, leading to an iterative algorithm that alternates between the solution of a linear fractional elliptic problem and a pointwise projection onto the unit sphere.

Finally, in Section 3.4, we perform several numerical experiments in rectangular and circular domains. These experiments aim to explore the qualitative behavior of the computed solutions and the performance of the method.

3.1 Oseen-Frank model

In this section we briefly describe the classical Oseen-Frank model and summarize some of the results of [38] regarding the existence of minimizers, with the aim to extend them to the non-local case. We assume that the liquid crystal occupies a Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$ for our purposes). The Oseen-Frank model proposes that the equilibrium configuration of a nematic liquid crystal can be model as a field $n : \Omega \to \mathbb{R}^d$ that minimizes the **Oseen-Frank energy**

$$E(v) := \frac{1}{2} \int_\Omega k_1(\operatorname{div} v)^2 + k_2(v \cdot \mathbf{curl}\, v)^2 + k_3|v \times \mathbf{curl}\, v|^2 + (k_2 + k_4)(\operatorname{tr}([\nabla v]^2) - (\operatorname{div} v)^2)\, dx,$$

over the set of admissible fields

$$\mathcal{A} = \{v \in [H^1(\Omega)]^d : |v| = 1,\, v|_{\partial\Omega} = g\},$$

for some given $g \in [H^{\frac{1}{2}}(\partial\Omega)]^d$ with $|g| = 1$. Such regularity of g ensures that the admissible class $\mathcal{A}$ is nonempty. This may not always be case as Remark 3.1.4 shows. The constants k_i are known as *Frank's constants*, and their associated terms are called splay (k_1), twist (k_2), bend (k_3) and saddle-splay ($k_2 + k_4$). In order for E to be positive semi-definite, the Frank's constants are assumed to satisfy

$$k_1 \geq 0, \quad k_2 \geq |k_4|, \quad k_3 \geq 0, \quad 2k_1 \geq k_2 + k_4.$$

The main ingredient to prove the existence of minimizers is to note that the saddle-splay term $\operatorname{tr}([\nabla v]^2) - (\operatorname{div} v)^2$ is a null Lagrangian. Indeed,

$$\operatorname{tr}([\nabla v]^2) - (\operatorname{div} v)^2 = \operatorname{div}\, [(\nabla v)v - (\operatorname{div} v)v],$$

and therefore its integral depends only on g, i.e., the term is constant over $\mathcal{A}$ (cf. [38, Lemma 1.2]). Let

$$C(g) := \int_\Omega \operatorname{tr}([\nabla v]^2) - (\operatorname{div} v)^2\, dx,$$

for any $v \in \mathcal{A}$, and $\alpha = \min\{k_1,\, k_2,\, k_3\}$. With this notation, we can define

$$\widetilde{E}(v) := E(v) + \frac{1}{2}(\alpha - k_2 - k_4)C(g).$$

At first glance we did not do much, but $\widetilde{E}$ is more manageable than E. First, obviously, the minimizers of E and $\widetilde{E}$ are the same. Second, now the quartic terms can be bounded without any extra assumptions over the constants.

Lemma 3.1.1. $\frac{1}{2}\alpha\|\nabla v\|^2_{L^2(\Omega)} \le \widetilde{E}(v) \le 3(k_1+k_2+k_3)\|\nabla v\|^2_{L^2(\Omega)}$ *for all* $v \in H^1(\Omega;\mathbb{R}^d)$ *with* $|v|=1$.

Proof. For $v \in [H^1(\Omega)]^d$, we have

$$|\nabla v|^2 = \operatorname{tr}([\nabla v]^2) + |\mathbf{curl}\, v|^2.$$

Thus, by the definition of α,

$$\begin{aligned}\widetilde{E}(v) &= \frac{1}{2}\int_\Omega k_1(\operatorname{div} v)^2 + k_2(v\cdot\mathbf{curl}\, v)^2 + k_3|v\times\mathbf{curl}\, v|^2 + \alpha(\operatorname{tr}([\nabla v]^2) - (\operatorname{div} v)^2)\,dx \\ &\ge \frac{1}{2}\alpha\int_\Omega \operatorname{tr}([\nabla v]^2) + |\mathbf{curl}\, v|^2\,dx.\end{aligned}$$

The other inequality follows upon observing that $(\operatorname{div} v)^2 \le |\nabla v|^2$ and $k_1-\alpha\ge 0$. ☐

Lemma 3.1.2. $\widetilde{E}$ *is lower semicontinuous in the weak topology of* $\mathcal{A}$.

Proof. Let $u\in\mathcal{A}$ and $\{u_n\}\subset\mathcal{A}$ be a sequence that converges weakly to u in $H^1(\Omega;\mathbb{R}^d)$. Let $\gamma=\min\{k_2,k_3\}$, adding and subtracting $(\gamma-\alpha)|\mathbf{curl}\, u|^2$, we can rewrite $\widetilde{E}$ as

$$\begin{aligned}\widetilde{E}(u) &= \frac{1}{2}\int_\Omega k_1(\operatorname{div} u)^2 + k_2(u\cdot\mathbf{curl}\, u)^2 + k_3|u\times\mathbf{curl}\, u|^2 + \alpha(\operatorname{tr}([\nabla u]^2) - (\operatorname{div} u)^2)\,dx \\ &= \frac{1}{2}\int_\Omega (k_1-\alpha)(\operatorname{div} u)^2 + (k_2-\gamma)(u\cdot\mathbf{curl}\, u)^2 + (k_3-\gamma)|u\times\mathbf{curl}\, u|^2 + (\gamma-\alpha)|\mathbf{curl}\, u|^2 + \alpha|\nabla u|^2.\end{aligned}$$

Recall that $u_n \rightharpoonup u$ in $H^1(\Omega;\mathbb{R}^d)$ if and only if $u_n\to u$ in $L^2(\Omega;\mathbb{R}^d)$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega;\mathbb{R}^d)$. Therefore,

$$\begin{aligned}\operatorname{div} u_n &\rightharpoonup \operatorname{div} u,\\ \mathbf{curl}\, u_n &\rightharpoonup \mathbf{curl}\, u,\\ u\cdot\mathbf{curl}\, u_n &\rightharpoonup u\cdot\mathbf{curl}\, u,\\ u\times\mathbf{curl}\, u_n &\rightharpoonup u\times\mathbf{curl}\, u,\end{aligned}$$

as $n\to\infty$, all in $L^2(\Omega)$. For the quartic terms we used that the product of a strongly convergent sequence with a weakly convergent one in L^2, converges in weakly in L^2; see Lemma 1.3.8. ☐

With the lower semicontinuity and the coercivity, we can deduce the existence of minimizers.

Theorem 3.1.3. *In case* $\mathcal{A}\neq\emptyset$, *there exist a minimizer of* $\widetilde{E}$ *in* $\mathcal{A}$.

Remark 3.1.4 (Empty admissible set). *For a general boundary condition* g *satisfying* $|g|=1$, *the admissible class* $\mathcal{A}$ *may be empty. Consider* $\Omega=B(0,1)\subset\mathbb{R}^d$, *denote* $S^{d-1}=\partial B(0,1)$, *and define* $g(x)=x$ *for all* $x\in S^{d-1}$. *Let*

$$u(x) = \frac{x}{|x|}, \qquad x\in\Omega\setminus\{0\}.$$

A direct computation shows that

$$|\nabla u(x)| = \frac{\sqrt{d-1}}{|x|}.$$

Therefore, $\|\nabla u\|_{L^2(\Omega)} < \infty$ *if and only if* $d \geq 3$. *In particular, the admissible class* $\mathcal{A}$ *is nonempty when* $d \geq 3$, *while this is no longer true for* $d = 2$.

To justify the latter statement, recall that the integral definition of the degree (or index) of a smooth field $w = (w_1, w_2) : B(0,1) \to S^1$ *reads*

$$\deg(w) = \frac{1}{2\pi} \int_{S^1} w^\perp \cdot \partial_\tau w \, dS,$$

where $w^\perp = (-w_2, w_1)$, *and* ∂_τ *denotes the tangential derivative along* S^1. *Since* $\partial_\tau w \in L^2(S^1)$ *and* $w^\perp \in L^\infty(S^1)$, *the integrand belongs to* $L^1(S^1)$. *Therefore, this formula can be extended to unit-length fields in* $[H^1(\Omega)]^2$. *Note that Green's Theorem implies*

$$\int_{S^1} w^\perp \cdot \partial_\tau w \, dS = \int_{S^1} w^\perp \cdot ([\nabla w]^t \nu) \, dS = \int_{S^1} ([\nabla w]^t w^\perp) \cdot \nu \, dS = \int_\Omega \mathbf{curl}\,([\nabla w]^t w^\perp) \, dx.$$

Additionally, a direct calculation shows that $\mathbf{curl}\,([\nabla w]^t w^\perp) = 2\det(\nabla w)$. *Therefore,*

$$\deg(w) = \frac{1}{\pi} \int_\Omega \det(\nabla w) \, dx.$$

Suppose now that there exists $w \in \mathcal{A}$. *On the one hand,* $|w| = 1$ *implies that* $[\nabla w]^t w = 0$, *i.e,* $\det(\nabla w) = 0$ *and thus* $\deg(w) = 0$. *On the other hand,* $\deg(g) = \deg(w|_{S^1}) = 0$, *which yields a contradiction since* $\deg(g) = 1$.

3.2 Fractional Oseen-Frank energy

We propose to study a non-local version of the Oseen-Frank model by using fractional operators. The base idea is to replace all the local differential operators involved in the energy E by their fractional counterparts. As mentioned at the beginning of the chapter, the interest in such an energy is, by lowering the differentiability of requirements of the orientation field, to allow the model to capture the presence of defects on the liquid crystal, thereby alleviating the emptiness issue shown in Remark 3.1.4.

Let $s \in (0,1)$, $\Omega \subset \mathbb{R}^d$ a Lipschitz domain and $g \in [C^\infty(\mathbb{R}^d)]^d$ with $|g| = 1$ is a fixed vector field that defines a unit-length exterior Dirichlet condition on Ω^c. We define the set of admissible fields as

$$\mathcal{A} = \{v \in [\widetilde{H}^s(\Omega)]^d + g : \|\nabla^s v\|_{L^2(\Omega)} < \infty, \, |v| = 1\}. \tag{3.2.1}$$

As usual, in non-local problems the Dirichlet conditions are imposed in the whole complement of the domain. For the same reason, when defining the fractional Oseen-Frank energy we will consider integration on the whole $\mathbb{R}^d$.

Let us recall the definition of E,

$$E(v) = \frac{1}{2} \int_\Omega k_1 (\operatorname{div} v)^2 + k_2 (v \cdot \mathbf{curl}\, v)^2 + k_3 |v \times \mathbf{curl}\, v|^2 + (k_2 + k_4)(\operatorname{tr}([\nabla v]^2) - (\operatorname{div} v)^2) \, dx,$$

and also note that the modified energy $\widetilde{E}$ can be written as

$$\widetilde{E}(v) = \frac{1}{2} \int_\Omega \alpha |\nabla v|^2 + c_1 (\operatorname{div} v)^2 + c_2 (v \cdot \mathbf{curl}\, v)^2 + c_3 |v \times \mathbf{curl}\, v|^2 \, dx,$$

with $c_i := k_i - \alpha$. If one were to replace the differential operators by their fractional counterparts, it would not be clear that the term $\mathrm{tr}([\nabla^s v]^2) - (\mathrm{div}^s v)^2$ depends only on $v|_{\Omega^c}$ (or $v|_{\partial\Omega}$ for that matter). Consequently, there are **two, a priori distinct, natural extensions** of the Oseen-Frank energy to the non-local framework.

Let k_1, k_2, k_3, $k_4 \in \mathbb{R}$ be such that

$$k_1 \geq 0, \quad k_2 \geq |k_4|, \quad k_3 \geq 0, \quad 2k_1 \geq k_2 + k_4. \tag{3.2.2}$$

We define the functionals over $\mathcal{A}$,

$$E_s(v) = \frac{1}{2}\int_{\mathbb{R}^d} k_1(\mathrm{div}^s v)^2 + k_2(v\cdot\mathbf{curl}^s v)^2 + k_3|v\times\mathbf{curl}^s v|^2 + (k_2+k_4)(\mathrm{tr}([\nabla^s v]^2) - (\mathrm{div}^s v)^2)\,dx,$$

and

$$I_s(v) = \frac{1}{2}\int_{\mathbb{R}^d} \alpha|\nabla^s v|^2 + c_1(\mathrm{div}^s v)^2 + c_2(v\cdot\mathbf{curl}^s v)^2 + c_3|v\times\mathbf{curl}^s v|^2\,dx,$$

with $c_i := k_i - \alpha$ and $\alpha = \min\{k_1, k_2, k_3\}$. As shown in Chapter 1, each term under the integral is lower semicontinuous with respect to the weak topology of $[\widetilde{H}^s(\Omega)]^d$; see Lemma 1.3.9. Hence, the direct method of the calculus of variations yields the existence of minimizers by an argument analogous to the local case. The main difficulty arises when analyzing E_s: the argument used in Lemma 3.1.2 cannot be applied directly, since the coefficient multiplying the term $v\cdot\mathbf{curl}^s v$ effectively becomes $-k_4$. In order to guarantee coercivity, additional structural conditions on the Frank's constants are required; see Proposition 3.2.1 below. For this reason, we shall restrict our analysis to the functional I_s.

We want to show the existence of minimizers over the admissible set (3.2.1). Thus, we need to show that E_s and I_s are lower weakly semicontinuous and coercive.

