# Orbifolds and Thurston maps

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**Abstract**

In this thesis, we study orbifolds associated with Thurston maps. A Thurston map is a branched covering map $f : S^2 \to S^2$ on a 2-sphere such that each of its critical points has a finite future orbit. The key contribution of our work is establishing the strong relationship between Thurston maps with parabolic orbifolds and quotients of torus endomorphisms (QOTEs). A QOTE is a branched covering map $f : S^2 \to S^2$ such that there exists a degree $d \geq 2$ self-covering map $F : T^2 \to T^2$ on a torus that is semiconjugated to $f$ by a branched covering map $\rho : T^2 \to S^2$. We prove that every QOTE has a parabolic orbifold, addressing a question initially posed in [**1**]. Additionally, we show that Thurston maps with parabolic orbifolds and no periodic critical points are QOTEs. For Thurston maps with hyperbolic orbifolds, we develop a new framework that involves lifting these maps to covering maps on higher genus surfaces. This generalization leads to the introduction of a new definition that extends the concept of QOTE, and raises new questions.



**Resumen**

En esta tesis, estudiamos orbifolds associados a mapas de Thurston. Un mapa de Thurston es un cubrimiento ramificado $f : S^2 \to S^2$ en una 2-esfera tal que cada uno de sus puntos críticos tiene una órbita futura finita. La contribución clave de nuestro trabajo es establecer la fuerte relación entre los mapas de Thurston con orbifolds parabólicos y los cocientes de endomorfismos del toro (QOTEs). Un QOTE es un cubrimiento ramificado $f : S^2 \to S^2$ tal que existe un cubrimiento $F : T^2 \to T^2$ de grado $d \geq 2$ en un toro, que es semiconjugado a $f$ por un cubrimiento ramificado $\rho : T^2 \to S^2$. Demostramos que todo QOTE tiene un orbifold parabólico, abordando una pregunta inicialmente planteada en [**1**]. Además, mostramos que los mapas de Thurston con orbifolds parabólicos y sin puntos críticos periódicos son QOTEs. Para los mapas de Thurston con orbifolds hiperbólicos, desarrollamos un nuevo marco que implica levantar estos mapas a cubrimientos ramificados en superficies de mayor género. Esta generalización lleva a la introducción de una nueva definición que amplía el concepto de QOTE, y plantea nuevas preguntas.

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# Contents

CHAPTER 1

# Introduction

In this work we study orbifolds associated with Thurston maps. A *Thurston map* is a branched covering map $f : S^2 \to S^2$ on a 2-sphere $S^2$ such that the postcritical set

$$\mathrm{post}(f) = \bigcup_{n\geq 1} f^n(\mathrm{crit}(f))$$

is finite. Here $\mathrm{crit}(f)$ denotes the set of critical points of $f$.

As a simple example of Thurston map, consider the polynomial $f(z) = 1 + (\omega - 1)z^3$ extended as a map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ on the Riemann sphere, where $\omega = e^{2i\pi/3}$. Then, $f$ has exactly two critical points, namely 0 and $\infty$, each of which has local degree equal to 3. Also, we have $f(\infty) = \infty$, $f(0) = 1$ and $f(1) = f(\omega) = \omega$. Hence, $\mathrm{post}(f) = \{1, \omega, \infty\}$.

All this information related to $\mathrm{crit}(f)$ and $\mathrm{post}(f)$ can be summarized in what we call the *ramification portrait* of $f$:

$$3{:}1 \circlearrowright \infty \qquad 0 \xrightarrow{3:1} 1 \longrightarrow \omega \circlearrowleft$$

The *orbifold* $\mathcal{O}_f$ associated to a Thurston map $f : S^2 \to S^2$ has $S^2$ as underlying surface, and $\mathrm{post}(f)$ as the set of singular points. The weight at each $x \in S^2$ is computed as

$$\nu_f(x) = \mathrm{lcm}\left\{\deg(f^n, y) : \ y \in f^{-n}(x),\ n \geq 1\right\}.$$

Note that this value is always an integer number, except when $x$ is in a periodic orbit that contains a critical point of $f$. In this case, the weight equals to $\infty$.

The Euler characteristic of $\mathcal{O}_f$ is the number

$$\chi(\mathcal{O}_f) = 2 - \sum_{x\in \mathrm{post}(f)} \left(1 - \frac{1}{\nu_f(x)}\right).$$

In the given example, we have $\nu_f(\infty) = \infty$ and $\nu_f(1) = \nu_f(\omega) = 3$. That is, $\mathcal{O}_f$ is the orbifold over the sphere with signature $(3, 3, \infty)$ and $\chi(\mathcal{O}_f) = -1/3$. In general, by *signature* we mean the tuple $(m_1, \ldots, m_s)$ of weights $m_i \geq 2$, increasingly ordered.

Another example of Thurston map is the rational map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by $g(z) = 1 - 2/z^2$. The ramification portrait is the following:

$$0 \xrightarrow{2:1} \infty \xrightarrow{2:1} 1 \longrightarrow -1 \circlearrowleft$$

Hence, $\mathrm{post}(g) = \{-1, 1, \infty\}$, the signature of $\mathcal{O}_g$ is $(2, 4, 4)$, and $\chi(\mathcal{O}_g) = 0$.

The Euler characteristic of the orbifold of a Thurston map $f : S^2 \to S^2$ is always non-positive (see Proposition 4.2). We classify the orbifold as *parabolic* when $\chi(\mathcal{O}_f) = 0$, and *hyperbolic* when $\chi(\mathcal{O}_f) < 0$. In this thesis, we study Thurston maps based on whether their orbifolds are parabolic or hyperbolic.

We begin our work by studying two-dimensional orbifolds in Chapter 2, exploring their properties and showing their connection to group actions on surfaces.

The most significant contribution of our research is establishing the strong relationship between Thurston maps with parabolic orbifolds and quotients of torus endomorphisms, a connection we describe below.

A *quotient of torus endomorphism (QOTE)* is a branched covering map $f : S^2 \to S^2$ such that there is a torus endomorphism $F : T^2 \to T^2$ of degree $\deg(F) \geq 2$ and a branched covering map $\rho : T^2 \to S^2$ such that $f \circ \rho = \rho \circ F$. We have the following commutative diagram:

$$\begin{array}{ccc} T^2 & \xrightarrow{F} & T^2 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ S^2 & \xrightarrow[f]{} & S^2 \end{array}$$

Recall that a torus endomorphism is a self-covering map of a torus. It can be shown that every QOTE is a Thurston map with no periodic critical points. See Proposition 3.2 and Proposition 3.5. The most important contribution of this thesis is the following:

Theorem 3.1 Let $f$ be a QOTE. Then, $\mathcal{O}_f$ is parabolic.

The question of whether every quotient of torus endomorphism has a parabolic orbifold was originally posed in [**1**]. A positive answer to this question was provided in [**2**] for specific cases of QOTEs, namely for those that are *expanding*. We do not need to define this last condition here, as the proof we present is independent of it.

All the work leading to the proof of Theorem 3.1 is detailed in Chapter 3. Additionally, this chapter includes examples and other notable properties of quotients of torus endomorphisms.

Reciprocally, in Chapter 4 we begin with a Thurston map $f : S^2 \to S^2$ with a parabolic orbifold and no periodic critical points, and show that it is a QOTE. The fact that such a map $f$ can be lifted to a torus endomorphism was previously established in [**4**]. However, here we provide a different proof of this result. See Theorem 4.3.

Afterwards, we address Thurston maps $f : S^2 \to S^2$ with hyperbolic orbifolds. Our goal is to lift such a map $f$ to a covering map as well. However, in this case, higher genus surfaces must be considered, which significantly alters the situation since these surfaces do not admit self-covering maps like the torus does. Consequently, a Thurston map with hyperbolic orbifold does not admit a self-covering map that is semiconjugated to it, in the same manner as a QOTE does; see formula (2.2). This leads to the introduction of a new definition that generalizes the concept of QOTE, involving towers of coverings instead of just one covering map. The details of this concept are provided in Chapter 4.

We conclude this document with a discussion of problems and related questions in Chapter 5.

CHAPTER 2

# Orbifolds and branched covering maps

In this chapter we recall some definitions and results from covering space theory that we will use throughout this work. This will also motivate the theory of branched covering maps, which generalises the former one.

Related to covering maps and fundamental groups of surfaces, we would like to obtain similar results for branched covering maps. This is where the concept of a (two-dimensional) *orbifold* comes in to generalise the idea of a surface. For us, an orbifold will be a surface with additional structure. In short, points in the surface will have a number associated that we call the *weight* of the point.

The results concerning orbifolds presented here are likely standard, but clear references for these results can sometimes be difficult to find. Consequently, some of the proofs are original and provided without specific citations.

In Section 2.2, we introduce the concept of an *orbifold covering map* between orbifolds, which generalizes the concept of covering maps between surfaces. We then introduce the definition of the *Euler characteristic* of an orbifold, highlighting its multiplicative property with respect to finite orbifold covers, similar to the case for surfaces. See Lemma 2.7.

Next, we recall that a *good* orbifold is one that can be *orbifold-covered* by a surface. Using covering space theory, one can show that any orbifold whose underlying surface $S$ has Euler characteristic $\chi(S) \leq 1$ is good. See Corollary 2.9. The case of orbifolds over the sphere is more intricate because some of them are not good.

The *universal branched cover* for orbifolds is the analogous to the universal cover for surfaces. This leads to the study of its group of *deck transformations*, which we will refer to as the *fundamental group* of the orbifold. Unlike the case for surfaces, this group does not act freely on the universal branched cover. This non-free action results in branching points of the universal branched covering map, consistent with the structure of the orbifold.

Motivated by the concept of the fundamental group of an orbifold, we further explore group actions on surfaces and maps *induced* by such actions. Specifically, we study quotient spaces by group actions. We then establish that a good orbifold can be regarded as the quotient space of its universal branched cover by the action of its fundamental group. See Corollary 2.17.

Additionally, we examine the relationship between orbifold covering maps and subgroups of the fundamental group. In particular, the relationship between the degree of the orbifold covering map and the index of the subgroup. See Proposition 2.18.

In section 2.5, we provide an extended definition of orbifolds, allowing them to include points with weight equal to $\infty$. This extension is necessary for studying the orbifolds associated with *Thurston maps*, which will be covered in subsequent chapters.

To conclude, we present two examples of orbifolds at the end of this chapter.

## 2.1. Branched coverings maps

**2.1.1. Covering maps.** We will start by briefly reviewing some definitions and results from covering space theory. For more details, refer to [**3**]. Let $Y$ and $Z$ be connected topological spaces. We say a map $\rho : Z \to Y$ is a *covering map* if for each $y \in Y$ there is an open neighborhood $V \subset Y$ containing $y$ such that $\rho^{-1}(V)$ is a union of disjoint open sets each of which is mapped homeomorphically onto $V$ by $\rho$.

We will say $\rho : Z \to Y$ is a *finite covering map* if $|\rho^{-1}(y)| < \infty$ for some $y \in Y$. In this case, the cardinality of $\rho^{-1}(y)$ is independent of $y$. We call this number the *degree* of $\rho$, an denote it by $\deg(\rho)$.

The *fundamental group* $\pi_1(Y, y_0)$ of a space $Y$ based at $y_0 \in Y$ is the group of homotopy classes of loops based at $y_0$ with the product induced by the concatenation of loops. In what follows, if $\gamma_0, \gamma_1 : [0,1] \to Y$ are two path with $\gamma_0(1) = \gamma_1(0)$, we will denote by $\gamma_0 * \gamma_1$ their concatenation. Recall that any continuous map $f : X \to Y$ with $f(x_0) = y_0$ induces a group morphism between the fundamental groups, that we will denote by $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

A *lift* of a continuous map $f : X \to Y$ with respect to a covering map $\rho : Z \to Y$ is a continuous map $\hat{f} : X \to Z$ such that $\rho \circ \hat{f} = f$. About existence and uniqueness of lifts, we have the propositions below. See [**3**, Proposition 1.33] and [**3**, Proposition 1.34].

Proposition 2.1. Let $\rho : Z \to Y$ be a covering map and $f : X \to Y$ a continuous map where $X$ is path-connected and locally path-connected. Suppose $z_0 \in Z$ and $x_0 \in X$ are such that $\rho(z_0) = f(x_0)$. Then, a lift $\hat{f} : X \to Z$ exists iff $f_*\pi_1(X, x_0) \subset \rho_*\pi_1(Z, z_0)$.

Proposition 2.2. Let $\rho : Z \to Y$ be a covering map and $f : X \to Y$ a continuous map. Suppose $\hat{f}_0, \hat{f}_1 : X \to Z$ are lifts of $f$ and there is some $x_0 \in X$ such that $\hat{f}_0(x_0) = \hat{f}_1(x_0)$. Then, $\hat{f}_1 = \hat{f}_0$.

On the other hand, associated to each subgroup $\Lambda < \pi_1(Y, y_0)$, there is a covering map $\rho : Z \to Y$ such that $\rho_*\pi_1(Z, z_0) = \Lambda$. Besides, this covering map is unique up to precomposition with a homeomorphism. This works under few assumptions, namely $Y$ must be path-connected, locally path-connected, and semilocally simply-connected. See [**3**, Theorem 1.38].

A covering map $\rho : X \to Y$ associated to the trivial subgroup is called a *universal covering map* of $Y$. As the morphism induced in the fundamental groups is an injection for the case of covering maps, the space $X$ will be simply connected.

**2.1.2. Branched coverings maps between surfaces.** Let $X, Y$ be two connected and oriented surfaces. All surfaces shall be assumed to be of finite type and without boundary, unless otherwise stated. A continuous and surjective map $f : X \to Y$ is called *branched covering map* if for every point $y \in Y$ there is a topological disk $V \subset Y$

containing $y$ such that $f^{-1}(V)$ can be written as a disjoint union

$$f^{-1}(V) = \bigcup_{i \in I} U_i$$

of open sets $U_i \subset X$ with the following property. For every $U = U_i$ there is a unique $x \in U \cap f^{-1}(y)$ and an integer $d = d_i \geq 1$ such that the restriction $f : U \backslash \{x\} \to V \backslash \{y\}$ is a covering map of degree $d$.

The number $d \geq 1$ is called the *local degree* of $f$ at $x$, and we will denote it by $\deg(f, x)$. The set of *critical points* of $f$ is

$$\mathrm{crit}(f) = \{x \in X : \deg(f, x) \geq 2\},$$

and the image of a critical point is called *critical value* of $f$. When $Y = X$, a *postcritical point* of $f$ is a point $y \in Y$ such that $y = f^n(x)$ for some $n \geq 1$ and $x \in \mathrm{crit}(f)$. Consequently, the set of postcritical points of $f$ is

$$\mathrm{post}(f) := \bigcup_{n \geq 1} f^n(\mathrm{crit}(f)).$$

On the other hand, one can consider the spaces $Y_\star := Y \backslash f(\mathrm{crit}(f))$ and $X_\star := X \backslash f^{-1}(f(\mathrm{crit}(f)))$, and the restriction $f : X_\star \to Y_\star$ results in a covering map. We say the branched covering map $f : X \to Y$ is of *degree* $\deg(f)$ if the covering map $f : X_\star \to Y_\star$ has degree $\deg(f)$.

Observe that the disk $U \subset X$ as in the definition of branched covering map contains at most one critical point and so the set $\mathrm{crit}(f)$ is discrete in $X$. Moreover, the following holds for all $y \in Y$:

$$\sum_{x \in f^{-1}(y)} \deg(f, x) = \deg(f). \tag{2.1}$$

The *Riemann-Hurwitz formula* states in addition that

$$\chi(X) + \sum_{x \in \mathrm{crit}(f)} \big(\deg(f, x) - 1\big) = \deg(f)\chi(Y), \tag{2.2}$$

where $\chi(X)$ is the Euler characteristic of a compact surface $X$.

It is easy to see that composing branched covering maps $f : X \to Y$ and $g : Y \to Z$ leads in another branched covering map $g \circ f : X \to Z$ such that

$$\deg(g \circ f, x) = \deg(g, f(x)) \deg(f, x)$$

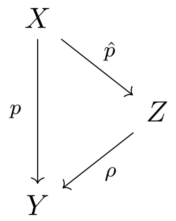
for all $x \in X$.

All these formulas can be found in [**1**, Section 2.1].

About lifting respect to branched covering maps, we have the statements below. Refer to [**1**, Lemma A.19].

LEMMA 2.3. (Lifting branched covering maps). Let $\rho : Z \to Y$ and $p : X \to Y$ be branched covering maps. Suppose that $X$ is simply connected and, whenever $x \in X$ and $z \in Z$ verify $p(x) = \rho(z)$, one has $\deg(\rho, z) \mid \deg(p, x)$. Then, for all $x_0 \in X$ and $z_0 \in Z$ with $p(x_0) = \rho(z_0)$, there exists a branched covering map $\hat{p} : X \to Z$ such that $\hat{p}(x_0) = z_0$ and $p = \rho \circ \hat{p}$.

So the following diagram commutes:




LEMMA 2.4. (Uniqueness of lifts). Let $\rho : Z \to Y$ be a branched covering map. Suppose $\hat{p}_0, \hat{p}_1 : X \to Z$ are discrete maps (the inverse image of a point is discrete ) such that $\rho \circ \hat{p}_0 = \rho \circ \hat{p}_1$ and there is some $x_0 \in X$ such that $\hat{p}_0(x_0) = \hat{p}_1(x_0) =: z_0$ and $\rho(z_0) \notin \rho(\text{crit}(\rho))$. Then, $\hat{p}_1 = \hat{p}_0$.

To finish, we prove the following proposition that relates finite covering maps in punctured surfaces with finite branched covering maps in compact surfaces.

PROPOSITION 2.5. Let $X$ and $Y$ be compact surfaces, $x_1, \dots, x_r \in X$ and $y_1, \dots, y_s \in Y$. Consider the punctured surfaces $X_\star := X \backslash \{x_1, \dots, x_r\}$ and $Y_\star := Y \backslash \{y_1, \dots, y_s\}$. Then, any finite covering map $f : X_\star \to Y_\star$ can be extended to a branched covering map $f : X \to Y$.

PROOF. We will call "dist" a distance in $X$ and consider the restriction of it to $X_\star$. The same name will be used for the distance in $Y$ and $Y_\star$.

We first claim the map $f : X_\star \to Y_\star$ is proper. This means the inverse image of a compact set $K \subset Y_\star$ is compact in $X_\star$. Note that the compact sets in $X_\star$ are the compact sets in $X$ that do not contain any of the punctures $x_1, \dots, x_r$. The same holds for $Y_\star$. To prove $f : X_\star \to Y_\star$ is proper, we will show any sequence $z_n \in f^{-1}(K)$ has an accumulation point in $f^{-1}(K)$.

Since $K$ is compact, the sequence $f(z_n) \in K$ has an accumulation point $z \in K$. For $\ell \geq 1$, consider the ball $B(z, 1/\ell) \subset Y_\star$, and extract a subsequence $f(z_{n_\ell}) \in B(z, 1/\ell)$. Then, $z_{n_\ell} \in f^{-1}(B(z, 1/\ell))$ and $\text{dist}(f^{-1}(B(z, 1/\ell)), f^{-1}(z)) \to 0$ as $\ell \to \infty$. Since $f^{-1}(z)$ is a finite set, the sequence $z_{n_\ell}$ must accumulate at some point $\hat{z} \in f^{-1}(z) \in f^{-1}(K)$, which proves the claim.

Now we will show $f : X_\star \to Y_\star$ extends continuously to any $x \in \{x_1, \dots, x_r\}$.

Let $\varepsilon_0 > 0$ be such that the connected components of $B(\{y_1, \dots, y_s\}, \varepsilon_0)$ are the balls $B(y_1, \varepsilon_0), \dots, B(y_s, \varepsilon_0)$, and consider the set $K := Y \backslash B(\{y_1, \dots, y_s\}, \varepsilon_0)$. Since $K$ is a compact set of $Y_\star$, we know that $f^{-1}(K)$ is a compact set of $X_\star$. This implies $\text{dist}(x, f^{-1}(K)) > 0$, and so there is some $\delta_0 > 0$ such that $B(x, \delta_0) \subset X \backslash K$. Since $f(B(x, \delta_0) \backslash \{x\}) \subset B(\{y_1, \dots, y_s\}, \varepsilon_0)$ and $B(x, \delta_0) \backslash \{x\}$ is a connected set, there must be some puncture $y \in \{y_1, \dots, y_s\}$ such that $f(B(x, \delta_0) \backslash \{x\}) \subset B(y, \varepsilon_0)$.

We claim that for all $\varepsilon < \varepsilon_0$ there is some $\delta > 0$ with $f(B(x, \delta) \backslash \{x\}) \subset B(y, \varepsilon)$. The same argument as the one given above shows that there is some $\delta > 0$ such that $f(B(x, \delta) \backslash \{x\}) \subset B(\{y_1, \dots, y_s\}, \varepsilon)$. We may assume $\delta < \delta_0$. Then, $f(B(x, \delta) \backslash \{x\}) \subset f(B(x, \delta_0) \backslash \{x\}) \subset B(y, \varepsilon_0)$, an so

$$f(B(x, \delta) \backslash \{x\}) \subset B(\{y_1, \dots, y_s\}, \varepsilon) \cap B(y, \varepsilon_0) = B(y, \varepsilon)$$

as claimed. This proves $f : X_\star \to Y_\star$ extends continuously to $x$, when defining $f(x) = y$.

On the other hand, we also claim that the extension $f : X \to Y$ is onto. Indeed, recall that the covering map $f : X_\star \to Y_\star$ is already onto. This gives $Y_\star \subset f(X)$. Moreover, $X$ is compact by hypothesis, and so it is $f(X)$. It follows $f(X) = Y$.

To finish, we must show the local property of branched covering maps, at any puncture $x \in \{x_1, \ldots, x_r\}$. Let $y = f(x)$ and $D \subset Y$ be a small disk with $y$ in its interior. We may assume each point of $f^{-1}(y)$ belongs to a different connected component of $f^{-1}(D)$. Call $\hat{D}$ the connected component of $f^{-1}(D)$ containing $x$. Then, the restriction $f : \hat{D}\backslash\{x\} \to D\backslash\{y\}$ is a finite covering map. Since $D\backslash\{y\}$ is an annulus, we deduce that $\hat{D}\backslash\{x\}$ is also an annulus, and the covering $f : \hat{D}\backslash\{x\} \to D\backslash\{y\}$ is $d$ to 1 for some $d \geq 1$. □

## 2.2. Orbifolds and orbifold covering maps

To define an orbifold, we will use the approach presented in [**1**], which employs *ramification functions*. Alternative definitions, often utilizing atlases, can be found in the literature; see, for example, [**8**] and [**7**].

Let $S$ be a surface. A map $\nu : S \to \mathbb{N}$ is said to be a *ramification function* on $S$ if its support

$$\operatorname{supp}(\nu) := \{x \in S : \ \nu(x) \geq 2\}$$

is a discrete set, and $\nu = 1$ away from it. An *orbifold* is a pair $\mathcal{O} = (S, \nu)$, where $S$ is an orientable surface (of finite type) and $\nu : S \to \mathbb{N}$ is a ramification function on $S$. We call $S$ the *underlying surface* of $\mathcal{O}$, and the points $x \in \operatorname{supp}(\nu)$ are called *singular points* of *weight* $\nu(x)$. The *signature* of the orbifold is the tuple $(m_1, m_2, \ldots)$ of weights $m_i \geq 2$, increasingly ordered.

Surfaces can be regarded as orbifolds without singular points. In what follows, we will treat them as orbifolds and omit the orbifold notation.

Suppose $\hat{\mathcal{O}} = (M, \hat{\nu})$ and $\mathcal{O} = (S, \nu)$ are two orbifolds. An *orbifold covering map* $\rho : \hat{\mathcal{O}} \to \mathcal{O}$ is a branched covering map $\rho : M \to S$ such that

$$\nu(\rho(\hat{x})) = \hat{\nu}(\hat{x}) \deg(\rho, \hat{x}), \text{ for all } \hat{x} \in M.$$

Here we say that $\hat{\mathcal{O}}$ *orbifold covers* $\mathcal{O}$. An *orbifold homeomorphism* $\varphi : \hat{\mathcal{O}} \to \mathcal{O}$ is a homeomorphism $\varphi : M \to S$ such that $\nu(\varphi(\hat{x})) = \hat{\nu}(\hat{x})$ for all $\hat{x} \in M$.

Observe that an orbifold covering map $\rho : M \to S$ between surfaces is simply a covering map in the usual sense since $\rho$ must have constant local degree $\deg(\rho, \hat{x}) = 1$ when the ramification function $\nu$ has empty support. Thus, this definition generalises the one of covering maps between surfaces.

An orbifold $\mathcal{O}$ is called *good* if it can be orbifold covered by a surface, that is, if there exists an orbifold covering map $\rho : M \to \mathcal{O}$ from a surface $M$. Figure 2.1 illustrates a degree-two orbifold covering map from a genus-two surface to an orbifold $\mathcal{O}$ over the torus of signature $(2, 2)$. This shows $\mathcal{O}$ is a good orbifold.

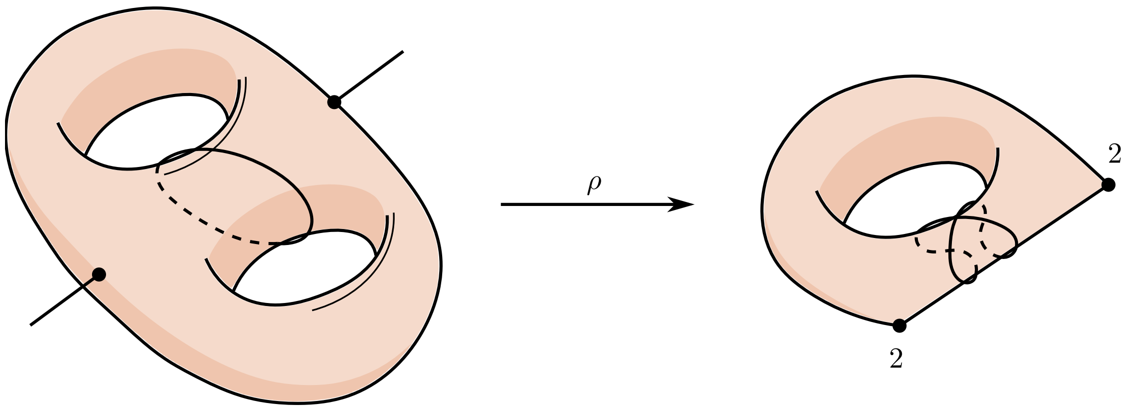



FIGURE 2.1. Illustration of a branched covering map $\rho : M \to T^2$ from a genus-two surface $M$ onto a torus $T^2$. The map $\rho$ is induced by identifying points on $M$ through a rotation of angle $\pi$ around the given axis. The map can be through as an orbifold covering map $\rho : M \to \mathcal{O}$, where $\mathcal{O}$ is an orbifold over $T^2$ with two singular points of weight two.

Note that in the definition of a good orbifold we do not require the orbifold covering map to be finite-sheeted. However, Theorem 2.11 asserts that it can be chosen to be finite-sheeted.

An example of an orbifold that cannot be orbifold covered by a surface is $\mathcal{O} = (S^2, \nu)$, where $S^2$ is a sphere and $\operatorname{supp}(\nu)$ has only one point $x_0$. To see this, note that, if $\rho : M \to \mathcal{O}$ is an orbifold covering map from a surface, then $\rho : M \setminus \rho^{-1}(\rho(\operatorname{crit}(\rho))) \to S^2 \setminus \{x_0\}$ would be a non-trivial covering map. However, this cannot be the case as $S^2 \setminus \{x_0\}$ has a trivial fundamental group.

Another example of bad orbifold is the following. If $\mathcal{O}$ is an orbifold over $S^2$ with exactly two singular points of different weight, then $\mathcal{O}$ is not good. Refer to [**7**, Theorem 2.3], where the author lists all bad orbifolds.

A *deck transformation* of an orbifold covering map $\rho : \hat{\mathcal{O}} \to \mathcal{O}$ is an orbifold homeomorphism $\varphi : \hat{\mathcal{O}} \to \hat{\mathcal{O}}$ such that $\rho \circ \varphi = \rho$. Note that the set $\Gamma$ of deck transformations form a group under composition. We say $\rho$ is a *regular* orbifold covering map if the following holds: $\rho(\hat{x}_1) = \rho(\hat{x}_0)$ iff there exists $\varphi \in \Gamma$ such that $\varphi(\hat{x}_0) = \hat{x}_1$.

PROPOSITION 2.6. Let $\rho : \hat{\mathcal{O}} \to \mathcal{O}$ be a regular orbifold covering map. Then, for all $x \in S$, the degree $\deg(\rho, \cdot)$ is constant on the fiber $\rho^{-1}(x)$.

PROOF. Let $x \in S$ and $\hat{x}_0, \hat{x}_1 \in \rho^{-1}(x)$. Then there is some deck transformation $\varphi : \hat{\mathcal{O}} \to \hat{\mathcal{O}}$ such that $\varphi(\hat{x}_0) = \hat{x}_1$. We have $\hat{\nu}(\hat{x}_1) = \hat{\nu}(\varphi(\hat{x}_0)) = \hat{\nu}(\hat{x}_0)$ because deck transformation are orbifold homeomorphisms, hence

$$\hat{\nu}(\hat{x}_1) \deg(\rho, \hat{x}_1) = \nu(x) = \hat{\nu}(\hat{x}_0) \deg(\rho, \hat{x}_0) = \hat{\nu}(\hat{x}_1) \deg(\rho, \hat{x}_0).$$

It follows $\deg(\rho, \hat{x}_1) = \deg(\rho, \hat{x}_0)$. □

Now we generalise the notion of Euler characteristic from that of surfaces. Recall that the Euler characteristic of a genus $g \geq 1$ surface with $p \geq 0$ punctures is $2-2g-p$. The *Euler characteristic* of an orbifold $\mathcal{O} = (S, \nu)$ with ramification function of finite support is defined as

$$\chi(\mathcal{O}) = \chi(S) - \sum_{x \in \operatorname{supp}(\nu)} \left(1 - \frac{1}{\nu(x)}\right). \tag{2.3}$$

Note that $\chi(\mathcal{O})$ is a rational number. In this work, we focus exclusively on orbifolds with non-positive Euler characteristics. We classify such an orbifold $\mathcal{O}$ as *parabolic* if $\chi(\mathcal{O}) = 0$ and *hyperbolic* if $\chi(\mathcal{O}) < 0$.

LEMMA 2.7. Suppose $\rho : \hat{\mathcal{O}} \to \mathcal{O}$ is a finite orbifold covering map. Then, $\chi(\hat{\mathcal{O}}) = \deg(\rho)\chi(\mathcal{O})$.

PROOF. We will follow the proof given in [**1**, Lemma 2.11]. Call $d = \deg(\rho)$. Using formulas (2.1) and (2.2), we get

$$\begin{aligned}
\chi(M) - \chi(\hat{\mathcal{O}}) &= \sum_{\hat{x} \in M} \left(1 - \frac{1}{\hat{\nu}(\hat{x})}\right) = \sum_{\hat{x} \in M} \left(1 - \frac{\deg(\rho, \hat{x})}{\nu(\rho(\hat{x}))}\right) \\
&= \sum_{\hat{x} \in M} (1 - \deg(\rho, \hat{x})) + \sum_{\hat{x} \in M} \deg(\rho, \hat{x}) \left(1 - \frac{1}{\nu(\rho(\hat{x}))}\right) \\
&= \chi(M) - d\chi(S) + \sum_{x \in S} \sum_{\hat{x} \in \rho^{-1}(x)} \deg(\rho, \hat{x}) \left(1 - \frac{1}{\nu(\rho(\hat{x}))}\right) \\
&= \chi(M) - d\chi(S) + \sum_{x \in S} \left(1 - \frac{1}{\nu(x)}\right) \sum_{\hat{x} \in \rho^{-1}(x)} \deg(\rho, \hat{x}) \\
&= \chi(M) - d\chi(S) + \Big(\chi(S) - \chi(\mathcal{O})\Big) d \\
&= \chi(M) - d\chi(\mathcal{O})
\end{aligned}$$

and the statement follows. □

## 2.3. Universal branched covering map

A *universal branched covering map* for an orbifold $\mathcal{O} = (S, \nu)$ is a branched covering map $p : X \to S$ where $X$ is a simply connected surface and

$$\deg(p, \tilde{x}) = \nu(p(\tilde{x})), \text{ for all } \tilde{x} \in X.$$

In other words, $\rho : X \to \mathcal{O}$ is an orbifold covering map from a simply connected surface. We call the space $X$ a *universal branched cover* for $\mathcal{O}$. In particular, an orbifold admitting a universal branched cover is good.

Observe also that any good orbifold $\mathcal{O}$ admits a universal branched cover. Indeed, suppose $\rho : Y \to \mathcal{O}$ is an orbifold covering map, where $Y$ is a surface. Let $X$ be the universal cover of $Y$, and let $\hat{p} : X \to Y$ be a covering map. Then, $p := \rho \circ \hat{p}$ is a universal branched covering map for $\mathcal{O}$.