Proposition 3.2.1. *Let v be a measurable function such that $|v| = 1$ and $\|\nabla^s v\|_{L^2(\mathbb{R}^d)} < \infty$. The estimate*

$$\frac{1}{2}\alpha\|\nabla^s v\|^2_{L^2(\mathbb{R}^d)} \leq I_s(v) \leq \frac{3}{2}(k_1+k_2+k_3)\|\nabla^s v\|^2_{L^2(\mathbb{R}^d)},$$

holds with $\alpha = \min\{k_1, k_2, k_3\}$. Additionally, if $k_1 \geq k_2 + k_4$ then we have

$$\frac{1}{2}\beta\|\nabla^s v\|^2_{L^2(\mathbb{R}^d)} \leq E_s(v) \leq \frac{3}{2}(k_1+k_2+k_3)\|\nabla^s v\|^2_{L^2(\mathbb{R}^d)}$$

with $\beta = \min\{k_2, k_3, k_2 + k_4\}$.

Proof. The proof follows by rearranging terms and using (3.2.2). First, by Lemma 1.1.18 we have,

$$\begin{aligned}
\frac{1}{2}\alpha\|\nabla^s v\|^2_{L^2(\mathbb{R}^d)} &\leq I_s(v) \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \alpha|\nabla^s v|^2 + (k_1-\alpha)(\mathrm{div}^s v)^2 + (k_2-\alpha)(v\cdot\mathbf{curl}^s v)^2 + (k_3-\alpha)|v\times\mathbf{curl}^s v|^2\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \alpha\mathrm{tr}([\nabla^s v]^2) + (k_1-\alpha)(\mathrm{div}^s v)^2 + k_2(v\cdot\mathbf{curl}^s v)^2 + k_3|v\times\mathbf{curl}^s v|^2\,dx \\
&\leq \frac{3}{2}(k_1+k_2+k_3)\int_{\mathbb{R}^d} |\nabla^s v|^2\,dx,
\end{aligned}$$

which proves the first estimate. Now, assuming $k_1 \geq k_2 + k_4$, we have

$$\begin{aligned}E_s(v) &= \frac{1}{2}\int_{\mathbb{R}^d} k_1(\mathrm{div}^s v)^2 + k_2(v \cdot \mathbf{curl}^s v)^2 + k_3|v \times \mathbf{curl}^s v|^2 + (k_2+k_4)(\mathrm{tr}([\nabla^s v]^2) - (\mathrm{div}^s v)^2)\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^d} (k_1 - k_2 - k_4)(\mathrm{div}^s v)^2 + \beta(v \cdot \mathbf{curl}^s v)^2 + \beta|v \times \mathbf{curl}^s v|^2 + \beta \mathrm{tr}([\nabla^s v]^2\,dx \\ &\geq \frac{1}{2}\beta\|\nabla^s v\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

On the other hand, we deduce

$$\begin{aligned}E_s(v) &= \frac{1}{2}\int_{\mathbb{R}^d} (k_1 - k_2 - k_4)(\mathrm{div}^s v)^2 + k_2(v \cdot \mathbf{curl}^s v)^2 + k_3|v \times \mathbf{curl}^s v|^2 + (k_2+k_4)\mathrm{tr}([\nabla^s v]^2)\,dx \\ &\leq \frac{3}{2}(k_1 + k_2 + k_3)\int_{\mathbb{R}^d} |\nabla^s v|^2\,dx.\end{aligned}$$

□

As mentioned before, the lower semicontinuity of E_s is not guaranteed, at least by similar arguments to those used in the local case. However, the lower semicontinuity of I_s is an immediate consequence of Lemma 1.3.9.

Proposition 3.2.2. *The energy I_s is lower semicontinuous with respect to the weak topology of $[\widetilde{H}^s(\Omega)]^d$.*

The minimization problem we will concerned with in the rest of this chapter reads:

$$\text{Find } u \in \mathcal{A} \text{ such that } I_s(u) \leq I_s(w) \text{ for all } w \in \mathcal{A}. \tag{3.2.3}$$

Now we can prove the existence of minimizers for I_s.

Theorem 3.2.3. *Assuming $\mathcal{A} \neq \emptyset$ and $\|\nabla^s g\|_{L^2(\mathbb{R}^d)} < \infty$, there exists a minimizer of I_s in $\mathcal{A}$.*

Proof. Let $\{u_n\} \subset \mathcal{A}$ be a minimizing sequence. By Proposition 3.2.1, we deduce

$$\|\nabla^s(u_n - g)\|_{L^2(\mathbb{R}^d)} \leq \frac{2}{\alpha} I_s(u_n) + \|\nabla^s g\|_{L^2(\mathbb{R}^d)}.$$

Therefore, by Corollary 1.3.6, the sequence $u_n - g \in \widetilde{H}^s(\Omega)$ possesses a weakly convergent subsequence to some $u_0 \in \widetilde{H}^s(\Omega)$. Define $u = u_0 + g$. The strong convergence in $L^2(\mathbb{R}^d)$ implies $|u| = 1$, i.e, $u \in \mathcal{A}$. Finally, the lower semicontinuity of I_s implies

$$I_s(u) \leq \liminf_n I_s(u_n) = \min_{v \in \mathcal{A}} I_s(v).$$

□

Remark 3.2.4 (Simplifications of I_s)**.** *If $k_1 = k_2 = k_3 = 1$, then I_s takes on the form*

$$I_s(v) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla^s v|^2 dx,$$

which is a fractional Dirichlet energy; see [4] for a full analysis of this case.

If $k_2 = k_3 > k_1$ (so that $\alpha = k_1$), then I_s takes on the form

$$I_s(v) = \frac{1}{2}\int_{\mathbb{R}^d} k_1|\nabla^s v|^2 + c_2|\mathbf{curl}^s v|^2\,dx.$$

We numerically explore this case in the next section; see [36] for the local setting.

3.3 Operator splitting method

In order to approximate minimizers of the fractional Oseen–Frank energy introduced in the previous section, we adopt an operator-splitting strategy. The method is inspired by the classical approach developed in [36], and we adapt it to the present nonlocal setting. The main difficulty stems from the nonlinear pointwise constraint $|u| = 1$, which makes the admissible set $\mathcal{A}$ nonconvex. Rather than enforcing the constraint directly within a fully coupled nonlinear scheme, we reformulate the problem as the minimization of a sum of two functionals: a quadratic energy term and the indicator functional of the unit-length constraint. This structure naturally lends itself to a splitting procedure, where each subproblem can be treated separately.

More precisely, we consider the fractional energy

$$I_s(v) = \frac{1}{2}\int_{\mathbb{R}^d} k_1|\nabla^s v|^2 + c_2|\mathbf{curl}^s v|^2 \, dx,$$

and rewrite the constrained minimization problem in a form suitable for time evolution schemes. The resulting algorithm alternates between a linear fractional elliptic step and a pointwise projection onto the unit sphere, leading to a simple and implementable iterative method. In that spirit, we can reformulate problem (3.2.3) as

$$\text{find } u \in \mathcal{A} \text{ such that } I_s(u) + P_\Sigma(u) \leq I_s(v) + P_\Sigma(v) \text{ for all } v \in [\widetilde{H}^s(\Omega)]^d + g, \tag{O}$$

where $\Sigma := \{v \in [\widetilde{H}^s(\Omega)]^d + g : |v| = 1 \text{ a.e.}\}$ and

$$P_\Sigma(v) = \begin{cases} 0 & \text{if } v \in \Sigma, \\ +\infty & \text{otherwise.} \end{cases}$$

We write $F(v) = I_s(v) + P_\Sigma(v)$. Operator-splitting methods decompose the original problem into a number of simpler subproblems associated with each sub-operator of the original problem and solve subproblem separately. This idea is easy to implement under a time-evolution framework. Consider the time evolution formulation of problem (O): find $u \in L^2([0,T]; \widetilde{H}^s(\Omega) + g)$ such that $|u(t,x)| = 1$ almost everywhere in $(0,T) \times \Omega$ and

$$\begin{cases} \langle -F'(u), v\rangle = \langle u_t, v\rangle_{L^2(\mathbb{R}^d)}, & \text{for all } v \in \widetilde{H}^s(\Omega), \\ u(0) = u_0, \end{cases}$$

for some given $u_0 \in \mathcal{A}$. In order to solve (O), we seek stationary states of this evolution problem as $t \to \infty$.

3.3.1 Discrete problem

We use the same notation as in Chapter 2. Concretely, let B_H be an auxiliary domain containing Ω such that $H = d(\Omega, B_H^c)$. For $h > 0$, we consider a simplicial mesh $\mathcal{T}_h(B_H)$ of $\overline{B_H}$, whose elements $\{T\}_{T\in\mathcal{T}_h(B_H)}$ are assumed to be closed, and such that the set

$$\mathcal{T}_h(\Omega) := \{T \in \mathcal{T}_h(B_H) : T \cap \Omega \neq \emptyset\}$$

is a simplicial triangulation of $\overline{\Omega}$. We denote by $\mathcal{N}_h = \{z_i : i = 1, \dots, N_h\}$ the set of nodes of $\mathcal{T}_h(B_H)$. We consider first-order Lagrange elements,

$$V_{gh} := \{v_h \in \mathcal{P}^1(\mathcal{T}_h(B_H)) : v_h(z_i) = g(z_i) \text{ for all } z_i \in \mathcal{N}_h \cap \Omega^c\}.$$

Finally, we denote the standard Lagrange piecewise linear basis as $\{\mathbf{\Phi}_i\}_{i=1}^{2N_h}$.

Remark 3.3.1 (Marchuk-Yanenko operator-splitting method)**.** *The Marchuk–Yanenko operator-splitting method is a time-discretization technique designed for evolution problems whose governing operator can be decomposed into a sum of simpler sub-operators. Instead of solving the fully coupled problem at each time step, the method advances the solution by successively solving subproblems associated with each operator. More precisely, if an evolution equation can be written in the form*

$$u_t + (A+B)u = 0,$$

the scheme replaces it by two consecutive substeps: first solving a problem involving only A, and then a problem involving only B, each over the same time interval; see [44] for further details.

In our context, the method allows us to separate the fractional operator arising from the energy I_s and the nonlinear projection associated with the constraint functional P_Σ.

Let $\tau > 0$. Applying the Marchuk–Yanenko operator-splitting scheme to problem (O), we obtain the following iterative procedure.

For $n \geq 0$:

(i) Given $u^n \in L^2(\mathbb{R}^d)$, compute $u^{n+\frac{1}{2}} \in V_{gh}$ as the solution of the time-evolution problem without the projection step, discretized by an implicit Euler method:

$$\langle -I_s'(u^{n+\frac{1}{2}}), v\rangle = \left\langle \frac{u^{n+\frac{1}{2}} - u^n}{\tau}, v \right\rangle_{L^2(\mathbb{R}^d)}, \qquad v \in V_{0h}.$$

This is equivalent to

$$\int_{\mathbb{R}^d} \frac{u^{n+\frac{1}{2}} - u^n}{\tau} \cdot v\, dx + k_1 \int_{\mathbb{R}^d} \nabla^s u^{n+\frac{1}{2}} \cdot \nabla^s v\, dx + c_2 \int_{\mathbb{R}^d} \mathbf{curl}^s u^{n+\frac{1}{2}} \cdot \mathbf{curl}^s v\, dx = 0, \quad v \in V_{0h}.$$

To impose the boundary condition, we proceed as follows: let $\widetilde{u}^{n+\frac{1}{2}} \in V_{0h}$ be the solution of

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\widetilde{u}^{n+\frac{1}{2}} - u^n}{\tau} \cdot v\, dx + k_1 \int_{\mathbb{R}^d} \nabla^s \widetilde{u}^{n+\frac{1}{2}} \cdot \nabla^s v\, dx + c_2 \int_{\mathbb{R}^d} \mathbf{curl}^s \widetilde{u}^{n+\frac{1}{2}} \cdot \mathbf{curl}^s v\, dx \\ = -k_1 \int_{\mathbb{R}^d} \nabla^s g \cdot \nabla^s v\, dx - c_2 \int_{\mathbb{R}^d} \mathbf{curl}^s g \cdot \mathbf{curl}^s v\, dx, \quad v \in V_{0h}.\end{aligned} \tag{3.3.1}$$

We then define

$$u^{n+\frac{1}{2}} = \widetilde{u}^{n+\frac{1}{2}} + g.$$

This corresponds to the standard lifting technique for imposing Dirichlet boundary conditions, where the solution is decomposed into a homogeneous part and a prescribed boundary extension.

(ii) Given $u^{n+\frac{1}{2}}$, compute $u^{n+1} \in L^2(\mathbb{R}^d)$ as the solution of

$$\left\langle \frac{u^{n+1} - u^{n+\frac{1}{2}}}{\tau}, v \right\rangle_{L^2(\mathbb{R}^d)} + \langle P_\Sigma'(u^{n+1}), v\rangle = 0.$$

This step reduces to the pointwise projection

$$u^{n+1}(x) := \begin{cases} \dfrac{u^{n+\frac{1}{2}}(x)}{|u^{n+\frac{1}{2}}(x)|}, & \text{if } u^{n+\frac{1}{2}}(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

In practice, this projection is performed only at the nodes of the mesh.