As a simple example of universal branched cover, consider an orbifold $\mathcal{O} = (D, \nu)$, where $D$ is an open disk with only one singular point $x_0 \in D$ of weight $\nu(x_0) = n \geq 2$.

Recall the fundamental group of $D_\star := D\backslash\{x_0\}$ can be identified with $\mathbb{Z}$. Associated to the subgroup $n\mathbb{Z} < \mathbb{Z}$, there is a covering map $p : D_\star \to D_\star$ with $\deg(p) = [\mathbb{Z} : n\mathbb{Z}] = n$. Moreover, this map extends to a branched covering map $p : D \to D$, with $\deg(p, x_0) = n$ and $\deg(p, z) = 1$ for all $z \in D_\star$. Then, $p : D \to D$ is a universal branched covering map for $\mathcal{O}$.

In the following theorem, we extend the approach used in the previous example to obtain a universal branched covering map for more general cases.

THEOREM 2.8. Let $D$ be an open disk. Then, any orbifold $\mathcal{O} = (D, \nu)$ is good.

PROOF. Let $\{y_i\}_{i\in I}$ be the points in $\mathrm{supp}(\nu)$. Since it is a discrete set in $D$, we may take balls $B_i = B(y_i, \varepsilon_i)$ of disjoint closure. Let $N$ be the disk $D$ minus those balls. For each $i \in I$, let $\sigma_i : [0, 1] \to N$ be a simple closed curve parametrizing $\partial B_i$.

Fix a basepoint $b_0 \in N$ and let $\delta_i : [0, 1] \to N$ be a path starting at $b_0$ and ending at $\sigma_i(0)$. Then, each loop $\gamma_i := \delta_i * \sigma_i * \delta_i^{-1}$ has an associated element in the fundamental group $\pi_1(N, b_0)$. Moreover, the homotopy classes of these loops generate the group.

Let $F < \pi_1(N, b_0)$ be the normal subgroup generated by the homotopy classes of all the curves $\gamma_i^{\nu_i}$, where $\nu_i := \nu(y_i)$. Associated to this subgroup, there is a covering map $\rho : M \to N$. More specifically, there is some $\hat{b}_0 \in \rho^{-1}(b_0)$ such that $\rho_*\pi_1(M, \hat{b}_0) = F$.

Since $\rho$ is a local homeomorphism, $M$ is a surface with boundary and $\partial M = \rho^{-1}(\partial N)$. Since $\rho$ is a regular covering map, and since and there is some loop in $M$ lifting $\gamma_i^{\nu_i}$, it follows every $\rho$-lift of $\gamma_i^{\nu_i}$ is a loop. We claim that this also implies that any path $\hat{\sigma} : [0, 1] \to M$ lifting $\sigma_i^{\nu_i}$ is a loop. Indeed, let $\hat{\delta} : [0, 1] \to M$ be the lift of $\delta_i$ ending at $\hat{\sigma}(0)$ and $\overline{\delta} : [0, 1] \to M$ be the one ending at $\sigma(1)$. Then, the path $\hat{\delta} * \hat{\sigma} * \overline{\delta}^{-1}$ lifts $\delta_i * \sigma_i^{\nu_i} * \delta_i^{-1} = \gamma_i^{\nu_i}$ and, consequently, it must be a loop. It follows $\hat{\delta}(0) = \overline{\delta}(0)$ and, by uniqueness of path lifts, this gives $\hat{\delta}(1) = \overline{\delta}(1)$. Thus, $\hat{\sigma}(0) = \hat{\sigma}(1)$.

This proves that every boundary component $\mathcal{C} \subset \partial M$ of $M$ can be parametrized by a simple loop $\hat{\sigma} : [0, 1] \to \mathcal{C}$. Let $i \in I$ be such that $\rho(\mathcal{C}) \subset \partial B_i$. We want to show $\rho \circ \hat{\sigma}$ is homotopic to $\sigma_i^{\nu_i}$. We know there exist some $k \geq 1$ such that $\rho \circ \hat{\sigma}$ is homotopic to $\sigma_i^k$. This is because the restriction of $\rho$ to $\mathcal{C}$ is a covering map from $\mathcal{C}$ onto $\partial B_i$. Note then that $\gamma_i^k$ belongs to $F$. Since $\pi_1(M, \hat{b}_0)$ is the free group generated by the homotopy clasess of the $\gamma_i$'s, it follows that $k \geq \nu_i$. Otherwise $\gamma_i^k \in F$ would give a relation in the group. What is more, it cannot be the case that $k > \nu_i$ as this would imply that $\gamma_i^{\nu_i}$ lifts to an open path. We conclude that $k = \nu_i$.

We have shown that any boundary component $\mathcal{C}_{ij} \subset \rho^{-1}(\partial B_i)$ is a circle that is mapped $\nu_i$-to-one onto $\partial B_i$ by $\rho$. Then, we can cap a closed disk $\overline{D_{ij}}$ by its boundary in $\mathcal{C}_{ij}$ and extend $\rho$ to it in such a way that $\rho$ maps the interior disk $D_{ij}$ onto $B_i$, and the restriction $\rho : D_{ij}\backslash\{\hat{y}_{ij}\} \to B_i\backslash\{y_i\}$ is a covering map of degree $\nu_i$ between annuli, for some $\hat{y}_{ij} \in D_{ij}$.

Let $Y$ be the surface obtained by capping disks at the boundary components of $M$, and $\rho : Y \to D$ be the extension defined above. Then, $\rho : Y \to D$ is a branched covering map and $\deg(\rho, \hat{y}) = \nu_i$ for all $\hat{y} \in \rho^{-1}(y_i)$ and $i \in I$. Then, $\rho : Y \to \mathcal{O}$ is an orbifold covering map. □

COROLLARY 2.9. Any orbifold $\mathcal{O} = (S, \nu)$ with $\chi(S) \leq 1$ is good.

Proof. Since $\chi(S) \leq 1$, the universal cover of $S$ is the open disk. Let $p : D \to S$ be a covering map. Then, $p$ induces an orbifold $\mathcal{O}' = (D, \nu')$ by defining $\nu'(\hat{x}) = \nu(p(\hat{x}))$. By Theorem 2.8, there is some orbifold covering map $\rho : Y \to \mathcal{O}'$, where $Y$ is a surface. Then, $p \circ \rho : Y \to \mathcal{O}$ is an orbifold covering map. □

For the following theorem, refer to [**1**, Theorem A.26].

Theorem 2.10. Let $\mathcal{O} = (S^2, \nu)$ be a parabolic or hyperbolic orbifold over a sphere $S^2$. Then, $\mathcal{O}$ has a universal branched cover.

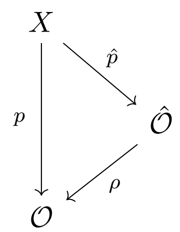
To conclude, we refer to [**7**, Theorem 2.5], emphasizing that the orbifold covering map mentioned in the statement is finite:

Theorem 2.11. Every good, compact orbifold is finitely covered by a surface.

**2.3.1. Uniqueness of the universal branched covering map.** Same as in the case of surfaces, one can show that the universal branched covering map is unique up to precomposition with an orientation preserving homeomorphism. See Corollary 2.13. Also, the universal branched covering map is universal in the following sense:

Theorem 2.12. Let $\rho : \hat{\mathcal{O}} \to \mathcal{O}$ be an orbifold covering map and suppose $p : X \to \mathcal{O}$ is a universal branched covering map for $\mathcal{O}$. Then, whenever $\tilde{x}_0 \in X$ and $\hat{x}_0 \in M$ verify $p(\tilde{x}_0) = \rho(\hat{x}_0) =: x_0$, there exists an orbifold covering map $\hat{p} : X \to \hat{\mathcal{O}}$ such that $\hat{p}(\tilde{x}_0) = \hat{x}_0$ and $p = \rho \circ \hat{p}$. Moreover, if $\nu(x_0) = 1$, then the map $\hat{p}$ is unique.

So the following diagram commutes:




Proof. This proof is an adaptation of the one given for [**1**, Theorem A.28].
Observe that whenever $\tilde{x} \in X$ and $\hat{x} \in M$ verify $p(\tilde{x}) = \rho(\hat{x})$, we have

$$\deg(\rho, \hat{x})\hat{\nu}(\hat{x}) = \nu(\rho(\hat{x})) = \nu(p(\tilde{x})) = \deg(p, \tilde{x}).$$

Then, $\deg(\rho, \hat{x}) \mid \deg(p, \tilde{x})$ and Lemma 2.3 gives the existence of a branched covering map $\hat{p} : X \to M$ such that $\hat{p}(\tilde{x}_0) = \hat{x}_0$ and $p = \rho \circ \hat{p}$. We only need to check $\hat{p}$ satisfies the orbifold covering condition. Given $\tilde{x} \in X$, call $\hat{x} = \hat{p}(\tilde{x})$ and $x = \rho(\hat{x})$. Then,

$$\deg(\hat{p}, \tilde{x}) \deg(\rho, \hat{x}) = \deg(p, \tilde{x}) = \nu(x) = \hat{\nu}(\hat{x}) \deg(\rho, \hat{x}) = \hat{\nu}(\hat{p}(\tilde{x})) \deg(\rho, \hat{x}),$$

and $\deg(\hat{p}, \tilde{x}) = \hat{\nu}(\hat{p}(\tilde{x}))$ as desired.
The uniqueness part comes from Lemma 2.4. □

Observe that the map $\hat{p} : X \to \hat{\mathcal{O}}$ given in Theorem 2.12 is a universal branched covering map for $\hat{\mathcal{O}}$. The following corollary can be found as [**1**, Corollary A.29].

Corollary 2.13. (Uniqueness of the universal branched covering map). Suppose $p : X \to \mathcal{O}$ and $p' : X' \to \mathcal{O}$ are universal branched covering maps. Then, whenever $p(\tilde{x}_0) = p'(\tilde{x}'_0) =: x_0$ there exists an orientation preserving homeomorphism $h : X \to X'$ such that $h(\tilde{x}_0) = \tilde{x}'_0$ and $p = p' \circ h$. Moreover, $h$ is unique if $\nu(x_0) = 1$.

Proof. By Theorem 2.12 there is a covering map $h : X \to X'$ such that $h(\tilde{x}_0) = \tilde{x}'_0$ and $p = p' \circ h$. Choose $x_1 \in S$ with $\nu(x_1) = 1$. Then the fiber $p^{-1}(x_1)$ has no critical points. Take $\tilde{x}_1 \in p^{-1}(x_1)$ and call $\tilde{x}'_1 := h(\tilde{x}_1)$. Note that $p'(\tilde{x}'_1) = p' \circ h(\tilde{x}_1) = p(\tilde{x}_1)$. Again by Theorem 2.12 there is a covering map $g : X' \to X$ such that $g(\tilde{x}'_1) = \tilde{x}_1$ and $p' = p \circ g$.

Since $p \circ g \circ h = p$ and $g \circ h(\tilde{x}_1) = g(\tilde{x}'_1) = \tilde{x}_1$ where $\nu(p(\tilde{x}_1)) = \nu(x_1) = 1$, uniqueness of Theorem 2.12 gives $g \circ h = \text{id}$. On the other hand, we also have $p' \circ h \circ g = p'$ and $h \circ g(\tilde{x}'_1) = h(\tilde{x}_1) = \tilde{x}'_1$ where $\nu(p'(\tilde{x}'_1)) = \nu(p(\tilde{x}_1)) = \nu(x_1) = 1$. Again, uniqueness of lifts implies $h \circ g = \text{id}$.

From $g \circ h = h \circ g = \text{id}$ one concludes $h$ is a homeomorphism with inverse $g$. Because $h$ is a covering map, it is necessarily orientation-preserving. □

Corollary 2.14. Let $\mathcal{O}$ be an orbifold and let $p : X \to \mathcal{O}$ be a universal branched covering map. Then, $p$ is a regular orbifold covering map.

Proof. If $p(\tilde{x}_1) = p(\tilde{x}_0)$, then by Corollary 2.13 there exists an orientation preserving homeomorphism $\varphi : X \to X$ such that $\varphi(\tilde{x}_0) = \tilde{x}_1$ and $p = p \circ \varphi$. In other words, there exists a deck transformation $\varphi : X \to X$ of $p$ such that $\varphi(\tilde{x}_0) = \tilde{x}_1$.

Conversely, if there exists a deck transformation $\varphi : X \to X$ of $p$ such that $\varphi(\tilde{x}_0) = \tilde{x}_1$, then $p(\tilde{x}_1) = (p \circ \varphi)(\tilde{x}_0) = p(\tilde{x}_0)$. □

**2.3.2. The fundamental group of an orbifold.** The group of deck transformations of a universal branched covering map $p : X \to \mathcal{O}$ will be denoted by $\pi_1(\mathcal{O}, p)$. As stated in the lemma below, up to isomorphism, this group does not depend on the map $p$, but only on the orbifold $\mathcal{O}$. We call this group the *fundamental group* of $\mathcal{O}$, and denote it by $\pi_1(\mathcal{O})$.

Lemma 2.15. Suppose $p : X \to \mathcal{O}$ and $p' : X' \to \mathcal{O}$ are universal branched covering maps. Then, the groups $\pi_1(\mathcal{O}, p)$ and $\pi_1(\mathcal{O}, p')$ are isomorphic.

Proof. Denote $\Gamma = \pi_1(\mathcal{O}, p)$ and $\Gamma' = \pi_1(\mathcal{O}, p')$. As in Corollary 2.13, let $h : X \to X'$ be a homeomorphism such that $p = p' \circ h$. For $\varphi \in \Gamma$, we define $\Phi(\varphi) := h \circ \varphi \circ h^{-1}$. Note that $\Phi(\varphi)$ is a deck transformation for $p'$:

$$p' \circ \Phi(\varphi) = p' \circ h \circ \varphi \circ h^{-1} = p \circ \varphi \circ h^{-1} = p \circ h^{-1} = p'.$$

It is straightforward to check that $\Phi : \Gamma \to \Gamma'$ is a group isomorphism. □

## 2.4. Group actions on surfaces

Let $X$ be a surface and $\Gamma$ be a group of orientation-preserving homeomorphisms acting on $X$. Then, $\Gamma$ induces an equivalence relation in $X$ given by $\hat{x}_0 \sim \hat{x}_1$ iff there exists $\varphi \in \Gamma$ such that $\varphi(\hat{x}_0) = \hat{x}_1$. We denote the quotient space by $X/\Gamma$, and equip it with the quotient topology. Consider $\pi : X \to X/\Gamma$ the quotient map.

A map $p : X \to S$ is said to be *induced* by $\Gamma$ if the following holds: $p(\hat{x}_1) = p(\hat{x}_0)$ iff there exists $\varphi \in \Gamma$ such that $\varphi(\hat{x}_0) = \hat{x}_1$. In particular, $p \circ \varphi = p$ for all $\varphi \in \Gamma$.

Observe that Corollary 2.14 states that a universal branched covering map is induced by its group of deck transformations. The following result can be found stated more generally as [**1**, Corollary A.23].

LEMMA 2.16. Let $\Gamma$ be a group acting on a surface $X$ and $\pi : X \to X/\Gamma$ be the associated quotient map. Suppose $p : X \to S$ is a branched covering map induced by $\Gamma$. Then, there exists a unique homeomorphism $\theta : X/\Gamma \to S$ such that $p = \theta \circ \pi$.

Now suppose $p : X \to S$ is a branched covering map induced by a group $\Gamma$. We will equip $S$ with an orbifold structure as follows. Let $x \in S$ and $\hat{x}_0, \hat{x}_1 \in p^{-1}(x)$. Since there exists $\varphi \in \Gamma$ such that $\varphi(\hat{x}_0) = \hat{x}_1$, we see that

$$\deg(p, \tilde{x}_1) = \deg(p, \varphi(\hat{x}_0)) = \deg(p \circ \varphi, \hat{x}_0) = \deg(p, \tilde{x}_0).$$

This means the local degree $\deg(p, \cdot)$ is constant in each fiber $p^{-1}(x)$. One can then define an orbifold $\mathcal{O} = (S, \nu)$ with ramification function given by

$$\nu(x) := \deg(p, \hat{x}), \text{ for } \hat{x} \in p^{-1}(x).$$

In this way, the map $p : X \to \mathcal{O}$ is a regular orbifold covering map. Moreover, if $X$ is simply connected, then $p$ is a universal branched covering map for $\mathcal{O}$, and $\Gamma$ is the fundamental group of $\mathcal{O}$.

In particular, if $\Gamma$ is a group acting on a simply connected surface $X$ in such a way that the quotient map $\pi : X \to X/\Gamma$ is a branched covering map, we will automatically endow $X/\Gamma$ with the orbifold structure described above.

The following corollary tells us that every good orbifold can be thought as the quotient of its universal branched cover by the action of the fundamental group.

COROLLARY 2.17. Let $\mathcal{O} = (S, \nu)$ be a good orbifold and $p : X \to S$ be a universal branched covering map for $\mathcal{O}$. Consider the group $\Gamma = \pi_1(\mathcal{O}, p)$ and the orbifold $\mathcal{O}' = (X/\Gamma, \nu')$ induced by the quotient map $\pi : X \to X/\Gamma$. Then, there exists a unique orbifold homeomorphism $\theta : \mathcal{O}' \to \mathcal{O}$ such that $p = \theta \circ \pi$.

PROOF. By Corollary 2.14, we know $p$ is induced by $\Gamma$. Then, Lemma 2.16 gives the existence of a homeomorphism $\Theta : X/\Gamma \to S$ such that $p = \Theta \circ \pi$. Let $x \in X/\Gamma$ and $\hat{x} \in \pi^{-1}(x)$. Since $p(\hat{x}) = (\Theta \circ \pi)(\hat{x}) = \Theta(x)$ and $\deg(\Theta, \cdot) = 1$, we get

$$\nu(\Theta(x)) = \deg(p, \hat{x}) = \deg(\Theta \circ \pi, \hat{x}) = \deg(\pi, \hat{x}) = \nu'(x).$$

This proves that $\Theta$ is an orbifold homeomorphism. □

**2.4.1. Subgroup actions on surfaces.** Again suppose $p : X \to S$ is a branched covering map induced by a group $\Gamma$, and let $\Lambda < \Gamma$ be a subgroup. Consider the quotient space $M := X/\Lambda$ and the quotient map $\hat{p} : X \to M$. Note that $\hat{p}(\tilde{x}_1) = \hat{p}(\tilde{x}_0)$ implies $p(\tilde{x}_1) = p(\tilde{x}_0)$ since $\Lambda \subset \Gamma$. We can then define a natural map $\rho : M \to S$ as

$$\rho(\hat{x}) = p(\tilde{x}), \text{ for } \tilde{x} \in \hat{p}^{-1}(\hat{x}).$$

Clearly one has $p = \rho \circ \hat{p}$.

As discussed before, we have associated orbifold structures $\mathcal{O} = (S, \nu)$ and $\hat{\mathcal{O}} = (M, \hat{\nu})$ given by the maps $p$ and $\hat{p}$. We claim that the natural map $\rho : \hat{\mathcal{O}} \to \mathcal{O}$ is an orbifold covering map. Indeed, given $\hat{x} \in M$, call $x := \rho(\hat{x})$ and choose any $\tilde{x} \in \hat{p}^{-1}(\hat{x})$. We see that

$$\nu(x) = \deg(p, \tilde{x}) = \deg(\rho \circ \hat{p}, \tilde{x}) = \deg(\hat{p}, \tilde{x}) \deg(\rho, \hat{x}) = \hat{\nu}(\hat{x}) \deg(\rho, \hat{x})$$

as claimed.

Proposition 2.18. The map $\rho : M \to S$ defined above has $\deg(\rho) = [\Gamma : \Lambda]$.

Proof. Fix $x_0 \in X/\Gamma$ such that $\nu(x_0) = 1$ and $\tilde{x}_0 \in p^{-1}(x_0)$. We want to prove that $|\rho^{-1}(x_0)| = [\Gamma : \Lambda]$, and we will show there exists a bijection $\Phi : \Lambda\backslash\Gamma \to \rho^{-1}(x_0)$, where $\Lambda\backslash\Gamma = \{\Lambda \circ \varphi : \ \varphi \in \Gamma\}$ is the set of right cosets. Since $\hat{p} \circ \psi = \hat{p}$ for all $\psi \in \Lambda$, it makes sense to define $\Phi$ as $\Phi(\Lambda \circ \varphi) = (\hat{p} \circ \varphi)(\tilde{x}_0)$.

To prove injectivity, suppose $\Phi(\Lambda \circ \varphi) = \Phi(\Lambda \circ \varphi')$. Then $\hat{p}(\varphi'(\tilde{x}_0)) = \hat{p}(\varphi(\tilde{x}_0))$ and there exists $\psi \in \Lambda$ such that $\psi(\varphi(\tilde{x}_0)) = \varphi'(\tilde{x}_0)$. Then $\varphi' \circ (\psi \circ \varphi)^{-1} \in \Gamma_{\tilde{x}_0} = \{\mathrm{id}\}$, which gives $\varphi' = \psi \circ \varphi$ and $\Lambda \circ \varphi' = \Lambda \circ \varphi$. To prove $\Phi$ is onto, let $\hat{x}_1 \in \rho^{-1}(x_0)$ and choose any $\tilde{x}_1 \in \hat{p}^{-1}(\hat{x}_1)$. Since $p(\tilde{x}_1) = p(\tilde{x}_0)$, there is some $\varphi \in \Gamma$ such that $\varphi(\tilde{x}_0) = \tilde{x}_1$ and it follows $\Phi(\Lambda \circ \varphi) = \hat{x}_1$. ☐

We now prove a general fact about groups that we will use later.

Lemma 2.19. Let $\Gamma$ be a group and $\Lambda$ a subgroup of finite index. If $H$ is another subgroup of $\Gamma$, then $[H : H \cap \Lambda] \leq [\Gamma : \Lambda]$.

Proof. Let $H/(H \cap \Lambda) = \{(H \cap \Lambda) \circ h : \ h \in H\}$ and $\Gamma/\Lambda = \{\Lambda \circ \varphi : \ \varphi \in \Gamma\}$. It suffices to define an injective map $\Phi : H/(H \cap \Lambda) \to \Gamma/\Lambda$.

Suppose $h, h' \in H$ are such that $(H \cap \Lambda) \circ h' = (H \cap \Lambda) \circ h$. Then, $h' = \psi \circ h$ for some $\psi \in H \cap \Lambda$. In particular, $h' = \psi \circ h$ for some $\psi \in \Lambda$, and so $\Lambda \circ h' = \Lambda \circ h$. It follows there is a well defined map $\Phi : H/(H \cap \Lambda) \to \Gamma/\Lambda$ such that $\Phi((H \cap \Lambda) \circ h) = \Lambda \circ h$.

To prove injectivity, assume $\Phi((H \cap \Lambda) \circ h') = \Phi((H \cap \Lambda) \circ h)$. That is, $\Lambda \circ h' = \Lambda \circ h$, and so $h' = \psi \circ h$ for some $\psi \in \Lambda$. Since $\psi = h' \circ h^{-1} \in H$, we get $\psi \in H \cap \Lambda$, and conclude that $(H \cap \Lambda) \circ h' = (H \cap \Lambda) \circ h$. ☐

## 2.5. Extended orbifolds

The definition of an orbifold $\mathcal{O} = (S, \nu)$ can be extended by allowing the ramification function to take values in $\mathbb{N} \cup \infty$. In this case, points with weight equal to $\infty$ should be regarded as punctures in $S$. We will use the notation

$$S_* := S \backslash \{x \in S : \ \nu(x) = \infty\}$$

for the punctured surface, and also

$$\mathrm{supp}_*(\nu) := \mathrm{supp}(\nu) \backslash \{x \in S : \ \nu(x) = \infty\}.$$

The Euler characteristic of such an orbifold is defined in the same way as in 2.3, considering $1/\infty = 0$. For example, an orbifold over the sphere $\mathcal{O} = (S^2, \nu)$ with only two points in $\mathrm{supp}(\nu)$ of weights $\infty$ is simply an annulus and $\chi(\mathcal{O}) = 0$.

The universal branched covering map now is a branched covering map $p : X \to S_*$ where $X$ is a simply connected surface and $\deg(p, \tilde{x}) = \nu(p(\tilde{x}))$ for all $\tilde{x} \in X$. In the same manner, an orbifold covering map $\rho : \hat{\mathcal{O}} \to \mathcal{O}$ will be an orbifold covering but in the punctured surfaces $\rho : M_* \to S_*$.

## 2.6. Examples

**2.6.1. Parabolic example.** Here we will see an example of a parabolic orbifold and its universal branched covering map. For this, we will define a group of isometries of the complex plane, whose action in $\mathbb{C}$ is properly discontinuous but not free, and consider the induced orbifold structure over the quotient space $\mathbb{C}/\Gamma$.

Let $\Lambda$ be the group of automorphisms of the complex plane $\mathbb{C} \simeq \mathbb{R}^2$ of the form $z \mapsto z + k$, with $k \in \mathbb{Z} \oplus i\mathbb{Z}$. Let $\Gamma$ be the one of automorphisms of the form $z \mapsto \pm z + k$, with $k \in \mathbb{Z} \oplus i\mathbb{Z}$. Then, $\Lambda$ is a subgroup of $\Gamma$.

A fundamental domain for the action of $\Lambda$ is the square $[-\frac{1}{2}, \frac{1}{2}]^2$. Also, note that the map $z \mapsto -z$ preserves the lattice $\frac{1}{2}\mathbb{Z} \oplus \frac{1}{2}i\mathbb{Z}$, and sends the rectangle $[-\frac{1}{2}, 0] \times [-\frac{1}{2}, \frac{1}{2}]$ onto the rectangle $R := [0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. Then, $R$ is a fundamental domain for the action of $\Gamma$. The segment $\{0\} \times [0, \frac{1}{2}]$ is identified with the segment $\{0\} \times [-\frac{1}{2}, 0]$, and the segment $\{\frac{1}{2}\} \times [0, \frac{1}{2}]$ is identified with the segment $\{\frac{1}{2}\} \times [-\frac{1}{2}, 0]$. Finally, the segment $[0, \frac{1}{2}] \times \{-\frac{1}{2}\}$ is identified with the segment $[0, \frac{1}{2}] \times \{\frac{1}{2}\}$.

The quotient space $S^2 = \mathbb{C}/\Gamma$ is then a topological a 2-sphere, and we shall study the orbifold structure $\mathcal{O} = (S^2, \nu)$ induced by the quotient map $p : \mathbb{C} \to S^2$. The set of critical points of this map is

$$\operatorname{crit}(p) = \tfrac{1}{2}\mathbb{Z} \oplus \tfrac{1}{2}i\mathbb{Z}.$$

What is more, the stabilizer of a point $z_0 \in \operatorname{crit}(p)$ is generated by the order-two automorphism given by $z \mapsto -(z - 2z_0)$. Thus, $\deg(p, z_0) = 2$ for all $z_0 \in \operatorname{crit}(p)$, and the singular points of $\mathcal{O}$ are all of weight 2. Also, the critical values of $p$ are $x_1 := p(0)$, $x_2 := p(\frac{1}{2})$, $x_3 := p(\frac{1}{2}i)$ and $x_4 := p(\frac{1}{2} + \frac{1}{2}i)$. Then, $\operatorname{supp}(\nu) = \{x_1, \ldots, x_4\}$, and we that $\mathcal{O}$ is the orbifold $(2, 2, 2, 2)$ over the sphere. See Figure 2.2.

Note that $T^2 = \mathbb{C}/\Lambda$ is topologically a torus. As $\Lambda$ is a subgroup of $\Gamma$, there is an induced natural map $\rho : T^2 \to S^2$, which is an orbifold covering map over $\mathcal{O}$. This map is a degree-two branched covering map with four critical points, namely, the classes modulo $\Lambda$ of the complex numbers $0$, $\frac{1}{2}$, $\frac{1}{2}i$ and $\frac{1}{2} + \frac{1}{2}i$.

One can compute the Euler characteristic to check that $\chi(\mathcal{O}) = 0$. Note that this is consistent with the statement of Lemma 2.7 as $\chi(T^2) = 0$.

**2.6.2. Hyperbolic example.** Consider the example of an orbifold $\mathcal{O}'$ over the sphere, with signature $(2, 2, 2, \infty)$. We may take advantage of the degree-two branched covering map $\rho : T^2 \to S^2$ given above.

Recall the point $x_4 \in S^2$ already defined, and let $\hat{x}_4 \in T^2$ be the (only) point in $\rho^{-1}(x_4)$. Then, the surfaces $T_* := T^2 \backslash \{\hat{x}_4\}$ and $S_* := S^2 \backslash \{x_4\}$ are respectively a punctured torus and a punctured sphere. Let $\hat{p} : X \to T_*$ be the universal covering map of $T_*$. Then, $p' := \rho \circ \hat{p}$ is a branched covering map with the same critical values as the map $\rho : T_* \to S_*$, namely, $x_1, x_2$ and $x_3$. By placing the weight 2 to each of this points, and the weight $\infty$ to $x_4$, we obtain the orbifold $\mathcal{O}'$, and $p' : X \to S_*$ is its universal branched covering map. We will return to this example in section 4.3.2.

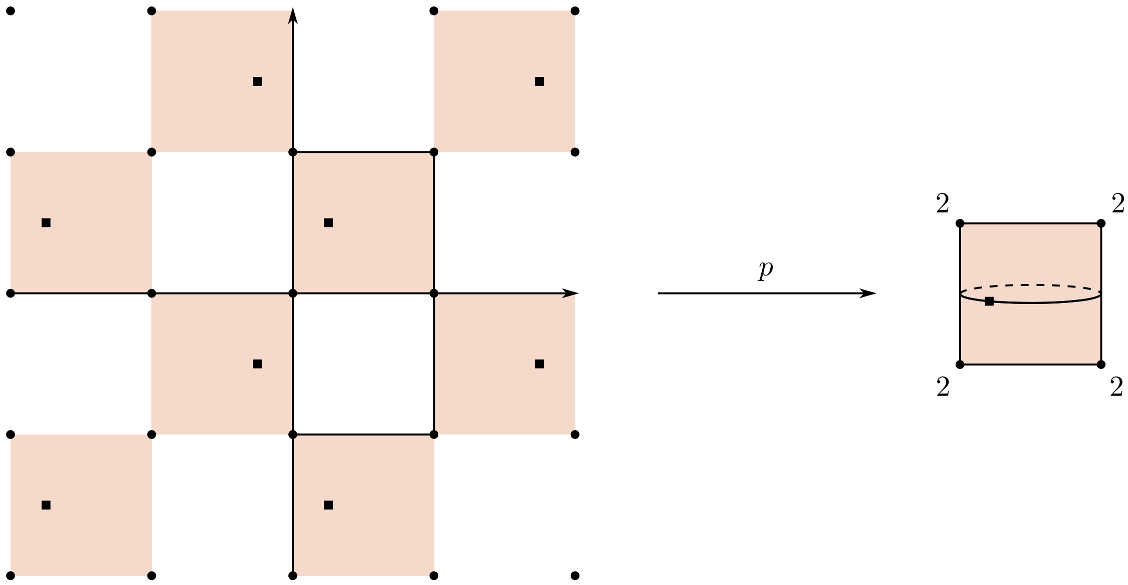



FIGURE 2.2.

An interesting part of this example is that we now know the orbifold $(2, 2, 2, \infty)$ over the sphere can be finitely-covered by a once punctured torus. Note that its Euler characteristic is

$$\chi(\mathcal{O}') = \chi(S^2) - 3\left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{\infty}\right) = 2 - \frac{3}{2} - 1 = -\frac{1}{2},$$

and this is consistent with the fact that $\chi(T_*) = -1$ and $\deg(\rho) = 2$.

CHAPTER 3

# Quotients of torus endomorphisms

The work presented in this thesis emerged from studying orbifolds associated with *Thurston maps*, as discussed in [**1**]. See the definition below. In that book, the authors introduce the concept of a *quotient of torus endomorphism (QOTE)*, which are specific instances of Thurston maps, and pose the question of whether these maps have parabolic orbifolds. After encountering this question, my advisor and I began to investigate it, ultimately providing a positive answer. Therefore, the ideas and results presented in this chapter are original and represent the mathematical core of this thesis. They can also be found in [**6**].

We will start by recalling what a Thurston map $f$ is and describing its associated orbifold $\mathcal{O}_f$. Then, we will also recall the notion of QOTEs and prove that they are indeed Thurston maps. See remarks after Proposition 3.2. After that, we will provide an example of these kinds of maps and establish some general properties of them. In Section 3.2, we state the key theorem of this chapter, that is Theorem 3.3, and demonstrate how it leads to the main contribution of this thesis:

THEOREM 3.1. Let $f$ be a QOTE. Then, $\mathcal{O}_f$ is parabolic.