Remark 3.3.2 (Projection-free method)**.** *In [6, Chapter 7], the projection method described above is analyzed for the one-constant approximation in the local case. In order to guarantee that step (ii) decreases the energy, the mesh must satisfy a discrete maximum principle, which in two space dimensions is equivalent to the triangulation being acute. In higher dimensions, however, this requirement becomes rather restrictive. For this reason, the possibility of omitting step (ii) is also explored in that work. It is shown that, although this modification imposes a restriction on the time-step size, the unit-length constraint is still satisfied in the limit $(\tau, h) \to (0, 0)$. More recently, this projection-free method was studied in [4] for the one-constant approximation in the nonlocal setting.*

3.4 Numerical experiments

In the present section we perform several experiments in dimension $d = 2$. Before proceeding, let us describe how step (i) of the method is solved. Note that the length constraint implies $u_t \cdot u = 0$. Consequently, we search for descent directions in

$$\mathcal{O}_h(u_n) := \{v_h \in V_{0h} : v_h(z_i) \cdot u_n(z_i) = 0 \quad \forall z_i \in \mathcal{N}_h \cap \Omega\}.$$

Therefore, we reformulate (3.3.1) as follows: find $v^{n+\frac{1}{2}} \in \mathcal{O}_h(u_n)$ such that

$$\begin{aligned} -\int_{\mathbb{R}^d} v^{n+\frac{1}{2}} \cdot v \, dx + k_1 \int_{\mathbb{R}^d} \nabla^s(u^n - \tau v^{n+\frac{1}{2}}) \cdot \nabla^s v \, dx + c_2 \int_{\mathbb{R}^d} \mathbf{curl}^{\,s}(u^n - \tau v^{n+\frac{1}{2}}) \cdot \mathbf{curl}^{\,s} v \, dx \\ = -k_1 \int_{\mathbb{R}^d} \nabla^s g \cdot \nabla^s v \, dx - c_2 \int_{\mathbb{R}^d} \mathbf{curl}^{\,s} g \cdot \mathbf{curl}^{\,s} v \, dx, \qquad \forall v \in \mathcal{O}_h(u_n). \end{aligned}$$

Then, the solution of the first step is given by

$$u^{n+\frac{1}{2}} = u^n - \tau v^{n+\frac{1}{2}} + g_h,$$

where g_h denotes the Lagrange interpolation of g.

To solve the discrete system we employ a classical Lagrange multiplier approach. To that end, let $L_n \in \mathbb{R}^{n_h \times 2n_h}$ be such that $v_h \in \mathcal{O}_h(u_n)$ if and only if $L_n v_h = 0$. Additionally, we define the matrices $B, K, C \in \mathbb{R}^{n_h \times n_h}$ by

$$B_{ij} = \int_{\mathbb{R}^d} \mathbf{\Phi}_i \cdot \mathbf{\Phi}_j \, dx, \qquad K_{ij} = \int_{\mathbb{R}^d} \nabla^s \mathbf{\Phi}_i \cdot \nabla^s \mathbf{\Phi}_j \, dx, \qquad C_{ij} = \int_{\mathbb{R}^d} \mathbf{curl}^s \mathbf{\Phi}_i \cdot \mathbf{curl}^s \mathbf{\Phi}_j \, dx.$$

Thus, the resulting discrete saddle-point problem reads: find $v^{n+\frac{1}{2}} \in V_{0h}$ such that

$$\begin{pmatrix} -(B + k_1 \tau K + c_2 \tau C) & L_n^t \\ L_n & 0 \end{pmatrix} \begin{pmatrix} v^{n+\frac{1}{2}} \\ \Lambda \end{pmatrix} = \begin{pmatrix} -(k_1 K + c_2 C)(u^n + g_h) \\ 0 \end{pmatrix}.$$

Here $\Lambda \in \mathbb{R}^{n_h}$ denotes the Lagrange multiplier associated with the constraint.

Remark 3.4.1. *We remark that most of the computational effort is concentrated in the assembly of the matrices K and C. These matrices are dense and their computation is not straightforward; see Appendix A for further details. On the other hand, they do not depend on the time step τ. Hence, for a fixed mesh $\mathcal{T}_h(B_H)$ and $s \in (0, 1)$, they need to be computed only once.*

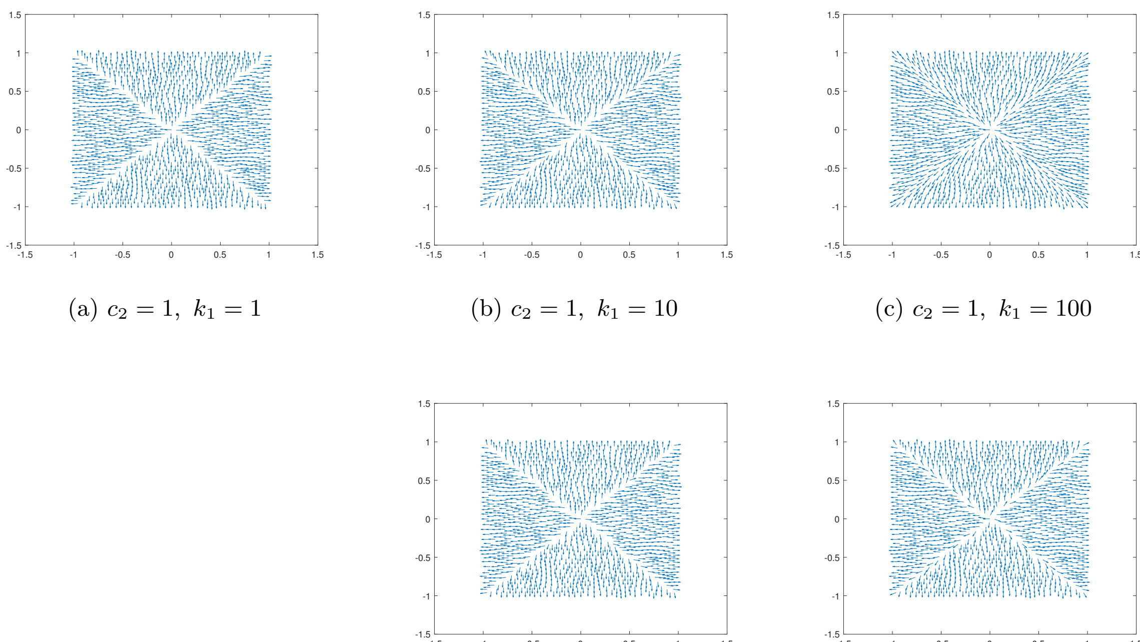
Remark 3.4.2. *In practice, it does not seem important for the initial vector field u_0 to satisfy the boundary condition.*

3.4.1 Square domain

We first consider Ω to be a square. For $H > 0$ and $h > 0$, we take a uniform triangulation $\mathcal{T}_h(B_H)$. In particular, we assume that $\Omega = [-1,1] \times [-1,1]$ and consider $g = \nu$, where ν denotes the outer normal vector of Ω, extended to $\mathbb{R}^2 \setminus \{|x| = |y|\}$ in a piecewise constant way. As initial value we fix a constant unit-length vector $u_0 \in S^1$. The stopping criterion is

$$\left\langle I_s'\left(\frac{u^{n+1}-u^n}{\tau}\right), \frac{u^{n+1}-u^n}{\tau}\right\rangle < 10^{-6}.$$

The computational results for the boundary condition are shown in Figure 3.1 for $s = \frac{1}{2}$. The defects tend to appear along the diagonal of the square, in agreement with the results reported in [36]. Moreover, increasing the value of k_1 (the coefficient of ∇^s) has a smoothing effect on the singularities, which is also observed in [36]. In contrast, increasing c_2 does not seem to produce the same smoothing effect. Finally, note that when $k_1 = 100$ and $c_2 = 1$ we are essentially in a approximation of the one-constant model.

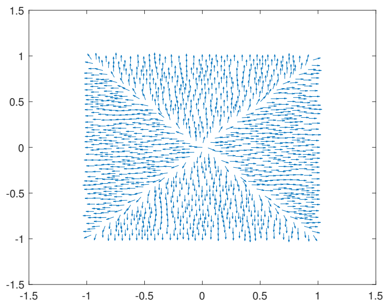


(a) $c_2 = 1,\ k_1 = 1$ (b) $c_2 = 1,\ k_1 = 10$ (c) $c_2 = 1,\ k_1 = 100$

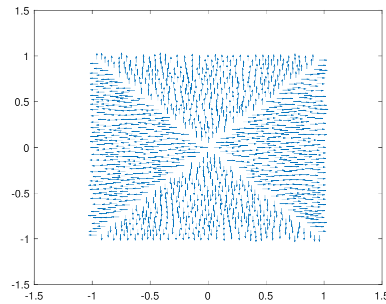
(d) $c_2 = 10,\ k_1 = 1$ (e) $c_2 = 100,\ k_1 = 1$

Figure 3.1: Computational results for different values of k_1 and c_2. The differentiability order is $s = \frac{1}{2}$, the time-step is $\tau = 0.001$ and the mesh size is $h = 0.07$.

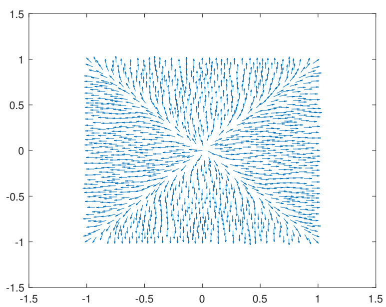
As expected, different values of s yield different behaviors of the method. As $s \to 1$, the model approaches the local case, and the smoothing effect observed previously appears already for smaller values of k_1. On the other hand, for smaller values of s the equilibrium configurations become more singular, even for large values of k_1. These results are shown in Figure 3.2.



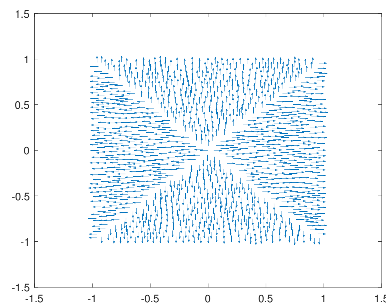
(a) $c_2 = 1,\ k_1 = 0.001,\ s = 0.999$



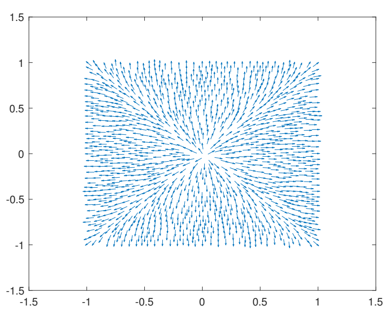
(b) $c_2 = 1,\ k_1 = 0.001,\ s = 0.2$



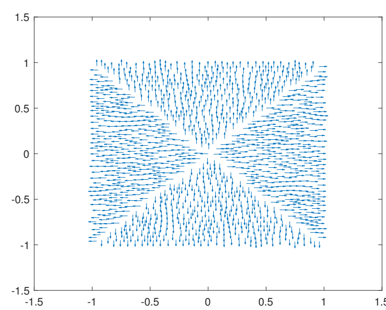
(c) $c_2 = 1,\ k_1 = 1,\ s = 0.999$



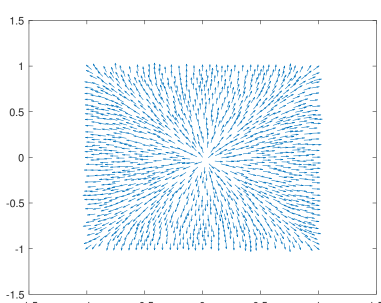
(d) $c_2 = 1,\ k_1 = 1,\ s = 0.2$



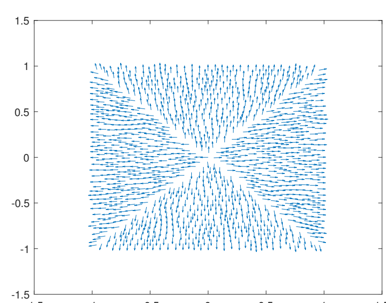
(e) $c_2 = 1,\ k_1 = 10,\ s = 0.999$



(f) $c_2 = 1,\ k_1 = 10,\ s = 0.2$



(g) $c_2 = 1,\ k_1 = 100,\ s = 0.999$



(h) $c_2 = 1,\ k_1 = 100,\ s = 0.2$

Figure 3.2: Computational results for different values of k_1 and s. The time-step is $\tau = 0.001$ and the mesh size is $h = 0.07$.

3.4.2 Circular domain

Now we consider the case $\Omega = B(0,1)$. Again, we use a uniform triangulation $\mathcal{T}_h(B_H)$. We impose the boundary condition

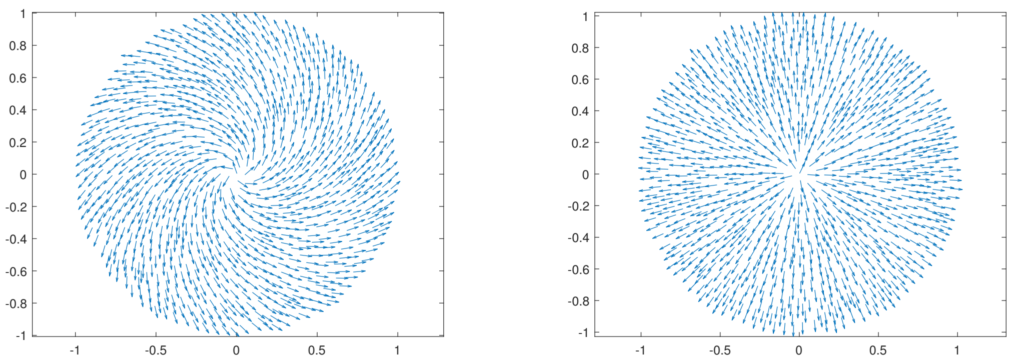
$$g(x,y) = R_\phi \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix},$$

where R_ϕ denotes the counterclockwise rotation matrix of angle ϕ, and θ is the polar angle that (x,y) forms with the x-axis. This corresponds to a rotation of the outer normal vector in $B(0,1)$. We use the same stopping criterion as in the previous section, and the initial condition is taken to be a normalized random vector field.

The computed nearly stationary configurations for $s = \frac{1}{2}$ are displayed in Figure 3.3. The solutions appear to approximate the expected stationary configuration in the one-constant approximation, namely

$$u(x) = R_\phi \frac{x}{|x|},$$

see [41]. However, we are unable to clearly identify a convergence behavior towards this canonical solution. Figure 3.4 shows the discrete energies of the iterates u^n for different values of $s \in (0,1)$ and a fixed mesh size $h = 0.07$. In particular, a rapid initial decay of the energy is followed by a slower decrease until the process becomes nearly stationary. Here, the initial configuration is taken to be a constant unit-length vector that coincides with g in Ω^c, which explains why the energy is large for $s = 0.8$. Figure 3.5 illustrates the evolution through snapshots of the iterates. We observe that the initial discontinuity of the initial function is quickly removed and a slightly diffused point defect develops, which moves towards the center of the domain during the evolution. It is also interesting to analyze the evolution when the method is initialized with a constant unit-length vector field. In this case, we observe that a line defect rapidly develops and moves towards the origin, eventually forming a point defect; see Figure 3.6.



(a) $\phi = \frac{\pi}{4}$ (b) $\phi = 0$

Figure 3.3: Computed stationary configurations, note the presence of a point defect in the origin. The time-step is $\tau = 0.0001$ and the mesh size is $h = 0.07$.

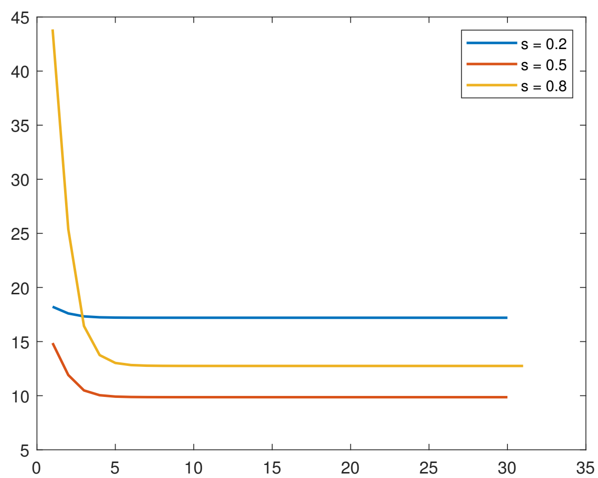
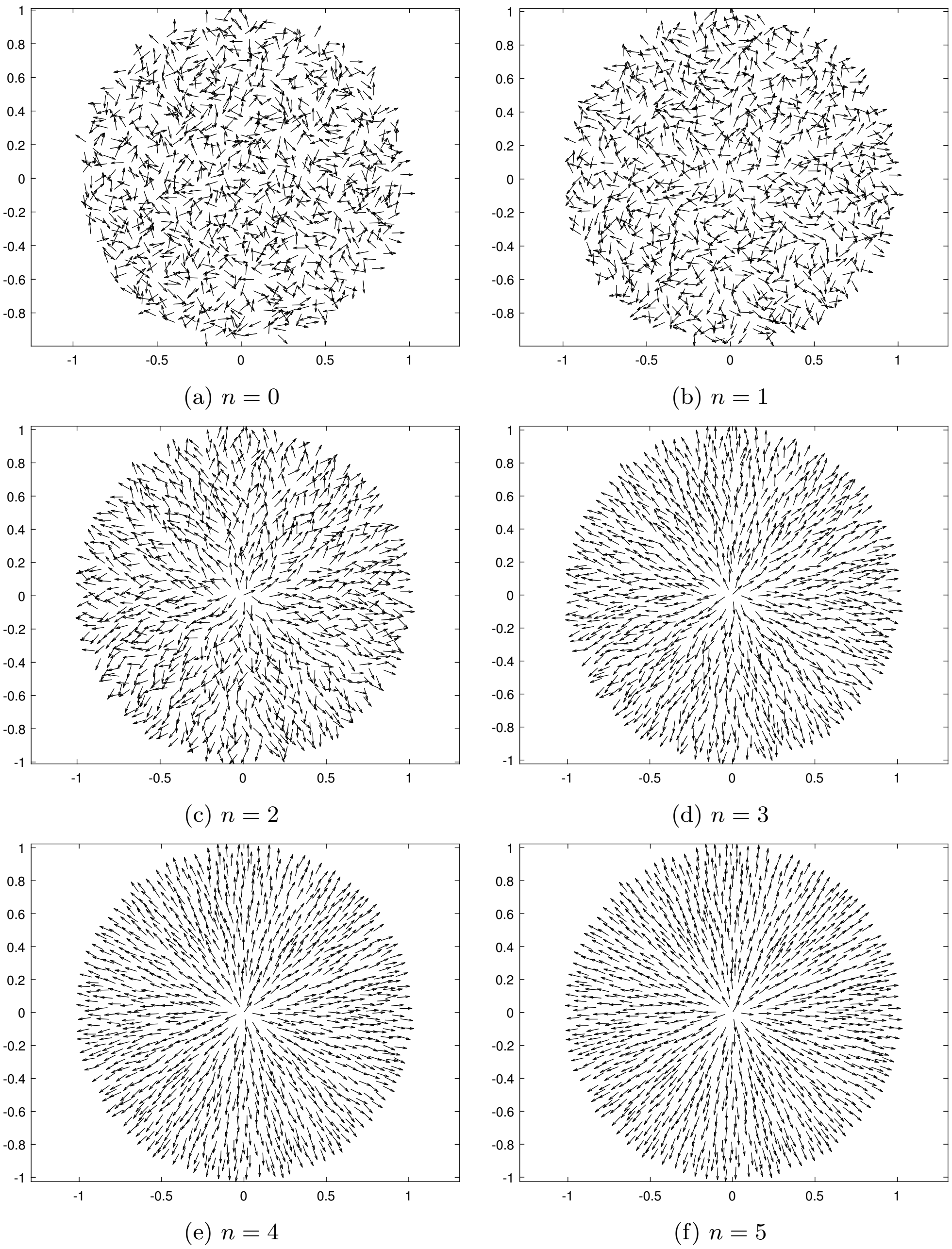


Figure 3.4: Discrete energies of the iterates for different values of s. The parameters are $k_1 = c_2 = 1$, $\tau = 10^{-5}$, and mesh size $h = 0.07$.

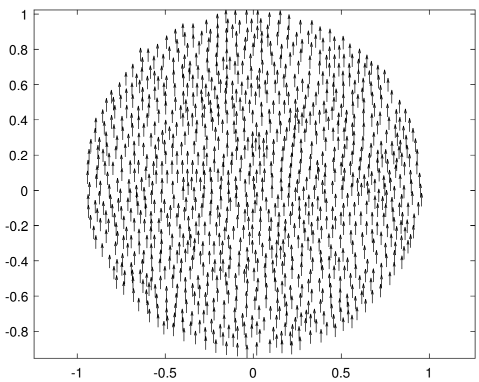


(a) $n = 0$ (b) $n = 1$

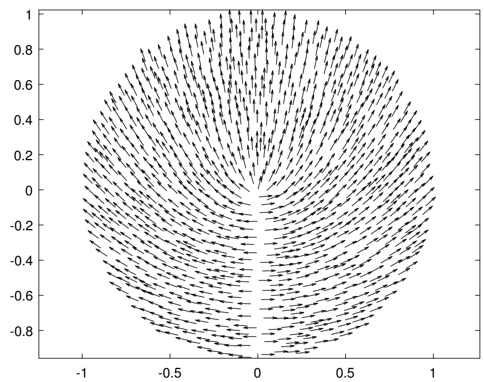
(c) $n = 2$ (d) $n = 3$

(e) $n = 4$ (f) $n = 5$

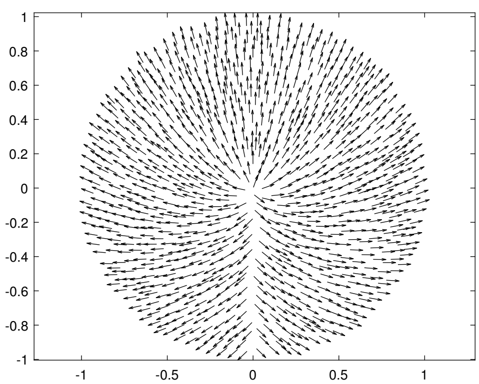
Figure 3.5: Evolution of the iterates with the initial condition given by a random unit-length vector field. The time-step is $\tau = 0.0001$, the mesh size is $h = 0.07$ and $s = 0.8$.



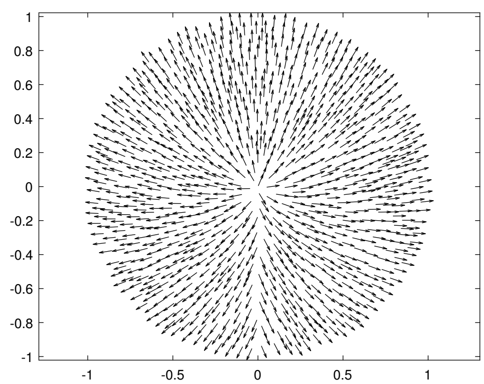
(a) $n = 0$



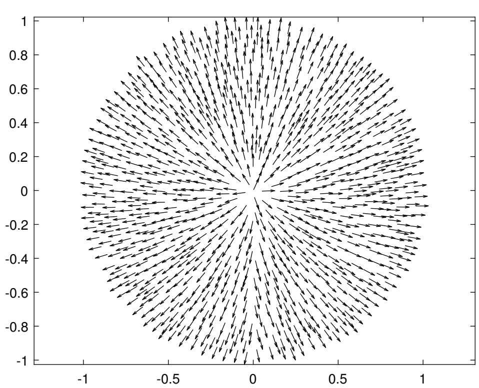
(b) $n = 1$



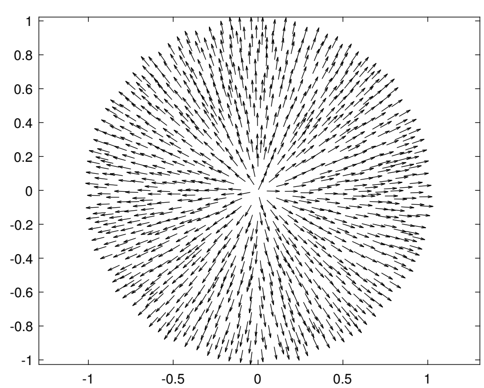
(c) $n = 2$



(d) $n = 3$



(e) $n = 4$



(f) $n = 5$

Figure 3.6: Evolution of the iterates with the initial condition given by $u_0 \equiv (0, 1)$. The time-step is $\tau = 0.0001$, the mesh size is $h = 0.07$ and $s = 0.8$.

Appendix A

Implementation details

The main challenges in computing solutions of problems (D) and (O) are the need to evaluate integrals over $\mathbb{R}^d$, the presence of a singular kernel which complicates the use of quadrature rules, and the nonlocal nature of the problem, that leads to dense system matrices; recall the example (2.2.2).

To tackle the integration in $\mathbb{R}^d$ issue, recall that we consider $H = H(h) \geq 1$ and our computational domains will be a family of balls B_H containing Ω and such that $H \simeq d(\overline{\Omega}, B_H^c)$. We extend our discrete functions to $\mathbb{R}^d$ by zero. In order to control the truncation error in the flux, the parameter H must be chosen appropriately; we refer to Lemma 2.4.5 for details. For $h > 0$, we consider a simplicial mesh $\mathcal{T}_h(B_H)$ of $\overline{B_H}$, whose elements $\{T\}_{T \in \mathcal{T}_h(B_H)}$ are assumed to be closed, and such that the set

$$\mathcal{T}_h(\Omega) = \{T \in \mathcal{T}_h(B_H) : T \cap \Omega \neq \emptyset\}$$

is a simplicial triangulation of $\overline{\Omega}$. We denote by $\mathcal{N}_h = \{z_i : i = 1, \ldots, N_h\}$ the set of nodes of $\mathcal{T}_h(B_H)$. We set $n_h = \#(\mathcal{N}_h \cap \Omega)$ and assume that the nodes are labeled so that those belonging to Ω come first, i.e., $\mathcal{N}_h \cap \Omega = \{z_1, \ldots, z_{n_h}\}$.