It was first demonstrated in [**1**] that proving Theorem 3.3 suffices to establish Theorem 3.1; see [**1**, Lemma 3.13]. In this work, we provide a slightly different proof of this implicance.

Theorem 3.3 is actually interesting in its own right, and it also implies that the projection associated with a QOTE $f$ is an orbifold covering map of $\mathcal{O}_f$ (see Theorem 3.4), which is noteworthy as well.

We close this chapter with the proof of Theorem 3.3.

A *Thurston map* is a self-branched covering map $f : S^2 \to S^2$ in a sphere with degree $\deg(f) \geq 2$ and such that $\mathrm{post}(f) = \bigcup_{n\geq 1} f^n(\mathrm{crit}(f))$ is a finite set.

For a Thurston map $f : S^2 \to S^2$ one can associate an orbifold $\mathcal{O}_f = (S^2, \nu_f)$ over the sphere, with ramification function given by

$$\nu_f(x) := \mathrm{lcm}\left\{\deg(f^n, y) : \ y \in f^{-n}(x), \ n \geq 1\right\}.$$

Note that $\mathrm{supp}(\nu_f) = \mathrm{post}(f)$.

## 3.1. QOTEs

A *quotient of torus endomorphism (QOTE)* is a branched covering map $f : S^2 \to S^2$ such that there is a torus endomorphism $F : T^2 \to T^2$ of degree $\deg(F) \geq 2$ and a branched covering map $\rho : T^2 \to S^2$ such that $f \circ \rho = \rho \circ F$.

So the following diagram commutes:

$$\begin{array}{ccc} T^2 & \xrightarrow{F} & T^2 \\ \rho\downarrow & & \downarrow\rho \\ S^2 & \xrightarrow[f]{} & S^2 \end{array}$$

Here by *endomorphism* we mean a self-branched covering map. However, formula (2.2) implies that a torus endomorphism cannot have critical points as $\chi(T^2) = 0$. In consequence, a torus endomorphism is simply a self-covering map of the torus. The fact that $\deg(F, \cdot) = 1$ gives the following for all $\hat{y} \in T^2$:

$$\deg(\rho, F(\hat{y})) = \deg(\rho, \hat{y}) \deg(f, \rho(\hat{y})). \tag{3.1}$$

We say $F$ and $\rho$ are *associated maps* for $f$ if they satisfy the conditions given above. We shall also say that $F$ and $\rho$ are respectively an *associated endomorphism* and an *associated projection* for $f$.

For a given QOTE $f$, the associated maps $F$ and $\rho$ are not unique. For example, if $F' : T^2 \to T^2$ and $G : T^2 \to T^2$ are torus endomorphisms such that $F \circ G = G \circ F'$, then $F'$ and $\rho' := \rho \circ G$ are associated maps for $f$. One could choose $F' = G = F$, which would preserve the same torus endomorphism but with a different projection.

PROPOSITION 3.2. Let $f : S^2 \to S^2$ be a QOTE with associated projection $\rho$. Then, $\text{post}(f) = \rho(\text{crit}(\rho))$. In particular, $\text{post}(f)$ is a finite set.

PROOF. We will first prove $\text{post}(f) \subset \rho(\text{crit}(\rho))$. Let $x \in \text{post}(f)$. Then, there is some $n \geq 1$ and $y \in f^{-n}(x)$ such that $\deg(f^n, y) \geq 2$. Let $\hat{y} \in \rho^{-1}(y)$ and $\hat{x} := F^n(\hat{y})$. We have $\hat{x} \in \rho^{-1}(x)$ and

$$\deg(\rho, \hat{x}) = \deg(\rho, \hat{y}) \deg(f^n, y) \geq 2,$$

and so $x \in \rho(\text{crit}(\rho))$.

We now prove $\rho(\text{crit}(\rho)) \subset \text{post}(f)$. Let $x \in \rho(\text{crit}(\rho))$. Then there is some $\hat{x} \in \rho^{-1}(x)$ such that $\deg(\rho, \hat{x}) \geq 2$. Since $\text{crit}(\rho)$ is a finite set and $|F^{-n}(\hat{x})| = \deg(F)^n \geq 2^n$ for all $n \geq 1$, there is some $N \geq 1$ and $\hat{y} \in F^{-N}(\hat{x})$ such that $\hat{y} \notin \text{crit}(\rho)$. Call $y := \rho(\hat{y})$. Then, $f^N(y) = x$ and

$$\deg(f^N, y) = \deg(\rho, \hat{y}) \deg(f^N, y) = \deg(\rho, \hat{x}) \geq 2.$$

This proves $x \in \text{post}(f)$. □

This proposition, toghether with the fact that $\deg(f) = \deg(F) \geq 2$, shows every QOTE $f : S^2 \to S^2$ is a Thurston map.

**3.1.1. Example of a QOTE.** Consider the (parabolic) orbifold $(2, 2, 2, 2)$ over the sphere given in Section 2.6, with the associated degree-two branched covering map $\rho : T^2 \to S^2$. Recall that here $T^2 = \mathbb{C}/\Lambda$ and $S^2 = \mathbb{C}/\Gamma$, where $\Lambda$ is the group of homeomorphisms of the form $\varphi_k^+(z) = z + k$, with $k \in \mathbb{Z} \oplus i\mathbb{Z}$, and to $\Gamma$ are added the homeomorphisms of the form $\varphi_k^-(z) = -z + k$, with $k \in \mathbb{Z} \oplus i\mathbb{Z}$.

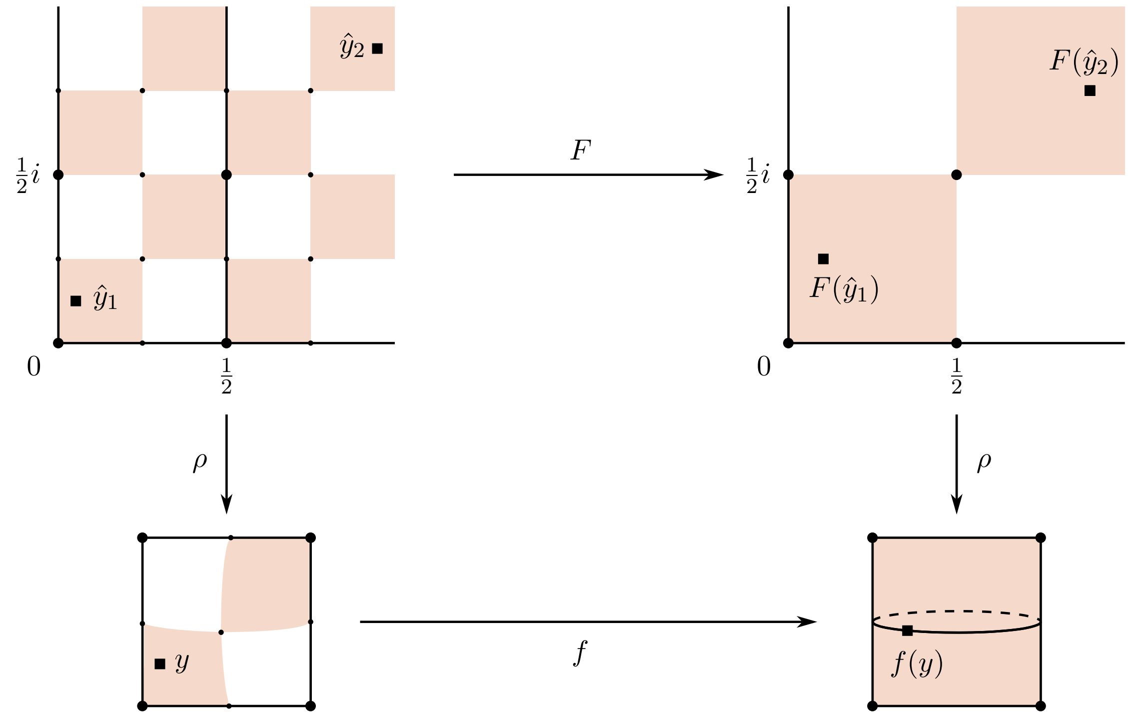



FIGURE 3.1. At the top, two squares $[0,1)^2$ represent the torus $T^2$. The illustration on the right square is obtained by applying the map $2z$ to the one on the left square. This represents a degree-four covering map on $T^2$. The vertical arrows represent a degree-two branched covering map onto the sphere. The self-covering map on $T^2$ descends to Thurston map.

If $A : \mathbb{C} \to \mathbb{C}$ is the homeomorphism given by $A(z) = 2z$, then we have

$$(A \circ \varphi_k^{\pm})(z) = 2(\pm z + k) = \pm 2z + 2k = (\varphi_{2k}^{\pm} \circ A)(z)$$

for all $z \in \mathbb{C}$ and $k \in \mathbb{Z} \oplus i\mathbb{Z}$. This shows that $A \circ \Lambda \subset \Lambda \circ A$ and $A \circ \Gamma \subset \Gamma \circ A$. As a consequence, $A$ induces a self-map of the torus $F : T^2 \to T^2$ and a self-map of the sphere $f : S^2 \to S^2$, satisfying $f \circ \rho = \rho \circ F$. See Figure 3.1.

The quotient map $\hat{p} : \mathbb{C} \to T^2$ associated to $\Lambda$ is a covering map and, since $F$ satisfies $F \circ \hat{p} = \hat{p} \circ A$, it follows $F$ is also a covering map. Besides, $\deg(F) = 4$ and so $\deg(f) = 4$ as well. This proves that $f$ is an example of a QOTE.

Note that this analysis applies to any linear map $A$ that is equivariant with respect to $\Lambda$ and $\Gamma$. Consequently, two QOTEs can share the same orbifold structure while exhibiting different dynamics on the sphere.

We now study the orbifold associated to this Thurston map. The map $f$ has six critical points that, seen in the fundamental domain $[0, \frac{1}{2}) \times [0, 1)$ of $\Gamma$'s action, are the complex numbers $\frac{1}{4}$, $\frac{1}{4}i$, $\frac{1}{4} + \frac{1}{4}i$, $\frac{1}{2} + \frac{1}{4}i$, $\frac{1}{4} + \frac{1}{2}i$ and $\frac{1}{4} + \frac{3}{4}i$. As stated in Proposition 3.2,

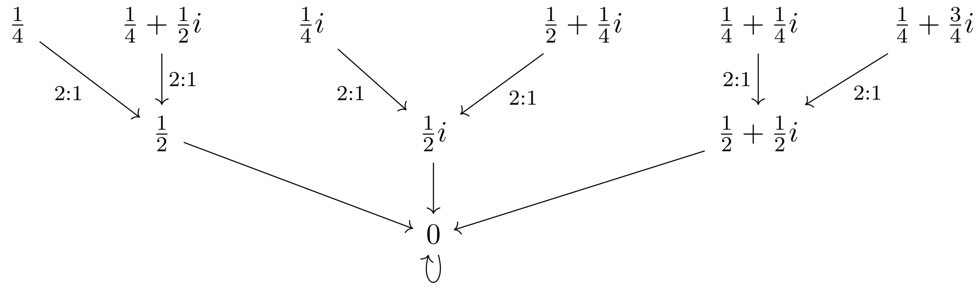



FIGURE 3.2.

the postcritical set of $f$ consists of the points $0$, $\frac{1}{2}$, $\frac{1}{2}i$ and $\frac{1}{2}+\frac{1}{2}i$ in $[0,\frac{1}{2})\times[0,1)$. The points in $\operatorname{crit}(f)\cup\operatorname{post}(f)$ are mapped by $f$ as shown in the diagram of Figure 3.2.

The orbifold $\mathcal{O}_f$ given in this example is then the orbifold $(2,2,2,2)$ over the sphere, and coincides with the orbifold we started with. This is no coincidence; we will show in general that the projection $\rho : T^2 \to S^2$ associated to a QOTE $f : S^2 \to S^2$ is an orbifold covering map of $\mathcal{O}_f$. See Theorem 3.4 below.

## 3.2. QOTEs have parabolic orbifolds

As already noted in the introduction to this chapter, the key theorem here is the following:

THEOREM 3.3. Let $f : S^2 \to S^2$ be a QOTE with associated projection $\rho$. Then, the degree $\deg(\rho,\cdot)$ is constant on $\rho^{-1}(x)$ for all $x \in S^2$.

Another way to say this is that there is a function $\alpha : S^2 \to \mathbb{N}$ such that $\alpha(x) = \deg(\rho,\hat{x})$ for all $\hat{x}\in\rho^{-1}(x)$. Before proving Theorem 3.3, we see some of its consequences.

THEOREM 3.4. Let $f : S^2 \to S^2$ be a QOTE and $\rho : T^2 \to S^2$ an associated projection for $f$. Then, $\rho : T^2 \to \mathcal{O}_f$ is an orbifold covering map. In other words,

$$\nu_f(x) = \deg(\rho,\hat{x})$$

for all $\hat{x}\in\rho^{-1}(x)$ and $x\in S^2$.

PROOF. By Theorem 3.3, there exists a function $\alpha : S^2 \to \mathbb{N}$ such that $\deg(\rho,\hat{x}) = \alpha(x)$ for all $\hat{x}\in\rho^{-1}(x)$. We want to show $\nu_f(x)=\alpha(x)$ for all $x\in S^2$. First note that this function satisfies

$$\begin{aligned}\alpha(y)\deg(f^n,y) &= \deg(\rho,\hat{y})\deg(f^n,y) = \deg(\rho,F^n(\hat{y})) = \alpha(\rho(F^n(\hat{y})))\\ &= \alpha((\rho\circ F^n)(\hat{y})) = \alpha((f^n\circ\rho)(\hat{y})) = \alpha(f^n(y))\end{aligned}$$

for all $y\in S^2$, where $\hat{y}\in\rho^{-1}(y)$.

Let $x\in S^2$. As pointed out before, $\alpha(y)\deg(f^n,y)=\alpha(x)$ for all $y\in f^{-n}(x)$. This gives $\deg(f^n,y)\mid\alpha(x)$ for all $y\in f^{-n}(x)$, and implies $\nu_f(x)\mid\alpha(x)$ by definition of $\nu_f$. It remains to prove that $\alpha(x)\mid\nu_f(x)$.

Choose any $\hat{x}\in\rho^{-1}(x)$. Since $\operatorname{crit}(\rho)$ is a finite set and $|F^{-n}(\hat{x})| = \deg(F)^n \geq 2^n$ for all $n\geq 1$, there is some $N\geq 1$ and $\hat{y}\in F^{-N}(\hat{x})$ such that $\hat{y}\notin\operatorname{crit}(\rho)$. Call $y := \rho(\hat{y})$.

Then, $f^N(y) = x$ and

$$\alpha(x) = \deg(\rho, \hat{x}) = \deg(\rho, \hat{y}) \deg(f^N, y) = \deg(f^N, y) \mid \nu_f(x)$$

as claimed. □

Given these results, the proof of Theorem 3.1 becomes straightforward:

PROOF. (Theorem 3.1) Let $\rho : T^2 \to S^2$ be an associated projection for $f$. By Theorem 3.4, we know that $\rho : T^2 \to \mathcal{O}_f$ is an orbifold covering map. By Lemma 2.7, we conclude that $\chi(\mathcal{O}_f) = \deg(\rho)\chi(T^2) = 0$. □

The following result can also be easily deduced using Theorem 3.4. However, a proof of this result was already provided in [**1**, Lemma 3.12].

PROPOSITION 3.5. Let $f$ be a QOTE. Then, $f$ has no periodic critical points.

PROOF. By Theorem 3.4, we know that $\nu_f(x) = \deg(\rho, \hat{x}) < \infty$ for all $x \in S^2$, where $\hat{x} \in \rho^{-1}(x)$. On the other hand, if $f$ had periodic critical points, there would exist some $x \in S^2$ such that $\nu_f(x) = \infty$, a contradiction. Therefore $f$ cannot have periodic critical points. □

Finally, another interesting fact about the associated projection of a QOTE is the one below:

COROLLARY 3.6. Let $f$ be a QOTE and $\rho$ an associated projection for $f$. Then, $\rho^{-1}(\rho(\mathrm{crit}(\rho))) = \mathrm{crit}(\rho)$.

PROOF. It is clear that $\rho^{-1}(\rho(\mathrm{crit}(\rho))) \supset \mathrm{crit}(\rho)$. On the other hand, suppose $\hat{x} \in \rho^{-1}(\rho(\mathrm{crit}(\rho)))$. Then, $x := \rho(\hat{x}) \in \rho(\mathrm{crit}(\rho))$ and there is some $\hat{x}_1 \in \mathrm{crit}(\rho)$ such that $\rho(\hat{x}_1) = x$. Since $\hat{x}, \hat{x}_1 \in \rho^{-1}(x)$, Theorem 3.3 gives $\deg(\rho, \hat{x}) = \deg(\rho, \hat{x}_1) \geq 2$. It follows $\hat{x} \in \mathrm{crit}(\rho)$. □

Let us look at an example of a finite branched covering map $\eta : T^2 \to \widehat{\mathbb{C}}$ not satisfying the condition of constant degree given in Theorem 3.3. Of course, the theorem itself tells us that $\eta$ will not be the associated projection of a QOTE.