A.1 Fractional Darcy problem

This section is devoted to the implementation of the matrices involved in the Darcy problem (D). According to definition (2.4.2), we consider the discrete spaces consisting of continuous functions $(q_h, \mathbf{\Psi}_h) \in \mathcal{P}_1(\mathcal{T}_h(B_H)) \times [\mathcal{P}_1(\mathcal{T}_h(B_H))]^d \subset \mathbb{V}$ such that q_h vanishes on Ω^c. For the pressure, we consider the standard Lagrange nodal basis $B_{\text{pressure}} = \{\varphi_i\}_{1 \leq i \leq n_h}$ defined on the interior nodes, while for the flux we employ the d-Lagrange nodal basis $B_{\text{flux}} = \{\mathbf{\Phi}_i\}_{1 \leq i \leq dN_h}$. The difficult exercise when implementing (2.4.3) lies in the computation of the matrices $K \in \mathbb{R}^{n_h \times n_h}$, $B \in \mathbb{R}^{n_h \times dN_h}$,

$$K_{ij} := \int_{\mathbb{R}^d} \nabla^s \varphi_i \cdot \nabla^s \varphi_j \, dx,$$

$$B_{ij} := \int_{\mathbb{R}^d} \nabla^s \varphi_i \cdot \mathbf{\Phi}_j \, dx.$$

We point out here that, due to the identity $\|\nabla^s u\|_{L^2(\mathbb{R}^d)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^d)}$ for $u \in \widetilde{H}^s(\Omega)$, the matrix K satisfies

$$K_{ij} = \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}} \varphi_i (-\Delta)^{\frac{s}{2}} \varphi_j \, dx = \frac{\nu(d,s)}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{d+2s}} dx \, dy,$$

so that it coincides with the stiffness matrix of the Poisson problem (P). Moreover, splitting

$$\mathbb{R}^d \times \mathbb{R}^d = (B_H \times B_H) \cup (B_H^c \times B_H) \cup (B_H \times B_H^c) \cup (B_H^c \times B_H^c)$$

we obtain

$$\begin{aligned} K_{ij} =& \frac{\nu(d,s)}{2} \iint_{B_H \times B_H} \frac{(\varphi_i(x) - \varphi_i(y))(\varphi_j(x) - \varphi_j(y))}{|x-y|^{d+2s}} dx\, dy \\ &+ 2 \int_{B_H \times B_H^c} \frac{\varphi_i(x)\varphi_j(x)}{|x-y|^{d+2s}} dx\, dy. \end{aligned}$$

Remark A.1.1 (∇^s formula for piecewise linear functions)**.** *It is also possible to compute K_{ij} directly as*

$$K_{ij} = \int_{\mathbb{R}^d} \nabla^s \varphi_i \cdot \nabla^s \varphi_j \, dx.$$

By Theorem 1.1.8, we deduce that

$$\nabla^s \varphi_i(x) = c(d, 1-s) \int_{\mathbb{R}^d} \frac{\nabla \varphi_i(y)}{|x-y|^{d+s-1}} dy$$

where

$$c(d, 1-s) = \frac{\Gamma(\frac{d+s-1)}{2}}{\pi^{\frac{d}{2}} 2^{1-s} \Gamma(\frac{1-s}{2})}.$$

If we denote by $\omega_i := \operatorname{supp} \mathbf{\Phi}_i$ (that is, the patch of the vertex z_i), we see that the integration in $\nabla^s \varphi_i$ takes place only on ω_i. Moreover, $\nabla \varphi_i$ is a constant vector on each element of ω_i. Therefore, for each $x \in \mathbb{R}^d$, it can be written as

$$\nabla^s \varphi_i(x) = c(d, 1-s) \sum_{T \colon T \subset \omega_i} \nabla(\varphi_i|_T) \int_T \frac{1}{|x-y|^{d+s-1}} \, dy.$$

Thus, one approach is to define

$$\alpha_T(x) := c(d, 1-s) \int_T \frac{1}{|x-y|^{d+s-1}} \, dy,$$

and therefore

$$\nabla^s \varphi_i(x) = \sum_{T \colon T \subset \omega_i} \nabla(\varphi_i|_T) \, \alpha_T(x),$$

so that

$$K_{ij} = \sum_{T, T' \colon T \subset \omega_i, \, T' \subset \omega_j} \nabla(\varphi_i|_T) \cdot \nabla(\varphi_j|_{T'}) \int_{\mathbb{R}^d} \alpha_T(x) \, \alpha_{T'}(x) \, dx. \tag{A.1.1}$$

In addition, observe that

$$B_{i,j} = c(d, 1-s) \sum_{T_l \in sop \, \mathbf{\Phi}_j} \sum_{T_m \in sop \, \varphi_i} \nabla \varphi_i|_{T_m} \cdot \iint_{T_l \times T_m} \frac{\mathbf{\Phi}_j(x)}{|x-y|^{d+s-1}} dy \, dx.$$

We postpone the computation of K via the representation formula (A.1.1) *to the next section and focus instead on the computation of B.*

Hence, we iterate over the elements of the mesh using a double loop. More precisely, for any $T_l, T_m \in \mathcal{T}_h(B_H)$ (possibly with $T_l = T_m$), we must compute

$$\nabla\varphi_i|_{T_m} \cdot \iint_{T_l\times T_m} \frac{\mathbf{\Phi}_j(x)}{|x-y|^{d+s-1}} dy\, dx,$$

for the entries of B, and

$$\iint_{T_l\times T_m} \frac{(\varphi_i(x)-\varphi_i(y))(\varphi_j(x)-\varphi_j(y))}{|x-y|^{d+2s}} dx\, dy, \quad \int_{T_l\times B_H^c} \frac{\varphi_i(x)\varphi_j(x)}{|x-y|^{d+2s}} dx\, dy,$$

for the entries of K. The one-dimensional case is relatively simple, as the entries of both matrices K and B can be explicitly computed even on non-uniform meshes. The computation of K in the two-dimensional case is thoroughly discussed in [1] and we employ the same ideas to compute $B \in \mathbb{R}^{n_h\times 2N_h}$. The evaluation of the required integrals is carried out by mapping each pair of elements to a reference configuration and applying suitable quadrature rules. In particular, the quadrature rules described in [49, Section 5.2] have the key advantage of transforming the integral over the product of two elements into an integral over $[0,1]^4$, where the singularities of the kernel can be explicitly computed. The extension of these techniques to the three-dimensional setting, for the matrix K, has been developed in [34].

Now, for $d = 2$, we describe the computation of matrix B following ideas from [49, Section 5.2]. We order the flux basis in such a way that the x_1-coordinate basis comes first and x_2-coordinate basis comes next. That is, for all $1 \le j \le N_h$, we set

$$\mathbf{\Phi}_j = \begin{pmatrix}\varphi_j\\ 0\end{pmatrix}, \quad \mathbf{\Phi}_{j+N_h} = \begin{pmatrix}0\\ \varphi_j\end{pmatrix}.$$

Define $\hat{T} = \{(\hat{x}_1,\hat{x}_2) : \hat{x}_1 \in [0,1],\, 0 \le \hat{x}_2 \le \hat{x}_1\}$ the reference triangle. Also, for an element T_l, we consider the affine diffeomorphism $\chi_l : \hat{T} \to T_l$ whose Jacobian matrix is $d\chi_l = B_l \in \mathbb{R}^{2\times 2}$. We now focus on the computation of

$$\begin{aligned} I^{i,j}_{l,m} &= \nabla\varphi_i|_{T_m} \cdot \iint_{T_l\times T_m} \frac{\mathbf{\Phi}_j(x)}{|x-y|^{d+s-1}} dy\, dx \\ &= 4|T_l||T_m|\, \nabla\varphi_i|_{T_m} \cdot \int_{\hat{T}}\int_{\hat{T}} \frac{\mathbf{\Phi}_j(\chi_l(\hat{x}))}{|\chi_l(\hat{x})-\chi_m(\hat{y})|^{1+s}}\, d\hat{y}\, d\hat{x} \\ &= 4|T_l||T_m| \int_{\hat{T}}\int_{\hat{T}} F_{ij}\, d\hat{y}\, d\hat{x}. \end{aligned}$$

Following [1], given two elements T_l and T_m, we introduce a local numbering of the nodes as follows: first, the nodes shared by both elements; next, the nodes in $T_l \backslash T_m$; and finally, the nodes in $T_m \backslash T_l$; see Figure A.1. The local numbering of the flux basis functions is defined analogously, with the additional convention that the basis functions associated with the x_1-component are listed first, followed by those associated with the x_2-component. As we shall see, given any elements T_l and T_m, the computation of $I^{i,j}_{l,m}$ depends on whether the intersection $T_l \cap T_m$ is empty, a vertex, an edge, or whether $T_l = T_m$.

Finally $\otimes$ denotes the Kronecker product: if $A, B \in \mathbb{R}^{n\times m}$ we define

$$(A\otimes B)_{ij} := A_{ij}B_{ij}.$$

Figure A.1: Local numbering for elements T_l and T_m.

A.1.1 Case $T_l \cap T_m = \emptyset$

In this case, the integrand is not singular, and we simply employ six-point Gaussian quadrature rules over the elements. Nevertheless, this is the most computationally intensive of all cases. Let $\{p_k\}_{k=1}^6$ be quadrature nodes and $\{w_k\}_{k=1}^6$ their corresponding weights on the reference triangle. This yields 36 quadrature points on $\hat{T} \times \hat{T}$. Then,

$$I_{l,m}^{i,j} \approx \sum_{k=1}^{6} \sum_{q=1}^{6} 4|T_l||T_m| \, \nabla\varphi_i|_{T_m} \cdot \frac{\mathbf{\Phi}_j(p_k)}{|\chi_l(p_k) - \chi_m(p_q)|^{1+s}} \, w_k w_q.$$

Note that $I_{l,m}^{i,j} = 0$ if $i = 1, 2, 3$ or $j = 4, 5, 6, 10, 11, 12$. Indeed, if φ_i is the basis function associated with the node $z_i \in T_l$, then $\varphi_i|_{T_m} \equiv 0$ since $T_l \cap T_m = \emptyset$. The same argument applies for $j = 4, 5, 6, 10, 11, 12$. Therefore, the local matrix $I_{l,m} \in \mathbb{R}^{6\times 12}$ has the structure

$$I_{l,m} = 4|T_l||T_m| \begin{pmatrix} \mathcal{O}_{3\times3} & \mathcal{O}_{3\times3} & \mathcal{O}_{3\times3} & \mathcal{O}_{3\times3} \\ A_{l,m} & \mathcal{O}_{3\times3} & A_{l,m} & \mathcal{O}_{3\times3} \end{pmatrix} \otimes C_m.$$

Here, $A_{l,m} \in \mathbb{R}^3$ is defined by

$$A_{l,m}^{i,j} := \sum_{k=1}^{6} \sum_{q=1}^{6} \frac{\theta_j(p_k)}{|\chi_l(p_k) - \chi_m(p_q)|^{1+s}} \, w_k w_q,$$

where $\theta_1(x, y) = 1 - x$, $\theta_2(x, y) = x - y$, and $\theta_3(x, y) = y$. The matrix C_m, which collects the gradients of the local basis functions on T_m, is given by

$$C_m := \begin{pmatrix} \mathcal{O}_{3\times3} & \mathcal{O}_{3\times3} & \mathcal{O}_{3\times3} & \mathcal{O}_{3\times3} \\ G_1 & \mathcal{O}_{3\times3} & G_2 & \mathcal{O}_{3\times3} \end{pmatrix},$$

where

$$G_1 := \begin{pmatrix} \partial_1\varphi_1|_{T_m} & \partial_1\varphi_1|_{T_m} & \partial_1\varphi_1|_{T_m} \\ \partial_1\varphi_2|_{T_m} & \partial_1\varphi_2|_{T_m} & \partial_1\varphi_2|_{T_m} \\ \partial_1\varphi_3|_{T_m} & \partial_1\varphi_3|_{T_m} & \partial_1\varphi_3|_{T_m} \end{pmatrix}, \qquad G_2 := \begin{pmatrix} \partial_2\varphi_1|_{T_m} & \partial_2\varphi_1|_{T_m} & \partial_2\varphi_1|_{T_m} \\ \partial_2\varphi_2|_{T_m} & \partial_2\varphi_2|_{T_m} & \partial_2\varphi_2|_{T_m} \\ \partial_2\varphi_3|_{T_m} & \partial_2\varphi_3|_{T_m} & \partial_2\varphi_3|_{T_m} \end{pmatrix}.$$

A.1.2 Case $T_l \cap T_m$ is a vertex

We choose the diffeomorphisms χ_l and χ_m in such a way that the point $(0, 0)$ is mapped to the common node of T_l and T_m respectively. Hence,

$$\chi_l - \chi_m = B_l - B_m.$$

We decompose the domain $\hat{T} \times \hat{T} = D_1 \cup D_2$, where

$$D_1 = \{\hat{z} \in \mathbb{R}^4 : 0 \le \hat{z}_1 \le 1,\ 0 \le \hat{z}_2 \le \hat{z}_1,\ 0 \le \hat{z}_3 \le 1,\ 0 \le \hat{z}_4 \le \hat{z}_3\},$$
$$D_2 = \{\hat{z} \in \mathbb{R}^4 : 0 \le \hat{z}_3 \le 1,\ 0 \le \hat{z}_4 \le \hat{z}_3,\ 0 \le \hat{z}_1 \le \hat{z}_3,\ 0 \le \hat{z}_2 \le \hat{z}_1\}.$$

We consider the transformations $T_i : [0,1]^4 \to D_i$ defined by

$$T_1 \begin{pmatrix} \xi \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \xi \\ \xi\mu_1 \\ \xi\mu_2 \\ \xi\mu_2\mu_3 \end{pmatrix}, \qquad T_2 \begin{pmatrix} \xi \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} \xi\mu_2 \\ \xi\mu_2\mu_3 \\ \xi \\ \xi\mu_1 \end{pmatrix},$$

with

$$|\det(dT_i)| = \xi^3\mu_2.$$

These change of coordinates allow us to explicitly compute the singularity under the integral. Indeed,

$$\begin{aligned}
\int_{\hat{T}}\int_{\hat{T}} F_{ij} &= \int_{D_1} F_{ij} + \int_{D_2} F_{ij} \\
&= \int_{[0,1]^4} F_{ij}(\xi, \xi\mu_1, \xi\mu_2, \xi\mu_2\mu_3)\xi^3\mu_2 \, d\xi d\mu \\
&\quad + \int_{[0,1]^4} F_{ij}(\xi\mu_2, \xi\mu_2\mu_3, \xi, \xi\mu_1)\xi^3\mu_2 \, d\xi d\mu \\
&= \nabla\varphi_i|_{T_m} \cdot \Big[(I) + (II)\Big],
\end{aligned}$$

for the terms (I) and (II) defined below Using the homogeneity of the affine maps B_l and B_m, we obtain

$$(I) = \int_{[0,1]^4} \frac{\mathbf{\Phi}_j(\xi, \xi\mu_1)}{|B_l(1,\mu_1) - B_m(\mu_2, \mu_2\mu_3)|^{1+s}} \xi^{2-s}\mu_2 \, d\xi d\mu,$$

where the basis functions in the new coordinates become

$$\mathbf{\Phi}_1(\xi, \xi\mu_1) = 1 - \xi, \qquad \mathbf{\Phi}_2(\xi, \xi\mu_1) = \xi(1-\mu_1), \qquad \mathbf{\Phi}_3(\xi, \xi\mu_1) = \xi\mu_1.$$

Remark A.1.2. *Strictly speaking, $\mathbf{\Phi}_j$ is vector-valued. More precisely, its components are given by*

$$\mathbf{\Phi}_1(\xi, \xi\mu_1) = (1-\xi, 0), \qquad \mathbf{\Phi}_2(\xi, \xi\mu_1) = (\xi(1-\mu_1), 0), \qquad \mathbf{\Phi}_3(\xi, \xi\mu_1) = (\xi\mu_1, 0),$$
$$\mathbf{\Phi}_4(\xi, \xi\mu_1) = (0, 1-\xi), \qquad \mathbf{\Phi}_5(\xi, \xi\mu_1) = (0, \xi(1-\mu_1)), \qquad \mathbf{\Phi}_6(\xi, \xi\mu_1) = (0, \xi\mu_1).$$

Here, for convenience, we slightly abuse notation by treating them as scalar quantities. We shall maintain this abuse of notation throughout this section.

Integrating first with respect to ξ, we obtain

$$(I) = \frac{1}{4-s} \int_{[0,1]^3} \frac{\theta_j^{(1)}(\mu)}{|d^{(1)}(\mu)|^{1+s}} \mu_2 \, d\mu,$$

where

$$\theta_1^{(1)}(\mu) = \frac{4-s}{3-s} - 1, \quad \theta_2^{(1)}(\mu) = 1 - \mu_1, \quad \theta_3^{(1)}(\mu) = \mu_1, \quad \theta_4^{(1)} = \theta_5^{(1)} \equiv 0,$$

and

$$d^{(1)}(\mu) = B_l(1, \mu_1) - B_m(\mu_2, \mu_2\mu_3).$$

Analogously,

$$(II) = \frac{1}{4-s}\int_{[0,1]^3} \frac{\theta_j^{(2)}(\mu)}{|d^{(2)}(\mu)|^{1+s}}\mu_2\, d\mu,$$

with

$$\theta_1^{(2)}(\mu) = \frac{4-s}{3-s} - \mu_2, \quad \theta_2^{(2)}(\mu) = \mu_2(1-\mu_3), \quad \theta_3^{(2)}(\mu) = \mu_2\mu_3, \quad \theta_4^{(2)} = \theta_5^{(2)} \equiv 0,$$

and

$$d^{(2)}(\mu) = B_l(\mu_2, \mu_2\mu_3) - B_m(1, \mu_1).$$

Let $p_1, \ldots, p_{27} \in [0,1]^3$ and $w_1, \ldots, w_{27} \in \mathbb{R}$ be the quadrature points and weights in the unit cube. For $h = 1, 2$, we approximate

$$\int_{[0,1]^3} \frac{\theta_j^{(h)}(\mu)}{|d^{(h)}(\mu)|^{1+s}}\mu_2\, d\mu \approx \sum_{k=1}^{27} w_k \frac{\theta_j^{(h)}(p_k)}{|d^{(h)}(p_k)|^{1+s}} p_{k,2}.$$

We now write the resulting approximation of the local matrix $I_{l,m}$ in a compact fashion. Following the strategy of [1], we define:

- $\Theta^{(h)} \in \mathbb{R}^{50\times 27}$ by
$$\Theta_{ij}^{(h)} = w_j\, \theta^{(h)}_{[i-1]_5+1}(p_j)\, p_{j,2},$$
- the vector $d^{(h)} \in \mathbb{R}^{27}$ by
$$d_i^{(h)} = |d^{(h)}(p_i)|^{-1-s}.$$

Let

$$\hat{I}_{l,m} = \Theta^{(1)} d^{(1)} + \Theta^{(2)} d^{(2)}.$$

We then define $V_{l,m} \in \mathbb{R}^{5\times 10}$ by reshaping $\hat{I}_{l,m}$, namely

$$V_{l,m}^{[i-1]_5+1, \lceil i/5 \rceil} = \hat{I}_{l,m}^i, \qquad i = 1, \ldots, 50.$$

Define $C_m \in \mathbb{R}^{5\times 10}$ by

$$C_m = \begin{pmatrix} \partial_1\varphi_1|_{T_m} & \partial_1\varphi_1|_{T_m} & \partial_1\varphi_1|_{T_m} & 0 & 0 & \partial_2\varphi_1|_{T_m} & \partial_2\varphi_1|_{T_m} & \partial_2\varphi_1|_{T_m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_1\varphi_4|_{T_m} & \partial_1\varphi_4|_{T_m} & \partial_1\varphi_4|_{T_m} & 0 & 0 & \partial_2\varphi_4|_{T_m} & \partial_2\varphi_4|_{T_m} & \partial_2\varphi_4|_{T_m} & 0 & 0 \\ \partial_1\varphi_5|_{T_m} & \partial_1\varphi_5|_{T_m} & \partial_1\varphi_5|_{T_m} & 0 & 0 & \partial_2\varphi_5|_{T_m} & \partial_2\varphi_5|_{T_m} & \partial_2\varphi_5|_{T_m} & 0 & 0 \end{pmatrix}.$$

With this notation, the local matrix $I_{l,m} \in \mathbb{R}^{5\times 10}$ is given by

$$I_{l,m} = \frac{4}{4-s}|T_l||T_m|\, V_{l,m} \otimes C_m.$$

A.1.3 Case $T_l \cap T_m$ is an edge

This time we choose χ_l and χ_m so that the common edge between T_l and T_m is mapped onto $[0,1]\times\{0\}$. Hence, if we set

$$\hat{z} = (\hat{z}_1, \hat{z}_2, \hat{z}_3) = (\hat{y}_1 - \hat{x}_1,\, \hat{y}_2,\, \hat{x}_2),$$

the singularity occurs at $z = 0$. Then,

$$\int_{\hat{T}}\int_{\hat{T}} F_{ij} = \int_0^1 \int_{-\hat{x}_1}^{1-\hat{x}_1} \int_0^{\hat{z}_1+\hat{x}_1} \int_0^{\hat{x}_1} F_{ij}(\hat{x}_1, \hat{z}_3, \hat{x}_1 + \hat{z}_1, \hat{z}_2)\, d\hat{z}_2\, d\hat{z}_3\, d\hat{z}_1\, d\hat{x}_1.$$

We decompose the integration domain as

$$\hat{T} \times \hat{T} = \bigcup_{k=1}^{5} D_k,$$

where the subdomains D_k are defined as follows:

$$\begin{aligned}
D_1 &= \{(\hat{x}_1, \hat{z}) : -1 \le \hat{z}_1 \le 0,\ 0 \le \hat{z}_2 \le 1 + \hat{z}_1,\ 0 \le \hat{z}_3 \le \hat{z}_2 - \hat{z}_1,\ \hat{z}_2 - \hat{z}_1 \le \hat{x}_1 \le 1\},\\
D_2 &= \{(\hat{x}_1, \hat{z}) : -1 \le \hat{z}_1 \le 0,\ 0 \le \hat{z}_2 \le 1 + \hat{z}_1,\ \hat{z}_2 - \hat{z}_1 \le \hat{z}_3 \le 1,\ \hat{z}_3 \le \hat{x}_1 \le 1\},\\
D_3 &= \{(\hat{x}_1, \hat{z}) : 0 \le \hat{z}_1 \le 1,\ 0 \le \hat{z}_2 \le \hat{z}_1,\ 0 \le \hat{z}_3 \le 1 - \hat{z}_1,\ \hat{z}_3 \le \hat{x}_1 \le 1 - \hat{z}_1\},\\
D_4 &= \{(\hat{x}_1, \hat{z}) : 0 \le \hat{z}_1 \le 1,\ \hat{z}_1 \le \hat{z}_2 \le 1,\ 0 \le \hat{z}_3 \le \hat{z}_2 - \hat{z}_1,\ \hat{z}_2 - \hat{z}_1 \le \hat{x}_1 \le 1 - \hat{z}_1\},\\
D_5 &= \{(\hat{x}_1, \hat{z}) : 0 \le \hat{z}_1 \le 1,\ \hat{z}_1 \le \hat{z}_2 \le 1,\ \hat{z}_2 - \hat{z}_1 \le \hat{z}_3 \le 1 - \hat{z}_1,\ \hat{z}_3 \le \hat{x}_1 \le 1 - \hat{z}_1\}.
\end{aligned}$$

We introduce five mappings $T_k : [0,1]^4 \to D_k$:

$$T_1(\xi, \mu) = \begin{pmatrix} \xi \\ -\xi\mu_1\mu_2 \\ \xi\mu_1(1-\mu_2) \\ \xi\mu_1\mu_3 \end{pmatrix}, \quad T_2(\xi, \mu) = \begin{pmatrix} \xi \\ -\xi\mu_1\mu_2\mu_3 \\ \xi\mu_1\mu_2(1-\mu_3) \\ \xi\mu_1 \end{pmatrix},$$

$$T_3(\xi, \mu) = \begin{pmatrix} \xi(1-\mu_1\mu_2) \\ \xi\mu_1\mu_2 \\ \xi\mu_1\mu_2\mu_3 \\ \xi\mu_1(1-\mu_2) \end{pmatrix},$$

$$T_4(\xi, \mu) = \begin{pmatrix} \xi(1-\mu_1\mu_2\mu_3) \\ \xi\mu_1\mu_2\mu_3 \\ \xi\mu_1 \\ \xi\mu_1\mu_2(1-\mu_3) \end{pmatrix},$$

$$T_5(\xi, \mu) = \begin{pmatrix} \xi(1-\mu_1\mu_2\mu_3) \\ \xi\mu_1\mu_2\mu_3 \\ \xi\mu_1\mu_2 \\ \xi\mu_1(1-\mu_2\mu_3) \end{pmatrix}.$$

Note that

$$|\det(dT_1)| = \xi^3\mu_1^2, \qquad |\det(dT_k)| = \xi^3\mu_1^2\mu_2, \quad k = 2, 3, 4, 5.$$

We explicit the calculations for D_1; The remaining cases follow analogously. We have

$$\begin{aligned}
\int_{D_1} F_{ij}(\hat{x}_1, \hat{z}_3, \hat{x}_1 + \hat{z}_1, \hat{z}_2) &= \int_{[0,1]^4} \frac{\mathbf{\Phi}_j(\xi, \xi\mu_1\mu_3)}{|B_l(\xi, \xi\mu_1\mu_3) - B_m(\xi(1-\mu_1\mu_2), \xi\mu_1(1-\mu_2)|^{1+s}} \xi^3\mu_1^2\, d\xi d\mu\\
&= \int_{[0,1]^4} \frac{\mathbf{\Phi}_j(\xi, \xi\mu_1\mu_3)}{|B_l(1, \mu_1\mu_3) - B_m(1-\mu_1\mu_2, \mu_1(1-\mu_2))|^{1+s}} \xi^{2-s}\mu_1^2\, d\xi d\mu.
\end{aligned}$$

Note that the singularity only depends on ξ and can be explicitly computed. Thus, integrating with respect to ξ we deduce

$$\int_{D_1} F_{ij}(\hat{x}_1, \hat{z}_3, \hat{x}_1 + \hat{z}_1, \hat{z}_2) = \frac{1}{4-s} \int_{[0,1]^3} \frac{\psi_j^{(1)}(\mu) J^{(1)}(\mu)}{|d^{(1)}(\mu)|^{1+s}} d\mu,$$

with

$$\psi_1^{(1)}(\mu) = \frac{4-s}{3-s} - 1, \quad \psi_2^{(1)}(\mu) = 1 - \mu_1\mu_3, \quad \psi_3^{(1)}(\mu) = \mu_1\mu_3, \quad \psi_4^{(1)}(\mu) \equiv 0.$$

$$d^{(1)}(\mu) = B_l \begin{pmatrix} 1 \\ \mu_1\mu_3 \end{pmatrix} - B_m \begin{pmatrix} 1-\mu_1\mu_2 \\ \mu_1(1-\mu_2) \end{pmatrix}, \quad J^{(1)}(\mu) = \mu_1^2.$$