Let $\rho : T^2 \to S^2$ be the branched covering map given in Figure 3.1. Consider any homeomorphism $h : S^2 \to \widehat{\mathbb{C}}$ taking $\rho(\mathrm{crit}(\rho))$ onto the set $\{0, 1, 2, 3\}$, and let $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the map $z \mapsto z^2$. Then, the composition $\eta := g \circ h \circ \rho : T^2 \to \widehat{\mathbb{C}}$ is a branched covering map.

We claim that $\deg(\eta, \cdot)$ is not constant on $\eta^{-1}(1)$. To see this, let $x^+, x^- \in S^2$ be the points such that $h(x^+) = 1$ and $h(x^-) = -1$. Note that $x^+ \in \rho(\mathrm{crit}(\rho))$ and $x^- \notin \rho(\mathrm{crit}(\rho))$. Then, there exist $\hat{x}^+ \in \rho^{-1}(x^+)$ and $\hat{x}^- \in \rho^{-1}(x^-)$ such that $\deg(\rho, \hat{x}^+) = 2$ and $\deg(\rho, \hat{x}^-) = 1$. Besides, $\eta(\hat{x}^+) = \eta(\hat{x}^-) = 1$ and

$$\begin{aligned}\deg(\eta, \hat{x}^+) &= \deg(\rho, \hat{x}^+) \deg(h, x^+) \deg(g, 1) = 2 \\ &> 1 = \deg(\rho, \hat{x}^-) \deg(h, x^-) \deg(g, -1) = \deg(\eta, \hat{x}^-).\end{aligned}$$

**3.2.1. Projection's local degree.** We now give a proof of Theorem 3.3. For this, we need to introduce some definitions and lemmas.

Suppose $f : S^2 \to S^2$ is a QOTE with associated endomorphism $F : T^2 \to T^2$ and associated projection $\rho : T^2 \to S^2$. We say $F$ is *$\rho$-injective* if it is injective when restricted to each fiber $\rho^{-1}(y)$ with $y \notin f^{-1}(\text{post}(f))$. In this case, we will also say $F$ and $\rho$ have the *injectivity property.*

Not every pair of associated maps for a QOTE $f$ have the injectivity property. Indeed, for a given pair $F$ and $\rho$, consider $F$ and $\rho' := \rho \circ F$ as associated maps for $f$. For any $y \notin f^{-1}(\text{post}(f))$ and $\hat{y} \in \rho^{-1}(y)$, there are two different points $\tilde{y}_0, \tilde{y}_1 \in F^{-1}(\hat{y})$ as $\deg(F) \geq 2$. But then, $\rho'(\tilde{y}_1) = (\rho \circ F)(\tilde{y}_1) = \rho(\hat{y}) = y = \rho(\hat{y}) = (\rho \circ F)(\tilde{y}_0) = \rho'(\tilde{y}_0)$. This shows that $\tilde{y}_0, \tilde{y}_1 \in (\rho')^{-1}(y)$ and $F(\tilde{y}_1) = \hat{y} = F(\tilde{y}_0)$.

On the other hand, the associated maps given in Figure 3.1 do satisfy the injectivity property. One way of seeing this is to note that Lemma 3.8 below implies that $F$ must be $\rho$-injective whenever $\deg(\rho)$ is a prime number. Indeed, the condition $\rho = \rho' \circ G$ gives $\deg(\rho) = \deg(\rho') \deg(G)$, and this cannot be as $\deg(G) \geq 2$.

Below we prove that every QOTE has a pair of associated maps with the injectivity property. See Theorem 3.9.

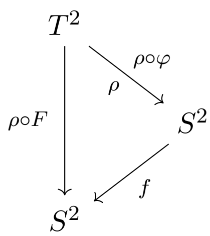
Lemma 3.7. Suppose $F$ is not $\rho$-injective. Then, the group

$$\Lambda(F, \rho) := \{\varphi \in \text{homeo}(T^2) : \ F \circ \varphi = F \text{ and } \rho \circ \varphi = \rho\}$$

is non-trivial.

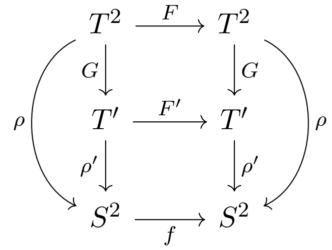
Proof. If $F$ is not $\rho$-injective, then there is some $y \notin f^{-1}(\text{post}(f))$ and two different points $\hat{y}_0, \hat{y}_1 \in \rho^{-1}(y)$ such that $F(\hat{y}_1) = F(\hat{y}_0)$. Because the fundamental group of $T^2$ is abelian, the covering $F : T^2 \to T^2$ is regular. Then, there is some $\varphi \in \text{homeo}(T^2)$ such that $F \circ \varphi = F$ and $F(\hat{y}_0) = \hat{y}_1$. Note that $\varphi \neq \text{id}$ as $\hat{y}_1 \neq \hat{y}_0$.

We claim that $\rho \circ \varphi = \rho$. To see this, apply Lemma 2.4 for the branched cover $f$. Then, it sufficies to show that $f \circ (\rho \circ \varphi) = f \circ \rho$ and that there is some $x_0 \in T^2$ such that $(\rho \circ \varphi)(x_0) = \rho(x_0) =: z_0$ with $f(z_0) \notin f(\text{crit}(f))$. First of all, we have $f \circ \rho \circ \varphi = \rho \circ F \circ \varphi = \rho \circ F = f \circ \rho$. Secondly, $(\rho \circ \varphi)(\hat{y}_0) = \rho(\hat{y}_1) = \rho(\hat{y}_0) = y$. Since $y \notin f^{-1}(\text{post}(f))$, we know that $f(y) \notin \text{post}(f) \supset f(\text{crit}(f))$.




This shows that $\varphi \in \Lambda(F, \rho)$ as desired. □

Lemma 3.8. Suppose $F$ is not $\rho$-injective. Then there is a torus $T'$, a pair of associated maps $F' : T' \to T'$ and $\rho' : T' \to S^2$ for $f$ and a covering map $G : T^2 \to T'$ with $\deg(G) \geq 2$ such that the following diagram commutes:




Proof. We know the group $\Lambda = \Lambda(F, \rho)$ defined in Lemma 3.7 is non-trivial. Since all elements in $\Lambda$ are deck transformations of the covering $F$, we get $\Lambda_{\tilde{x}} = \{\text{id}\}$ for all $\tilde{x} \in T^2$. It follows the quotient map $G : T^2 \to T^2/\Lambda$ is a covering map, and $\deg(G) \geq 2$. Moreover, since it is covered by a torus, $T' := T^2/\Lambda$ must be topologically a torus.

Observe that, whenever $\hat{x}_0, \hat{x}_1 \in T^2$ verify $G(\hat{x}_1) = G(\hat{x}_0) = \overline{x}$, there is some $\varphi \in \Lambda$ such that $\varphi(\hat{x}_0) = \hat{x}_1$ and so $F(\hat{x}_1) = (F \circ \varphi)(\hat{x}_0) = F(\hat{x}_0)$. In other words, $F$ is constant on each fiber $G^{-1}(\overline{x})$. Then, $F$ factors to a covering map $F' : T' \to T'$, meaning that it satisfies $F' \circ G = G \circ F$. Note also that $\deg(F') = \deg(F) \geq 2$.

In the same way, one can check that $\rho$ is constant on each fiber $G^{-1}(\overline{x})$. Then, there is a branched covering map $\rho' : T' \to S^2$ such that $\rho = \rho' \circ G$. Since $G$ is surjective and $f \circ \rho' \circ G = f \circ \rho = \rho \circ F = \rho' \circ G \circ F = \rho' \circ F' \circ G$, it follows $f \circ \rho' = \rho' \circ F'$. ☐

Theorem 3.9. Let $f : S^2 \to S^2$ be a QOTE. Then there exist associated maps for $f$ satisfying the injectivity property.

Proof. Let $F : T^2 \to T^2$ and $\rho : T^2 \to S^2$ be associated maps for $f$. If they do not satisfy the injectivity property, we can apply Lemma 3.8 to obtain the maps $F'$ and $\rho'$. Note that the conditions $\rho = \rho' \circ G$ and $\deg(G) \geq 2$ give $\deg(\rho') < \deg(\rho)$. If it is the case that $F'$ is not $\rho'$-injective, repeat the process and, in finitely many steps, obtain the maps with the desired property. ☐

Again, suppose $f : S^2 \to S^2$ is a QOTE with associated maps $F$ and $\rho$. Given $x \in S^2$ and $y \in f^{-1}(x)$, note that the inclusion $F(\rho^{-1}(y)) \subset \rho^{-1}(x)$ always holds. We say $F$ is *$\rho$-surjective* if $F(\rho^{-1}(y)) = \rho^{-1}(x)$ for all $x \in S^2$ and $y \in f^{-1}(x)$.

Lemma 3.10. *$F$ is $\rho$-injective if and only if $F$ is $\rho$-surjective.*

Proof. Assume $F$ is $\rho$-injective. Let $x \in S^2$ and $y \in f^{-1}(x)$. First suppose $x \in S^2 \backslash \text{post}(f)$. Since $f(\text{post}(f)) \subset \text{post}(f)$, we also have $y \in S^2 \backslash \text{post}(f)$. Using the fact that $\text{post}(f) = \rho(\text{crit}(\rho))$, we conclude both $y$ and $x$ are regular values of $\rho$. It follows $|\rho^{-1}(x)| = \deg(\rho) = |\rho^{-1}(y)|$. By hypotesis, $F : \rho^{-1}(y) \to \rho^{-1}(x)$ is injective, which implies it is also surjective. This proves the result for the case $x \in S^2 \backslash \text{post}(f)$.

Let now $x \in \text{post}(f)$. We will prove that for all $\hat{x} \in \rho^{-1}(x)$ there exists $\hat{y} \in \rho^{-1}(y)$ such that $F(\hat{y}) = \hat{x}$. Take a sequence $u^n \in S^2 \backslash \text{post}(f)$ with $u^n \to x$. Then there is a sequence $v^n \in f^{-1}(u^n)$ such that $v^n \to y$, and a sequence $\hat{u}^n \in \rho^{-1}(u^n)$ such that $\hat{u}^n \to \hat{x}$. Since $u^n \in S^2 \backslash \text{post}(f)$, we know $F(\rho^{-1}(v^n)) = \rho^{-1}(u^n)$. Then, there exists $\hat{v}^n \in \rho^{-1}(v^n)$ such that $F(\hat{v}^n) = \hat{u}^n$. Let $\hat{y}$ be a limit point of $\hat{v}^n$. Continuity of $\rho$ and $F$ gives $\rho(\hat{y}) = y$ and $F(\hat{y}) = \hat{x}$. This proves $F$ is $\rho$-surjective.

For the converse, assume $F$ is $\rho$-surjective. Let $y \in S^2 \backslash f^{-1}(\mathrm{post}(f))$ and call $x := f(y) \in S^2 \backslash \mathrm{post}(f)$. As seen in the first paragraph, we have $|\rho^{-1}(x)| = \deg(\rho) = |\rho^{-1}(y)|$. By hypotesis, $F : \rho^{-1}(y) \to \rho^{-1}(x)$ is surjective, which implies it is also injective. □

LEMMA 3.11. If $F$ is $\rho$-surjective, then the degree $\deg(\rho, \cdot)$ is constant on $\rho^{-1}(x)$ for all $x \in S^2$.

PROOF. Let $x \in S^2$ and $\hat{x}_0, \hat{x}_1 \in \rho^{-1}(x)$. Since $\rho^{-1}(\mathrm{post}(f))$ is a finite set and $|F^{-n}(\hat{x}_1)| = \deg(F)^n \geq 2^n$ for all $n \geq 1$, there is some $N \geq 1$ and $\hat{y}_1 \in F^{-N}(\hat{x}_1)$ such that $\hat{y}_1 \notin \rho^{-1}(\mathrm{post}(f))$. We then have $F^N(\hat{y}_1) = \hat{x}_1$ and $y := \rho(\hat{y}_1) \notin \mathrm{post}(f)$.

Observe that $f^N(y) = (f^N \circ \rho)(\hat{y}_1) = (\rho \circ F^N)(\hat{y}_1) = \rho(\hat{x}_1) = x$. Applying the $\rho$-surjectivity property of $F$ iteratively $N$ times, we arrive at:

$$F^N(\rho^{-1}(y)) = F^{N-1}(F(\rho^{-1}(y))) = F^{N-1}(\rho^{-1}(f(y))) = \cdots = \rho^{-1}(f^N(y)).$$

Then, there is some $\hat{y}_0 \in \rho^{-1}(y)$ such that $F^N(\hat{y}_0) = \hat{x}_0$. Note that $y \notin \mathrm{post}(f) = \rho(\mathrm{crit}(\rho))$ gives $\deg(\rho, \hat{y}_1) = \deg(\rho, \hat{y}_0) = 1$. Using formula (3.1) for $F^N$, we obtain

$$\begin{aligned}\deg(\rho, \hat{x}_1) &= \deg(\rho, F^N(\hat{y}_1)) = \deg(\rho, \hat{y}_1)\deg(f, y) \\ &= \deg(\rho, \hat{y}_0)\deg(f, y) = \deg(\rho, F^N(\hat{y}_0)) = \deg(\rho, \hat{x}_0)\end{aligned}$$

as desired. □

We are now ready to prove Theorem 3.3.

PROOF. (Theorem 3.3) Let $f : S^2 \to S^2$ be a QOTE with associated endomorphism $F : T^2 \to T^2$ and associated projection $\rho : T^2 \to S^2$. If $F$ is $\rho$-injective, then the statement follows from Lemma 3.10 and Lemma 3.11. Suppose on the contrary that $F$ is not $\rho$-injective. Recall that by Lemma 3.8 there exists an associated torus endomorphism $F'$ and an associated projection $\rho'$ for $f$ such that $F'$ is $\rho'$-injective. Moreover, there is a covering map $G$ such that $\rho = \rho' \circ G$.

Let $x \in S^2$ and $\hat{x}_0, \hat{x}_1 \in \rho^{-1}(x)$. Then, $\rho'(G(\hat{x}_1)) = \rho'(G(\hat{x}_0)) = x$, and it follows $\deg(\rho', G(\hat{x}_1)) = \deg(\rho', G(\hat{x}_0))$ by $\rho'$-injectivity. Since $\deg(G, \cdot) = 1$, we see that

$$\deg(\rho, \hat{x}_1) = \deg(\rho', G(\hat{x}_1)) = \deg(\rho', G(\hat{x}_0)) = \deg(\rho, \hat{x}_0)$$

as claimed. □

CHAPTER 4

# Lifting Thurston maps

We continue with the study of orbifolds associated with general Thurston maps. As established in Proposition 4.2, such an orbifold is either parabolic or hyperbolic.

In Chapter 3, we proved that every quotient of torus endomorphism (QOTE) has a parabolic orbifold (see Theorem 3.1). Conversely, Theorem 4.3 provides a reciprocal statement for Thurston maps without periodic critical points. This result was already demonstrated in [**4**] using algebraic topology tools, such as the lifting criterion given in Proposition 2.1. Here, we present an alternative proof, inspired by the approach used in [**1**, Theorem 3.1].

Thurston maps with parabolic orbifolds and periodic critical points are also explored in [**4**]. In this context, the authors show that the map can be lifted to an annulus covering map, similar to the treatment of QOTEs.

On the other hand, the scenario for Thurston maps $f$ with hyperbolic orbifolds is different. Following the definition of a QOTE, we aim to provide $f$ with analogous objects, such as the associated endomorphism and the associated projection.

The approach we present is the following. Just as Thurston maps with parabolic orbifolds can be lifted to covering maps of the torus or the annulus, we aim to show that Thurston maps with hyperbolic orbifolds can be lifted to covering maps of higher genus surfaces. Unlike the case of the torus or the annulus, we cannot expect to have a self-covering map of a higher genus surface. See formula (2.2).

What we will demonstrate is that for Thurston maps $f : S^2 \to S^2$ with hyperbolic orbifolds and no periodic critical points, there exist compact surfaces $M^0$ and $M^1$, a covering map $F_1 : M^0 \to M^1$, and finite branched covering maps $\rho_0 : M^0 \to S^2$ and $\rho_1 : M^1 \to S^2$ such that $f \circ \rho_1 = \rho_0 \circ F_1$.