Doing the same procedure for all the remaining domains yields

$$\int_{\hat{T}\times\hat{T}} F_{ij} = \frac{1}{4-s} \sum_{h=1}^{5} \int_{[0,1]^3} \frac{\psi_j^{(h)}(\mu)\, J^{(h)}(\mu)}{|d^{(h)}(\mu)|^{1+s}}\, d\mu,$$

where the shape functions $\psi_j^{(h)}$ are:

$$\psi_1^{(1)} = \frac{4-s}{3-s} - 1, \quad \psi_2^{(1)} = 1 - \mu_1\mu_3, \quad \psi_3^{(1)} = \mu_1\mu_3,$$

$$\psi_1^{(2)} = \frac{4-s}{3-s} - 1, \quad \psi_2^{(2)} = 1 - \mu_1, \quad \psi_3^{(2)} = \mu_1,$$

$$\psi_1^{(3)} = \frac{4-s}{3-s} - (1 - \mu_1\mu_2), \quad \psi_2^{(3)} = 1 - \mu_1, \quad \psi_3^{(3)} = \mu_1(1-\mu_2),$$

$$\psi_1^{(4)} = \frac{4-s}{3-s} - (1 - \mu_1\mu_2\mu_3), \quad \psi_2^{(4)} = 1 - \mu_1\mu_2, \quad \psi_3^{(4)} = \mu_1\mu_2(1-\mu_3),$$

$$\psi_1^{(5)} = \frac{4-s}{3-s} - (1 - \mu_1\mu_2\mu_3), \quad \psi_2^{(5)} = 1 - \mu_1, \quad \psi_3^{(5)} = \mu_1(1-\mu_2\mu_3).$$

The Jacobians are

$$J^{(1)}(\mu) = \mu_1^2, \qquad J^{(h)}(\mu) = \mu_1^2\mu_2, \quad h = 2, 3, 4, 5,$$

and the distance vectors $d^{(h)}(\mu)$ are defined as

$$d^{(1)}(\mu) = B_l \begin{pmatrix} 1 \\ \mu_1\mu_3 \end{pmatrix} - B_m \begin{pmatrix} 1-\mu_1\mu_2 \\ \mu_1(1-\mu_2) \end{pmatrix}, \quad d^{(2)}(\mu) = B_l \begin{pmatrix} 1 \\ \mu_1 \end{pmatrix} - B_m \begin{pmatrix} 1-\mu_1\mu_2\mu_3 \\ \mu_1\mu_2(1-\mu_3) \end{pmatrix},$$

$$d^{(3)}(\mu) = B_l \begin{pmatrix} 1-\mu_1\mu_2 \\ \mu_1(1-\mu_2) \end{pmatrix} - B_m \begin{pmatrix} 1 \\ \mu_1\mu_2\mu_3 \end{pmatrix}, \quad d^{(4)}(\mu) = B_l \begin{pmatrix} 1-\mu_1\mu_2\mu_3 \\ \mu_1\mu_2(1-\mu_3) \end{pmatrix} - B_m \begin{pmatrix} 1 \\ \mu_1 \end{pmatrix},$$

$$d^{(5)}(\mu) = B_l \begin{pmatrix} 1-\mu_1\mu_2\mu_3 \\ \mu_1(1-\mu_2\mu_3) \end{pmatrix} - B_m \begin{pmatrix} 1 \\ \mu_1\mu_2 \end{pmatrix}.$$

Let $p_k \in [0,1]^3$ and w_k be quadrature points and weights. Then

$$\int_{[0,1]^3} \frac{\psi_j^{(h)}(\mu) J^{(h)}(\mu)}{|d^{(h)}(\mu)|^{1+s}}\, d\mu \approx \sum_{k=1}^{27} w_k \frac{\psi_j^{(h)}(p_k) J^{(h)}(p_k)}{|d^{(h)}(p_k)|^{1+s}}.$$

To write the resulting quadrature in matrix form, we define:

- $\Psi^{(h)} \in \mathbb{R}^{32\times 27}$ by
$$\Psi^{(h)}_{ij} = w_j\, \psi^{(h)}_{[i-1]_4+1}(p_j)\, J^{(h)}(p_j),$$
- the vector $d^{(h)} \in \mathbb{R}^{27}$ by
$$d^{(h)}_k = |d^{(h)}(p_k)|^{-1-s}.$$

Let
$$\hat{I}_{l,m} = \sum_{h=1}^{5} \Psi^{(h)} d^{(h)}.$$
We reshape $\hat{I}_{l,m}$ into $E_{l,m} \in \mathbb{R}^{4\times 8}$ via
$$E^{[i-1]_4+1,\lceil i/4\rceil}_{l,m} = \hat{I}^i_{l,m}, \qquad i = 1,\dots,32.$$
To incorporate the gradient contribution, define
$$C_m = \begin{pmatrix} \partial_1\varphi_1|_{T_m} & \partial_1\varphi_1|_{T_m} & \partial_1\varphi_1|_{T_m} & 0 & \partial_2\varphi_1|_{T_m} & \partial_2\varphi_1|_{T_m} & \partial_2\varphi_1|_{T_m} & 0 \\ \partial_1\varphi_2|_{T_m} & \partial_1\varphi_2|_{T_m} & \partial_1\varphi_2|_{T_m} & 0 & \partial_2\varphi_2|_{T_m} & \partial_2\varphi_2|_{T_m} & \partial_2\varphi_2|_{T_m} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_1\varphi_4|_{T_m} & \partial_1\varphi_4|_{T_m} & \partial_1\varphi_4|_{T_m} & 0 & \partial_2\varphi_4|_{T_m} & \partial_2\varphi_4|_{T_m} & \partial_2\varphi_4|_{T_m} & 0 \end{pmatrix}.$$
Finally, the local matrix $I_{l,m} \in \mathbb{R}^{4\times 8}$ in this case is
$$I_{l,m} = \frac{4}{4-s}|T_l||T_m|\, E_{l,m} \otimes C_m.$$

A.1.4 Case $T_l = T_m$

In this case, we have
$$I_{l,l} = 4|T_l|^2 \int_{\hat{T}\times\hat{T}} F_{ij}.$$
Performing the change of variables $z = y - x$, we obtain
$$\int_{\hat{T}\times\hat{T}} F_{ij} = \int_0^1 \int_0^{\hat{x}_1} \int_{-\hat{x}_1}^{1-\hat{x}_1} \int_{-\hat{x}_2}^{\hat{z}_1+\hat{x}_1-\hat{x}_2} F_{ij}(\hat{x}, \hat{x}+\hat{z})\, d\hat{z}_2\, d\hat{z}_1\, d\hat{x}_2\, d\hat{x}_1.$$
We decompose the integration domain into six subdomains
$$\hat{T}\times\hat{T} = \bigcup_{k=1}^{6} D_k,$$
where
$$\begin{aligned} D_1 = \{(\hat{x},\hat{z}) :\ & -1 \le \hat{z}_1 \le 0,\ -1 \le \hat{z}_2 \le \hat{z}_1, \\ & -\hat{z}_2 \le \hat{x}_1 \le 1,\ -\hat{z}_2 \le \hat{x}_2 \le \hat{x}_1\}, \\ D_2 = \{(\hat{x},\hat{z}) :\ & 0 \le \hat{z}_1 \le 1,\ \hat{z}_1 \le \hat{z}_2 \le 1, \\ & \hat{z}_2 - \hat{z}_1 \le \hat{x}_1 \le 1 - \hat{z}_1,\ 0 \le \hat{x}_2 \le \hat{z}_1 - \hat{z}_2 + \hat{x}_1\}, \end{aligned}$$

$$D_3 = \{(\hat{x}, \hat{z}):\ -1 \le \hat{z}_1 \le 0,\ \hat{z}_1 \le \hat{z}_2 \le 0,$$
$$-\hat{z}_1 \le \hat{x}_1 \le 1,\ -\hat{z}_2 \le \hat{x}_2 \le \hat{x}_1 + \hat{z}_1 - \hat{z}_2\},$$

$$D_4 = \{(\hat{x}, \hat{z}):\ 0 \le \hat{z}_1 \le 1,\ 0 \le \hat{z}_2 \le \hat{z}_1,$$
$$0 \le \hat{x}_1 \le 1 - \hat{z}_1,\ 0 \le \hat{x}_2 \le \hat{x}_1\},$$

$$D_5 = \{(\hat{x}, \hat{z}):\ -1 \le \hat{z}_1 \le 0,\ 0 \le \hat{z}_2 \le 1 + \hat{z}_1,$$
$$\hat{z}_2 - \hat{z}_1 \le \hat{x}_1 \le 1,\ 0 \le \hat{x}_2 \le \hat{x}_1 + \hat{z}_1 - \hat{z}_2\},$$

$$D_6 = \{(\hat{x}, \hat{z}):\ 0 \le \hat{z}_1 \le 1,\ -1 + \hat{z}_1 \le \hat{z}_2 \le 0,$$
$$-\hat{z}_2 \le \hat{x}_1 \le 1 - \hat{z}_1,\ -\hat{z}_2 \le \hat{x}_2 \le \hat{x}_1\}.$$

The changes of variables $(\hat{x}, -\hat{z})$ in D_1 and $(\hat{x} + \hat{z}, \hat{z})$ in D_2 map both domains into the same region,

$$D_1' = \{(\hat{x}', \hat{z}') : 0 \le \hat{z}_1' \le 1,\ \hat{z}_1' \le \hat{z}_2' \le 1,\ \hat{z}_2' \le \hat{x}_1' \le 1, \hat{z}_2' \le \hat{x}_2' \le \hat{x}_1'\}.$$

Hence,

$$4|T_l|^2 \int_{D_1 \cup D_2} F_{ij} = 4|T_l|^2 \int_{D_1'} F_{ij}(\hat{x}', \hat{x}' - \hat{z}') + F_{ij}(\hat{x}' - \hat{z}', \hat{x}')\, d\hat{x}'\, d\hat{z}'$$
$$= 4|T_l|^2 \int_{D_1'} \frac{\mathbf{\Phi}_j(\hat{x}') + \mathbf{\Phi}_j(\hat{x}' - \hat{z}')}{|B_l(\hat{z}')|^{1+s}}\, d\hat{x}'\, d\hat{z}',$$

where in the last equality we used the symmetry of the kernel. The composition of the affine map

$$T_1(w) = \begin{pmatrix} w_1 \\ w_1 - w_2 + w_3 \\ w_4 \\ w_3 \end{pmatrix}$$

with the Duffy map

$$T(\xi, \mu) = \begin{pmatrix} \xi \\ \xi\mu_1 \\ \xi\mu_1\mu_2 \\ \xi\mu_1\mu_2\mu_3 \end{pmatrix}, \qquad |\det(dT)| = \xi^3 \mu_1^2 \mu_2, \tag{A.1.2}$$

yields the transformation from $[0,1]^4$ onto D_1':

$$x = \begin{pmatrix} \xi \\ \xi(1 - \mu_1 + \mu_1\mu_2) \end{pmatrix}, \qquad z = \begin{pmatrix} \xi\mu_1\mu_2\mu_3 \\ \xi\mu_1\mu_2 \end{pmatrix}.$$

Thus, we write

$$\int_{D_1'} \frac{\mathbf{\Phi}_j(\hat{x}') + \mathbf{\Phi}_j(\hat{x}' - \hat{z}')}{|B_l(\hat{z}')|^{1+s}}\, d\hat{x}'\, d\hat{z}' = \int_{[0,1]^4} \frac{\mathbf{\Phi}_j(x(\xi,\mu)) + \mathbf{\Phi}_j(x(\xi,\mu) - z(\xi,\mu))}{|B_l(z(\xi,\mu))|^{1+s}}\, \xi^3 \mu_1^2 \mu_2\, d\xi\, d\mu.$$

Integrating first with respect to ξ, we obtain

$$\int_{D_1'} \frac{\mathbf{\Phi}_j(\hat{x}') + \mathbf{\Phi}_j(\hat{x}' - \hat{z}')}{|B_l(\hat{z}')|^{1+s}}\, d\hat{x}'\, d\hat{z}' = \frac{1}{(4-s)(3-s)(2-s)} \int_0^1 \frac{\theta_j^{(1)}(\mu_3)}{|d^{(1)}(\mu_3)|^{1+s}}\, d\mu_3,$$

where

$$\theta_1^{(1)}(\mu_3) = \frac{2}{1-s} + \mu_3, \quad \theta_2^{(1)}(\mu_3) = \frac{4}{1-s} - 1 - \mu_3, \quad \theta_3^{(1)}(\mu_3) = \frac{2}{1-s} + 1,$$

and

$$d^{(1)}(\mu_3) = B_l \begin{pmatrix} \mu_3 \\ 1 \end{pmatrix}.$$

We proceed analogously for the pairs (D_3, D_4) and (D_5, D_6): we use the variable change $(\hat{x}, -\hat{z})$ in D_3, $(\hat{x}+\hat{z}, \hat{z})$ in D_4, $(\hat{x}+\hat{z}, \hat{z})$ in D_5 and $(\hat{x}, -\hat{z})$ in D_6, to map D_3 and D_4 into the same domain

$$D_2' = \{(\hat{x}', \hat{z}') : 0 \le \hat{z}_1' \le 1,\ 0 \le \hat{z}_2' \le \hat{z}_1',\ \hat{z}_1' \le \hat{x}_1' \le 1,\ \hat{z}_2' \le \hat{x}_2' \le \hat{x}_1' - \hat{z}_1' + \hat{z}_2'\},$$

and D_5 and D_6 into the same domain

$$D_3' = \{(\hat{x}', \hat{z}') : -1 \le \hat{z}_1' \le 0,\ 0 \le \hat{z}_2' \le 1 + \hat{z}_1',\ \hat{z}_2' \le \hat{x}_1' \le 1 + \hat{z}_1',\ \hat{z}_2' \le \hat{x}_2' \le \hat{x}_1'\}.$$

Therefore,

$$\int_{D_3 \cup D_4} F_{ij}(\hat{x}, \hat{x}+\hat{z}) = \int_{D_2'} F_{ij}(\hat{x}', \hat{x}'-\hat{z}') + F_{ij}(\hat{x}'-\hat{z}', \hat{x}') = \int_{D_2'} \frac{\mathbf{\Phi}_j(\hat{x}') + \mathbf{\Phi}_j(\hat{x}'-\hat{z}')}{|B_l(\hat{z}')|^{1+s}}\, d\hat{x}'\, d\hat{z}',$$

$$\int_{D_5 \cup D_6} F_{ij}(\hat{x}, \hat{x}+\hat{z}) = \int_{D_3'} F_{ij}(\hat{x}', \hat{x}'-\hat{z}') + F_{ij}(\hat{x}'-\hat{z}', \hat{x}') = \int_{D_3'} \frac{\mathbf{\Phi}_j(\hat{x}') + \mathbf{\Phi}_j(\hat{x}'-\hat{z}')}{|B_l(\hat{z}')|^{1+s}}\, d\hat{x}'\, d\hat{z}'$$

Again, the compositions of the maps

$$T_2 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 - w_3 + w_4 \\ w_3 \\ w_4 \end{pmatrix}, \quad T_3 \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_1 - w_4 \\ w_2 - w_4 \\ -w_4 \\ w_3 - w_4 \end{pmatrix},$$

and the map (A.1.2), transform D_2' and D_3' in $[0,1]^4$. Namely, in D_2' we perform the coordinate change

$$\hat{x}' = \begin{pmatrix} \xi \\ \xi(\mu_1 - \mu_1\mu_2 + \mu_1\mu_2\mu_3) \end{pmatrix}, \quad \hat{z}' = \begin{pmatrix} \xi\mu_1\mu_2 \\ \xi\mu_1\mu_2\mu_3 \end{pmatrix}, \quad \hat{x}' - \hat{z}' = \begin{pmatrix} \xi(1 - \mu_1\mu_2) \\ \xi(\mu_1 - \mu_1\mu_2) \end{pmatrix},$$

and in D_3' we perform

$$\hat{x}' = \begin{pmatrix} \xi(1 - \mu_1\mu_2\mu_3) \\ \xi(\mu_1 - \mu_1\mu_2\mu_3) \end{pmatrix}, \quad \hat{z}' = \begin{pmatrix} -\xi\mu_1\mu_2\mu_3 \\ \xi(\mu_1\mu_2 - \mu_1\mu_2\mu_3) \end{pmatrix}, \quad \hat{x}' - \hat{z}' = \begin{pmatrix} \xi \\ \xi(\mu_1 - \mu_1\mu_2) \end{pmatrix}.$$

We obtain two additional contributions:

$$\int_{D_2'} \frac{\mathbf{\Phi}_j(\hat{x}') + \mathbf{\Phi}_j(\hat{x}'-\hat{z}')}{|B_l(\hat{z}')|^{1+s}}\, d\hat{x}'\, d\hat{z}' = \frac{1}{(4-s)(3-s)(2-s)} \int_0^1 \frac{\theta_j^{(2)}(\mu_3)}{|d^{(2)}(\mu_3)|^{1+s}}\, d\mu_3,$$

$$\int_{D_3'} \frac{\mathbf{\Phi}_j(\hat{x}') + \mathbf{\Phi}_j(\hat{x}'-\hat{z}')}{|B_l(\hat{z}')|^{1+s}}\, d\hat{x}'\, d\hat{z}' = \frac{1}{(4-s)(3-s)(2-s)} \int_0^1 \frac{\theta_j^{(3)}(\mu_3)}{|d^{(3)}(\mu_3)|^{1+s}}\, d\mu_3,$$

with

$$\theta_1^{(2)} = \frac{2}{1-s} + 1, \quad \theta_2^{(2)} = \frac{2(2-s)}{1-s} - 1 - \mu_3, \quad \theta_3^{(2)} = \frac{2}{1-s} - 2 + \mu_3,$$

$$\theta_1^{(3)} = \frac{2}{1-s} + \mu_3, \quad \theta_2^{(3)} = \frac{2}{1-s} + 1, \quad \theta_3^{(3)} = \frac{2(2-s)}{1-s} - 1 - \mu_3,$$

and

$$d^{(2)}(\mu_3) = B_l \begin{pmatrix} 1 \\ \mu_3 \end{pmatrix}, \qquad d^{(3)}(\mu_3) = B_l \begin{pmatrix} \mu_3 \\ 1-\mu_3 \end{pmatrix}.$$

Collecting all terms in the preceding discussion, we have

$$4|T_l|^2 \int_{\hat{T}\times\hat{T}} F_{ij} = \frac{4|T_l|^2}{(4-s)(3-s)(2-s)} \sum_{h=1}^{3} \int_0^1 \frac{\theta_j^{(h)}(\mu_3)}{|d^{(h)}(\mu_3)|^{1+s}}\, d\mu_3,$$

with

$$\theta_1^{(1)}(\mu_3) = \frac{2}{1-s} + \mu_3, \quad \theta_2^{(1)}(\mu_3) = 2\frac{2}{1-s} - 1 - \mu_3, \quad \theta_3^{(1)}(\mu_3) = \frac{2}{1-s} + 1,$$

$$\theta_1^{(2)}(\mu_3) = \frac{2}{1-s} + 1, \quad \theta_2^{(2)}(\mu_3) = 2\frac{2-s}{1-s} - 1 - \mu_3, \quad \theta_3^{(2)}(\mu_3) = \frac{2}{1-s} - 2 + \mu_3,$$

$$\theta_1^{(3)}(\mu_3) = \frac{2}{1-s} + \mu_3, \quad \theta_2^{(3)}(\mu_3) = \frac{2}{1-s} + 1, \quad \theta_3^{(1)}(\mu_3) = 2\frac{2-s}{1-s} - 1 - \mu_3,$$

$$d^{(1)}(\mu_3) = B_l \begin{pmatrix} \mu_3 \\ 1 \end{pmatrix}, \quad d^{(2)}(\mu_3) = B_l \begin{pmatrix} 1 \\ \mu_3 \end{pmatrix}, \quad d^{(3)}(\mu_3) = B_l \begin{pmatrix} \mu_3 \\ 1-\mu_3 \end{pmatrix}.$$

Let $\{p_k, w_k\}_{k=1}^9$ be quadrature nodes and weights in $[0,1]$. Then

$$\int_0^1 \frac{\theta_j^{(h)}(\mu_3)}{|d^{(h)}(\mu_3)|^{1+s}}\, d\mu_3 \approx \sum_{k=1}^{9} w_k \frac{\theta_j^{(h)}(p_k)}{|d^{(h)}(p_k)|^{1+s}}.$$

As in the previous cases, we now describe how to write the resulting quadrature in matrix form. We define:

- $\Theta^{(h)} \in \mathbb{R}^{18\times 9}$ by
$$\Theta_{ij}^{(h)} = w_j\, \theta_{[i-1]_3+1}^{(h)}(p_j),$$
- $d^{(h)} \in \mathbb{R}^9$ by
$$d_k^{(h)} = |d^{(h)}(p_k)|^{-1-s}.$$

Set

$$\hat{I}_{l,l} = \sum_{h=1}^{3} \Theta^{(h)} d^{(h)},$$

and reshape into $Q_{l,l} \in \mathbb{R}^{3\times 6}$ via

$$Q_{l,l}^{[i-1]_3+1, \lceil i/3 \rceil} = \hat{I}_{l,l}^i, \qquad i = 1, \ldots, 18.$$

Define $C_l \in \mathbb{R}^{3\times 6}$ by

$$C_l = \begin{pmatrix} \partial_1\varphi_1|_{T_l} & \partial_1\varphi_1|_{T_l} & \partial_1\varphi_1|_{T_l} & \partial_2\varphi_1|_{T_l} & \partial_2\varphi_1|_{T_l} & \partial_2\varphi_1|_{T_l} \\ \partial_1\varphi_2|_{T_l} & \partial_1\varphi_2|_{T_l} & \partial_1\varphi_2|_{T_l} & \partial_2\varphi_2|_{T_l} & \partial_2\varphi_2|_{T_l} & \partial_2\varphi_2|_{T_l} \\ \partial_1\varphi_3|_{T_l} & \partial_1\varphi_3|_{T_l} & \partial_1\varphi_3|_{T_l} & \partial_2\varphi_3|_{T_l} & \partial_2\varphi_3|_{T_l} & \partial_2\varphi_3|_{T_l} \end{pmatrix}.$$

With this notation, the local matrix in this case reads

$$I_{l,l} = \frac{4|T_l|^2}{(4-s)(3-s)(2-s)}\, Q_{l,l} \otimes C_l.$$

A.2 Fractional Oseen-Frank model

In this section we describe the implementation of the matrices involved in Problem (O) in dimension $d = 2$. We need to compute the matrices

$$K_{ij} = \langle F_s \mathbf{\Phi}_i, F_s \mathbf{\Phi}_j \rangle_{L^2(\mathbb{R}^d)},$$

where $\{\mathbf{\Phi}_i\}_{i=1}^{dN_h} \subset [\mathcal{P}_1(\mathcal{T}_h(B_H))]^d$ is the usual $d-$Lagrange basis and F_s either ∇^s or $\mathbf{curl}^s$, we point out that the same ideas also work for $F_s = \mathrm{div}^s$. Note that, $\nabla \mathbf{\Phi}_i$ and $\mathbf{curl}\, \mathbf{\Phi}_i$ are constant on each element of the mesh. Therefore, in the same fashion as Remark A.1.1, we deduce

$$K_{ij} = \sum_{T,T' \colon T \subset \omega_i,\ T' \subset \omega_j} F(\mathbf{\Phi}_i|_T) \cdot F(\mathbf{\Phi}_j|_{T'}) \int_{\mathbb{R}^d} \alpha_T(x)\, \alpha_{T'}(x)\, dx,$$

where $F = F_1$ and

$$\alpha_T(x) = c(d, 1-s) \int_T \frac{1}{|x-y|^{d+s-1}} dy.$$

With this, we could use the Duffy type tranformations of the previous section combined with quadrature rules in a double loop over the elements. Still, we would need to deal with a triple integral. From a practical computational viewpoint, such an approach would be prohibitively expensive. For this reason, we adopt a different strategy. Instead of using the functions $\{\alpha_T\}$, we apply Fubini's theorem to the double integral:

$$K_{ij} = \sum_{T,T' \colon T \subset \omega_i,\ T' \subset \omega_j} F(\mathbf{\Phi}_i|_T) \cdot F(\mathbf{\Phi}_j|_{T'}) \int_T \int_{T'} \int_{\mathbb{R}^d} \frac{c(d,1-s)^2}{|x-y|^{d+s-1}\, |x-z|^{d+s-1}}\, dx\, dy\, dz,$$

and define $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

$$g(y,z) = c(d,1-s)^2 \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d+s-1}} \frac{1}{|x-z|^{d+s-1}}\, dx.$$

With this, we can write

$$K_{ij} = \sum_{T,T' \colon T \subset \omega_i,\ T' \subset \omega_j} F(\mathbf{\Phi}_i|_T) \cdot F(\mathbf{\Phi}_j|_{T'}) \int_T \int_{T'} g(y,z)\, dy\, dz.$$

The key idea is that we can give an explicit expression of $g(y, z)$. Recalling the Riesz kernel I_{1-s} defined by Formula (1.1.8), we have

$$g(y,z) = \int_{\mathbb{R}^d} I_{1-s}(x-y)\, I_{1-s}(x-z)\, dx = I_{1-s} * \Phi_z(y) = \mathcal{I}_{1-s} \mathbf{\Phi}_z(y),$$

where $\mathbf{\Phi}_z(x) = I_{1-s}(x - z)$, that is, $\mathbf{\Phi}_z = \tau_z I_{1-s}$ where $(\tau_z f)(x) = f(x - z)$. We recall that the Riesz kernels satisfy semi-group property

$$I_\alpha * I_\beta = I_{\alpha+\beta}, \quad \text{for } \alpha, \beta > 0 \text{ and } \alpha + \beta < d,$$

and that they commute with translations. Therefore, we deduce

$$g(y,z) = \mathcal{I}_{1-s} \Phi_z(y) = \mathcal{I}_{1-s}(\tau_z I_{1-s}) = \tau_z [I_{1-s} * I_{1-s}] = \tau_z I_{2-2s}.$$

Remark A.2.1. *The above derivation is not justified for $d = 1$ and $s \leq \frac{1}{2}$, because in that case the integrand in g decays too slowly; equivalently, $2-2s$ does not satisfy $0 < 2-2s < d$ if $d = 1$ and $s \leq \frac{1}{2}$.*

Therefore, we simply have

$$g(y,z) = c(d, 2-2s)\,\frac{1}{|y-z|^{d+2s-2}},$$

with

$$c(d, 2-2s) = \frac{\Gamma\big(\frac{d}{2}-1+s\big)}{\pi^{d/2}\,2^{\,2-2s}\,\Gamma(1-s)}.$$

Thus, let $n_T = \#\mathcal{T}_h(B_H)$ and $d=2$, it only remains to compute the matrix $G \in \mathbb{R}^{n_T \times n_T}$ given by

$$G_{l,m} := c(2, 2-2s) \iint_{T_l \times T_m} \frac{dydz}{|y-z|^{2s}}, \quad 1 \le l, m \le n_T.$$

We iterate over the elements of the mesh by means of a double loop and perform the computations using suitable quadrature rules. For $1 \le l \le n_T$ and $1 \le m \le l$ (note that G is a symmetric matrix), we first classify the pair (T_l, T_m) according to whether the intersection is empty, a vertex, an edge or T_l. We then compute $G_{l,m}$ following the same strategy as in the previous section. If the elements are disjoint, we simply apply a Gaussian quadrature rule, since no singularity is present in this case. In the remaining configurations, we employ appropriate Duffy-type transformations to treat the singularity explicitly, and subsequently apply quadrature rules.

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