The trade-off when seeking a covering map is that we require two associated projections rather than just one. Consequently, we do not have an associated projection for the iterates of $f$, unlike in the case of a QOTE, where $f^n \circ \rho = \rho \circ F^n$ holds. We will then provide a family of covering maps $F^{n+1} : M^{n+1} \to M^n$ and finite branched covering maps $\rho_n : M^n \to S^2$ such that $f \circ \rho_{n+1} = \rho_n \circ F_{n+1}$. See Theorem 4.5. This results in the commutative diagram shown in Figure 4.1.

In the case where $f : S^2 \to S^2$ has periodic critical points, $f$ will admit a similar diagram with the difference that the associated projections will be onto the punctured sphere $S^2_* = S^2 \backslash \{x \in S^2 :\ \nu_f(x) = \infty\}$.

To generalise the notion of a QOTE, we propose the following definition. We say a branched covering map $f : S^2 \to S^2$ is a *quotient of towers of coverings (QOTC)* if there

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & M^{n+1} & \xrightarrow{F_{n+1}} & M^n & \xrightarrow{F_n} & \cdots & \xrightarrow{F_2} & M^1 & \xrightarrow{F_1} & M^0 \\
 & & \downarrow{\scriptstyle \rho_{n+1}} & & \downarrow{\scriptstyle \rho_n} & & & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} \\
\cdots & \longrightarrow & S^2 & \xrightarrow[f]{} & S^2 & \xrightarrow[f]{} & \cdots & \xrightarrow[f]{} & S^2 & \xrightarrow[f]{} & S^2
\end{array}$$

FIGURE 4.1.

is a family of surfaces $M^n$, a family of covering maps $F_{n+1} : M^{n+1} \to M^n$, and a family of finite branched covering maps $\rho_n : M^n \to S^2$ such that $f \circ \rho_{n+1} = \rho_n \circ F_{n+1}$.

We can establish that $\mathrm{post}(f) \subset \rho_0(\mathrm{crit}(\rho_0))$ similarly to how it was demonstrated for QOTEs in Proposition 3.2. Therefore, $\mathrm{post}(f)$ is a finite set for every QOTC $f$.

A statement analogous to Theorem 3.3 is more challenging to address in this context. At the very least, some of the arguments presented in Chapter 3 cannot be applied here. Note that the degree of the projections is not asked to be constant in $n$, meaning the concepts of $\rho$-injectivity and $\rho$-surjectivity would not be equivalent here. Even if we assume this property and others, it remains unclear whether the methods and results from Chapter 3 can be adapted to this setting. We discuss this in Chapter 5.

## 4.1. The orbifold associated to a Thurston map

As already introduced, the orbifold associated to a Thurston map $f : S^2 \to S^2$ is the orbifold $\mathcal{O}_f = (S^2, \nu_f)$, where

$$\nu_f(x) := \mathrm{lcm}\left\{\deg(f^n, y) : \ y \in f^{-n}(x), \ n \geq 1\right\}.$$

In general, $f : S^2 \to S^2$ will not be an orbifold covering map putting the orbifold structure $\mathcal{O}_f$ on $S^2$. Actually, we will see the condition $\nu_f(y)\deg(f, y) = \nu_f(f(y))$ for all $y \in S^2$ holds iff $\mathcal{O}_f$ is parabolic. For every Thurston map, we have the following:

PROPOSITION 4.1. Let $f : S^2 \to S^2$ be a Thurston map. Then, $\nu_f(y)\deg(f, y)$ divides $\nu_f(f(y))$ for all $y \in S^2$.

PROOF. Let $y \in S^2$ and $x := f(y)$. For all $z \in f^{-n}(y)$ we have $f^{n+1}(z) = x$. Then, $\deg(f, y)\deg(f^n, z) = \deg(f^{n+1}, z) \mid \nu_f(x)$ by definition of $\nu_f(x)$. Also by definition of $\nu_f(x)$ it holds that $\deg(f, y) \mid \nu_f(x)$. As a consequence, we get

$$\deg(f^n, z) \mid \frac{\nu_f(x)}{\deg(f, y)}$$

for all $z \in f^{-n}(y)$. Since $\nu_f(y) := \mathrm{lcm}\left\{\deg(f^n, z) : \ z \in f^{-n}(y), \ n \geq 1\right\}$, we conclude that

$$\nu_f(y) \mid \frac{\nu_f(x)}{\deg(f, y)}$$

and the statement follows. □

The proposition below can be found as [**1**, Proposition 2.12]. The proof provided here follows essentially the same reasoning as the one in the cited reference.

PROPOSITION 4.2. Let $f : S^2 \to S^2$ be a Thurston map. Then $\chi(\mathcal{O}_f) \leq 0$, and $\chi(\mathcal{O}_f) = 0$ if and only if $f : \mathcal{O}_f \to \mathcal{O}_f$ is an orbifold covering map.

PROOF. We start by defining an orbifold $\mathcal{O}_f^1 = (S^2, \nu_f^1)$ as follows. The ramification function of is given by

$$\nu_f^1(y) = \frac{\nu_f(f(y))}{\deg(f,y)}$$

for all $y \in S^2$. Here $\nu_f^1(y) = \infty$ if $\nu_f(f(y)) = \infty$. Note that $\operatorname{supp}(\nu_f^1) \subset f^{-1}(\operatorname{supp}(\nu_f))$ is a finite set, and so $\nu_f^1$ is indeed a ramification function on $S^2$.

By Proposition 4.1, for all $y \in S^2$ there exists some $k = k(y) \geq 1$ such that $\nu_f(f(y)) = k\nu_f(y)\deg(f,y)$. Then, $\nu_f^1(y) = k\nu_f(y) \geq \nu_f(y)$ for all $y \in S^2$, and by definition (2.3) we know that $\chi(\mathcal{O}_f^1) \leq \chi(\mathcal{O}_f)$.

On the other hand, observe that $f : \mathcal{O}_f^1 \to \mathcal{O}_f$ is an orbifold covering map by definition of $\nu_f^1$. Applying Lemma 2.7, we see that $\chi(\mathcal{O}_f^1) = \deg(f)\chi(\mathcal{O}_f)$. Hence,

$$\Big(\deg(f) - 1\Big)\chi(\mathcal{O}_f) = \chi(\mathcal{O}_f^1) - \chi(\mathcal{O}_f) \leq 0.$$

From the condition $\deg(f) \geq 2$ it follows that $\chi(\mathcal{O}_f) \leq 0$.

If $\chi(\mathcal{O}_f) = 0$, then $\chi(\mathcal{O}_f^1) = 0$. If there were a point $y \in S^2$ such that $\nu_f^1(y) \neq \nu_f(y)$, then it would hold that $\nu_f^1(y) > \nu_f(y)$ and that $\chi(\mathcal{O}_f^1) < \chi(\mathcal{O}_f)$, a contradiction. Hence, $\chi(\mathcal{O}_f) = 0$ implies $\nu_f^1 = \nu_f$, which is the condition for $f : \mathcal{O}_f \to \mathcal{O}_f$ to be an orbifold covering map. □

## 4.2. Thurston maps with parabolic orbifolds

Suppose $f : S^2 \to S^2$ is a Thurston map with parabolic orbifold. It can be seen that $\mathcal{O}_f$ has one of the following signatures:

$$(\infty,\infty),\ (2,2,\infty),\ (2,2,2,2),\ (2,4,4),\ (3,3,3) \text{ or } (2,3,6).$$

Note that the first two occur if and only if $f$ has periodic critical points.

In this section we will see that Thurston maps having one of the last four signatures are QOTEs. See Theorem 4.3 below.

For what follows, we refer to [**1**, Section 3.1]. Recall that in Section 2.6 we described a group $\Gamma$ of isometries of the complex plane, whose action on $\mathbb{C}$ lead to the orbifold $(2,2,2,2)$ over the sphere $\mathbb{C}/\Gamma$. We will do the same description for the remaining parabolic orbifolds over the sphere. In each case, the groups we give are:

(2222) $z \mapsto \varphi(z) = \pm z + m + ni$, with $m, n \in \mathbb{Z}$;
(244) $z \mapsto \varphi(z) = i^k z + m + ni$, with $m, n \in \mathbb{Z}$, $k = 0,1,2,3$;
(333) $z \mapsto \varphi(z) = \omega^{2k} z + m + n\omega$, with $m, n \in \mathbb{Z}$, $k = 0,1,2$;
(236) $z \mapsto \varphi(z) = \omega^k z + m + n\omega$, with $m, n \in \mathbb{Z}$, $k = 0,1,2,3,4,5$.

Here $\omega = e^{i\pi/3}$.

If $\Gamma$ is any of these groups, then the quotient space $\mathbb{C}/\Gamma$ is topologically a sphere. Moreover, the quotient map is a universal branched covering map of the orbifold over $\mathbb{C}/\Gamma$ with the signature indicated in brackets. We will use this fact to prove the following:

THEOREM 4.3. Let $f : S^2 \to S^2$ be a Thurston map without periodic critical points. If $f$ has a parabolic orbifold, then $f$ is a QOTE.

Proof. We know $\mathcal{O}_f$ has one of the signatures $(2,2,2,2)$, $(2,4,4)$, $(3,3,3)$ or $(2,3,6)$. Consider the appropriate group $\Gamma$ of isometries (2222), (244), (333) or (236) described above. In the first two cases, let $\Lambda$ be the subgroup of $\Gamma$ consisting of maps of the form $z \mapsto z + m + ni$, with $m, n \in \mathbb{Z}$. In the last two cases, consider the maps of the form $z \mapsto z + m + n\omega$, with $m, n \in \mathbb{Z}$. Call $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ and $\hat{\pi} : \mathbb{C} \to \mathbb{C}/\Lambda$ the quotient maps.

Up to conjugating $f$ by a homeomorphism $\mathbb{C}/\Gamma \to S^2$, we may assume we have $f : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma$. As stated in Proposition 4.2, $f$ is an orbifold covering map, and so $f \circ \pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a universal branched covering map for $\mathcal{O}_f$. Hence, Corollary 2.13 gives a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ such that $f \circ \pi = \pi \circ h$.

The fact that $h$ satisfies $f \circ \pi = \pi \circ h$ implies that $h \circ \Gamma \circ h^{-1} \subset \Gamma$. In particular, if $\psi \in \Lambda$, then $h \circ \psi \circ h^{-1} \in \Gamma$. Since $\psi$ has no fixed points, nor does $h \circ \psi \circ h^{-1}$. We claim that this implies $h \circ \psi \circ h^{-1} \in \Lambda$. To prove this, it suffices to show that any element in $\Gamma \backslash \Lambda$ has a fixed point. This is true since a map of the form $z \mapsto az + b$ with $a \neq 1$ fixes the point $\frac{b}{1-a}$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{h} & \mathbb{C} \\ {\scriptstyle \hat{\pi}}\downarrow & & \downarrow{\scriptstyle \hat{\pi}} \\ \mathbb{C}/\Lambda & \xrightarrow{F} & \mathbb{C}/\Lambda \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ \mathbb{C}/\Gamma & \xrightarrow[f]{} & \mathbb{C}/\Gamma \end{array}$$

We conclude that $h \circ \Lambda \circ h^{-1} \subset \Lambda$, and so $h$ induces a map $F : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ such that $F \circ \hat{\pi} = \hat{\pi} \circ h$. Since $\hat{\pi}$ is a covering map and $h$ an homeomorphism, it follows $F$ is a covering map. Consider the natural map $\rho : \mathbb{C}/\Lambda \to \mathbb{C}/\Gamma$, that is, the one satisfying $\pi = \rho \circ \hat{\pi}$. The fact that $\hat{\pi} : \mathbb{C} \to \mathbb{C}/\Lambda$ is onto, together with the fact that $f \circ \rho \circ \hat{\pi} = f \circ \pi = \pi \circ h = \rho \circ \hat{\pi} \circ h = \rho \circ F \circ \hat{\pi}$ gives $f \circ \rho = F \circ \rho$. □

## 4.3. Thurston maps with hyperbolic orbifolds

Suppose $f : S^2 \to S^2$ is a Thurston map and call $\mathcal{O}_f^0 := \mathcal{O}_f$. For $n \geq 0$, we define an orbifold $\mathcal{O}_f^{n+1} = (S^2, \nu_f^{n+1})$ inductively as

$$\nu_f^{n+1}(y) = \frac{\nu_f^n(f(y))}{\deg(f, y)},$$

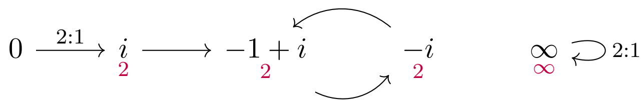
for all $y \in S^2$. We automatically have that $f : \mathcal{O}_f^{n+1} \to \mathcal{O}_f^n$ is an orbifold covering map for all $n \geq 0$. Also, $\mathcal{O}_f^n = \mathcal{O}_{f^n}^1$ for all $n \geq 1$. We will omit the subscript in $\mathcal{O}_f^n$ when there is no confusion.

$$\cdots \xrightarrow{f} \mathcal{O}^{n+1} \xrightarrow{f} \mathcal{O}^n \xrightarrow{f} \cdots \xrightarrow{f} \mathcal{O}^1 \xrightarrow{f} \mathcal{O}^0.$$

From Proposition 4.2 it follows that $f$ has a parabolic orbifold if and only if $\mathcal{O}^n = \mathcal{O}^0$ for all $n \geq 0$. On the other hand, when $\mathcal{O}_f$ is hyperbolic, we have $\chi(\mathcal{O}^{n+1}) = \deg(f)\chi(\mathcal{O}^n) = \deg(f)^n \chi(\mathcal{O}^0)$, which implies all orbifolds in the sequence are different.

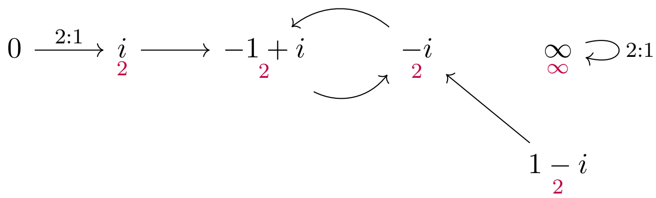
Let us illustrates this with an example. Consider the polynomial given by the expression $f(z) = z^2 + i$, extended as a map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ in the Riemann sphere. We will examine the associated sequence of orbifolds $\mathcal{O}^n = (\widehat{\mathbb{C}}, \nu^n)$, which will be used later in section 4.3.2.

The only two critical points of $f$ are 0 and $\infty$, and the ramification portrait is




Call $x_1 = i$, $x_2 = -1+i$ and $x_3 = -i$. Hence, $\text{supp}(\nu^0) = \{x_1, x_2, x_3, \infty\}$ and the signature of $\mathcal{O}^0$ is $(2, 2, 2, \infty)$. Since $\infty$ is a fixed critical point, we shall think we are working on the once punctured sphere $\mathbb{C} = \widehat{\mathbb{C}} \backslash \{\infty\}$, and consider then $\text{supp}_*(\nu^n) = \text{supp}(\nu^n)\backslash\{\infty\}$.

Let $x_4 = 1 - i$. To obtain $\mathcal{O}^1$, observe that $f^{-1}(\text{supp}_*(\nu^0)) = \text{supp}_*(\nu^0) \cup \{0, x_4\}$. Since $\deg(f, 0) = 2$ and $\deg(f, x_4) = 1$, it follows that $\text{supp}(\nu_1) = \text{supp}(\nu^0) \cup \{x_4\}$ and $\nu_1(x_4) = 2$. This means $\mathcal{O}^1$ has signature $(2, 2, 2, 2, \infty)$. Again, we may summarize this information with one diagram:




Now, since $x_4$ is a regular value of $f$, there are two different points $x_5$ and $x_6$ with $f(x_5) = f(x_6) = x_4$. We have $\text{supp}_*(\nu^2) = \text{supp}_*(\nu^1) \cup \{x_5, x_6\}$, and $\nu^2(x_5) = \nu^2(x_6) = 2$ as well. Moreover, $x_5$ and $x_6$ are both regular values of $f$. Again, there are four different points $x_7, x_8, x_9$ and $x_{10}$ such that $f(x_7) = f(x_8) = x_5$ and $f(x_9) = f(x_{10}) = x_6$. We have $\text{supp}_*(\nu^3) = \text{supp}_*(\nu^2) \cup \{x_7, x_8, x_9, x_{10}\}$, with $\nu^3(x_7) = \cdots = \nu^3(x_{10}) = 2$ as well.

Inductively, for $n \geq 0$ there are $2^n = 2 \cdot 2^{n-1}$ new points in $\text{supp}_*(\nu^{n+1})$, in addition to those already in $\text{supp}_*(\nu^n)$. As a consequence,

$$|\text{supp}_*(\nu^{n+1})| = 3 + \sum_{i=0}^{n} 2^i = 3 + \frac{2^{n+1} - 1}{2 - 1} = 2 + 2^{n+1}$$

for all $n \geq 0$.

**4.3.1. Lifting Thurston maps to covering maps.** Recall the definition of QOTC provided at the beginning of this chapter. In this section, we prove that Thurston maps with hyperbolic orbifolds are also QOTCs. Furthermore, the maps $\rho_n$ involved can be chosen with additional properties. See Theorem 4.5 for details.

The key lemma is the following:

LEMMA 4.4. *Let $f : S^2 \to S^2$ be a Thurston map and $n \geq 0$. Given a surface $M^n$ and a finite orbifold covering map $\rho_n : M^n \to \mathcal{O}^n_f$, there exists a surface $M^{n+1}$ and orbifold covering maps $F_{n+1} : M^{n+1} \to M^n$ and $\rho_{n+1} : M^{n+1} \to \mathcal{O}^{n+1}_f$ such that $f \circ \rho_{n+1} = \rho_n \circ F_{n+1}$ and $2 \leq \deg(\rho_{n+1}) \leq \deg(\rho_n)$.*

Proof. We will do a proof based on that of Theorem 4.3. Without loss of generality, we may assume $n = 0$ to simplify notation.

Let $\hat{p}_0 : X \to M_0$ be a universal covering map for $M_0$. Then, $p_0 := \rho_0 \circ \hat{p}_0$ is a universal branched covering map for $\mathcal{O}_0$. Let $\Gamma_0$ be the group of deck transformations of $p_0$, and $\Lambda_0$ be the one of $\hat{p}_0$. Let $p_1 : X' \to S^2$ be any universal branched covering map for $\mathcal{O}_1$, and call $\Gamma_1$ its group of deck transformations. Since $f : \mathcal{O}_1 \to \mathcal{O}_0$ is an orbifold covering map, we have that $f \circ p_1$ is a universal branched covering map for $\mathcal{O}_0$. Corollary 2.13 gives a homeomorphism $h : X' \to X$ such that $f \circ p_1 = p_0 \circ h$.

The fact that $h$ satisfies $f \circ p_1 = p_0 \circ h$ implies that $h \circ \Gamma_1 \circ h^{-1} \subset \Gamma_0$. Consider the group morphism $\Phi : \Gamma_1 \to \Gamma_0$ given by $\Phi(\varphi) = h \circ \varphi \circ h^{-1}$. Hence, $\Lambda_1 := \Phi^{-1}(\Lambda_0)$ is a subgroup of $\Gamma_1$ and satisfies $h \circ \Lambda_1 \circ h^{-1} \subset \Lambda_0$ by definition. As any $\psi_1 \in \Lambda_1$ is of the form $\psi_1 = h^{-1} \circ \psi_0 \circ h$ for some $\psi_0 \in \Lambda_0$, and elements in $\Lambda_0$ have no fixed points, it follows that elements in $\Lambda_1$ have no fixed points. Then, $M_1 := X'/\Lambda_1$ is a surface and the quotient map $\hat{p}_1 : X' \to M_1$ is a covering map.

Since $h \circ \Lambda_1 \circ h^{-1} \subset \Lambda_0$, we have that $h$ induces a map $F_1 : M_1 \to M_0$ such that $F_1 \circ \hat{p}_1 = \hat{p}_0 \circ h$. As $\hat{p}_0$ and $\hat{p}_1$ are covering maps, and $h$ is an homeomorphism, it follows $F_1$ is a covering map. Consider the natural map $\rho_1 : M_1 \to S^2$, that is, the one satisfying $p_1 = \rho_1 \circ \hat{p}_1$. The fact that $\hat{p}_1 : X' \to M_1$ is onto, together with the fact that $f \circ \rho_1 \circ \hat{p}_1 = f \circ p_1 = p_0 \circ h = \rho_0 \circ \hat{p}_0 \circ h = \rho_0 \circ F_1 \circ \hat{p}_1$, gives $f \circ \rho_1 = \rho_0 \circ F_1$.

$$\begin{array}{ccc} X' & \xrightarrow{\ h\ } & X \\ {\scriptstyle \hat{p}_1}\downarrow & & \downarrow{\scriptstyle \hat{p}_0} \\ M_1 & \xrightarrow{\ F_1\ } & M_0 \\ {\scriptstyle \rho_1}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ S^2 & \xrightarrow[\ f\ ]{} & S^2 \end{array}$$

We shall now prove that $\deg(\rho_1) \le \deg(\rho_0)$. In other words, we shall prove that $[\Gamma_1 : \Lambda_1] \le [\Gamma_0 : \Lambda_0]$. As $\Phi : \Gamma_1 \to \Gamma_0$ is an injective morphism, we have $[\Gamma_1 : \Lambda_1] = [\Phi(\Gamma_1) : \Phi(\Lambda_1)] = [\Phi(\Gamma_1) : \Phi(\Gamma_1) \cap \Lambda_0]$. By combining this equality with Lemma 2.19 for $H = \Phi(\Gamma_1)$, we arrive to $[\Gamma_1 : \Lambda_1] \le [\Gamma_0 : \Lambda_0]$.

To finish, note that $\Gamma_1$ has elements fixing points. This implies $\Lambda_1 \subsetneq \Gamma_1$ and $\deg(\rho_1) = [\Gamma_1 : \Lambda_1] \ge 2$. □

Theorem 4.5. Let $f : S^2 \to S^2$ be a Thurston map. Then, $f$ is a QOTC and the associated projections $\rho_n : M^n \to \mathcal{O}_f^n$ can be chosen in such a way that $\rho_n : M^n \to \mathcal{O}_f^n$ is an orbifold covering map and $2 \le \deg(\rho_{n+1}) \le \deg(\rho_n)$, for all $n \ge 0$.

Proof. Theorem 2.11 gives the existence of an initial orbifold covering map $\rho_0 : M_0 \to \mathcal{O}_0$ and, applying Lemma 4.4, one obtains the desired maps inductively. □

Corollary 4.6. In the context of Theorem 4.5, there exists some $N \ge 0$ and $r \ge 2$ such that $\deg(\rho_n) = r$ for all $n \ge N$.

PROOF. The condition $2 \leq \deg(\rho_{n+1}) \leq \deg(\rho_n)$ given in Theorem 4.5 tells us that the integers sequence $\{\deg(\rho_n)\}_{n\geq 0}$ is decreasing and bounded. The statement follows. $\square$

**4.3.2. The example $z^2 + i$.** Recall the example of the polynomial $f(z) = z^2 + i$ seen at the beginning of this section. We studied the associated orbifolds $\mathcal{O}^n$, and showed the support $\operatorname{supp}_*(\nu^n)$ has exactly $2 + 2^{n+1}$ points of weight 2, for all $n \geq 0$. We will refer these points as $x_1, \dots, x_{2+2^n}$.

For each $n \geq 0$, we will give a geometric description of a finite orbifold covering map $\rho_n : M^n \to \mathcal{O}^n$, where $M^n$ is a surface. We will then show we can obtain a diagram like the one shown in Figure 4.1, for this specific projections.

For $n \geq 1$, consider a genus $g_n := 2^{n-1}$ surface in $\mathbb{R}^3$ with an axis of symmetry as shown in Figure 4.2. More specifically, there is a line intersecting the surface in $2g_n + 2 = 2^n + 2$ points, and a rotation of angle $\pi$ respect to this line that induces an order-two homeomorphism $\varphi_n$ in the genus $g_n$ surface. We will denote by $\hat{x}_1^n, \dots, \hat{x}_{2^n+2}^n$ the fixed points of $\varphi_n$.

Choose a point $\hat{y}_\infty^n$ that is not fixed by $\varphi_n$, and call $\overline{y}_\infty^n = \varphi_n(\hat{y}_\infty^n)$. The surface $M^n$ that we will consider is the one obtained by removing the points $\hat{y}_\infty^n$ and $\overline{y}_\infty^n$ from the genus $g_n$ surface. Now, the restriction $\varphi_n : M^n \to M^n$ induces a quotient space $M^n/\langle\varphi_n\rangle$ with the equivalence relation given by $\hat{x} \sim \varphi_n(\hat{x})$ for all $\hat{x} \in M^n$. This space is topologically a punctured sphere, and the quotient map is a branched covering map. Moreover, the critical points of this map are the fixed points of $\varphi_n$, and the local degree is two for all of them.

We choose any homeomorphism $M^n/\langle\varphi_n\rangle \to \widehat{\mathbb{C}}$ sending the class of each $\hat{x}_i^n$ to the point $x_i$ in $\operatorname{supp}_*(\nu^n)$ defined above. The composition of this homeomorphism with the quotient map will be denoted by $\rho_n : M^n \to \widehat{\mathbb{C}}$, and is the desired orbifold covering map.

On the other hand, to obtain $M^0$ we consider a genus $g_0 = 1$ surface and a homeomorphism $\varphi_0$ defined in the same way as before. However, this time we puncture the genus $g_0$ surface in one of the fixed points $\hat{x}_1^0, \dots, \hat{x}_4^0$ of $\varphi_0$ to obtain $M^0$. Say we puncture in $\hat{x}_4^0$. Again, the quotient space $M^0/\langle\varphi_0\rangle$ is topologically a punctured sphere and we get an orbifold covering map $\rho_0 : M^0 \to \widehat{\mathbb{C}}$ in the same manner.

As already commented, we want to see that there is a family of covering maps $F_{n+1} : M^{n+1} \to M^n$ such that $\rho_n \circ F_{n+1} = f \circ \rho_{n+1}$ for all $n \geq 0$. In other words, we want to see each map $f \circ \rho_{n+1}$ has a lift with respect to $\rho_n$.

For this, we will puncture the surfaces involved in order to obtain truly covering maps. This will let us use the lifting criterion given in Proposition 2.1, and lift those maps to the punctured surfaces. Then, Proposition 2.5 will give an extension of these lifts to the original ones.

For $n \geq 0$, consider

$$\widehat{\mathbb{C}}_\star^n := \widehat{\mathbb{C}} \backslash f^{-n}(\operatorname{supp}(\nu^0)) \text{ and } M_\star^n := M^n \backslash \rho_n^{-1}(f^{-n}(\operatorname{supp}(\nu^0))).$$

Hence, the restrictions $f : \widehat{\mathbb{C}}_\star^{n+1} \to \widehat{\mathbb{C}}_\star^n$ and $\rho_n : M_\star^n \to \widehat{\mathbb{C}}_\star^n$ are covering maps. Note that $f^{-n}(\operatorname{supp}(\nu^0)) = \operatorname{supp}(\nu^n) \cup \{0\}$ for all $n \geq 1$.

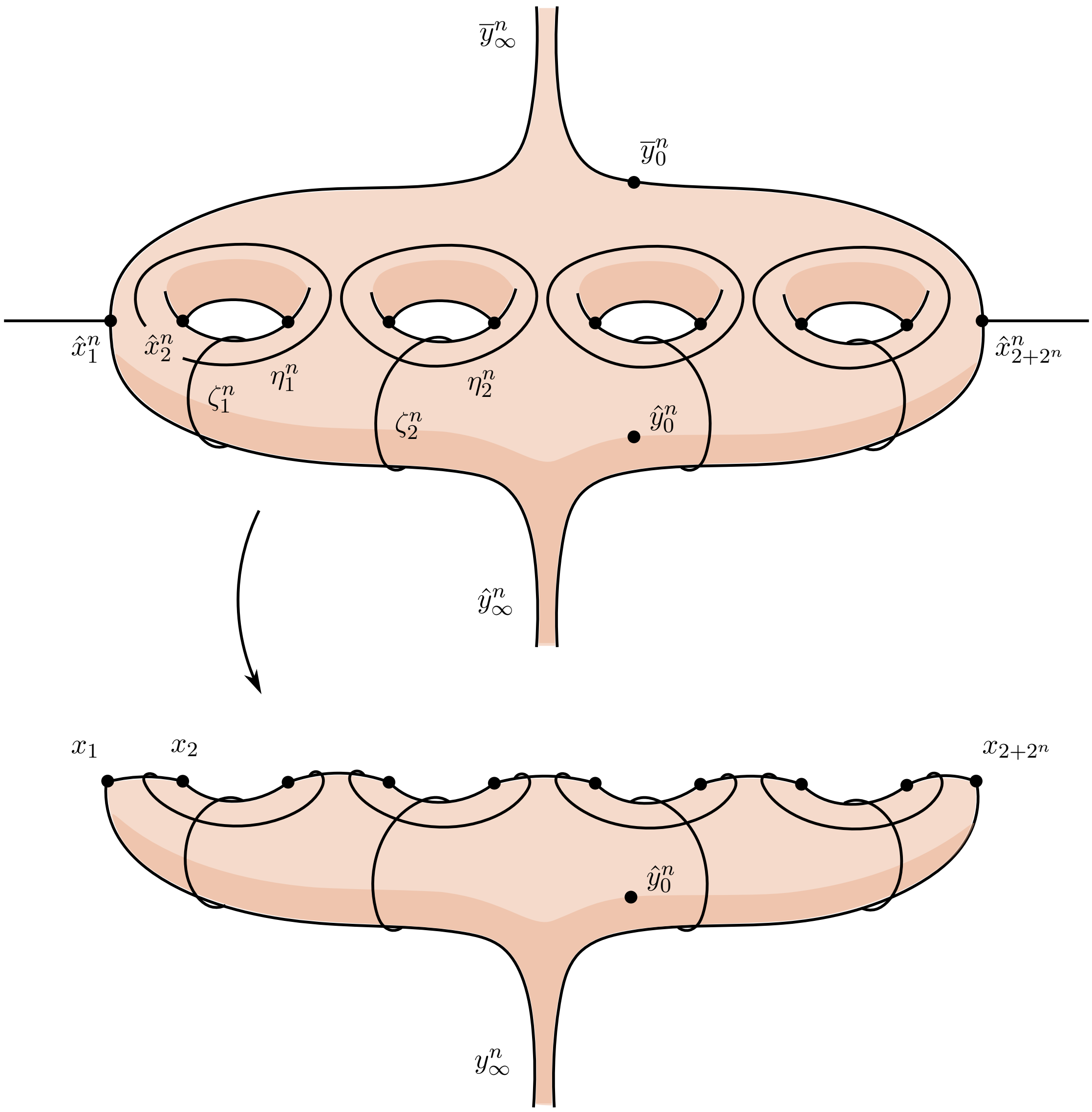



FIGURE 4.2.

In what follows, we will treat punctures as points in the surface indistinctly and, in particular, refer to them by the name of the points in the non-punctured surface. For a puncture $x$ in a punctured surface $X_\star$, let $D(x) \subset X$ be a small disk around $x$, and $\sigma(x) : [0, 1] \to \partial D(x)$ be a simple closed curve parametrizing the boundary of the disk.

By definition of a branched covering map, we may assume $f \circ \sigma(x) = \sigma(f(x))^d$ for punctures $x$ in $\hat{\mathbb{C}}^{n+1}_\star$, where $d = \deg(f, x)$. In the same manner, we may assume that $\rho_n \circ \sigma(x) = \sigma(\rho_n(x))^d$ for punctures $x$ in $M^n_\star$, where $d = \deg(\rho_n, x)$.

LEMMA 4.7. Let $n \geq 0$, $b \in \widehat{\mathbb{C}}^n_\star$ and $\hat{b} \in (\rho_n)^{-1}(b)$. Suppose $\gamma_1 * \gamma_2$ is a loop based at $b$ such that $\gamma_r = \delta_r * \sigma(x^n_{i_r}) * \delta_r^{-1}$ for some $i_r$ and some path $\delta_r : [0,1] \to M^n_\star$ with $\delta_r(1) = \sigma(x_{i_r})(0)$. Then, the homotopy class of $\gamma_1 * \gamma_2$ belongs to $(\rho_n)_* \pi_1(M^n_\star, \hat{b})$.

PROOF. We must show that the $\rho_n$-lift of $\gamma_1 * \gamma_2$ starting at $\hat{b}$ ends at $\hat{b}$. Recall that $\deg(\rho_n) = 2$, and therefore $b$ has exactly two lifts, say $\hat{b}$ and $\overline{b}$. We will strongly use this last fact to prove the statement.

Let $\hat{\delta}_1$ be the $\rho_n$-lift of $\delta_1$ starting at $\hat{b}$, and $\overline{\delta}_1$ be the one starting at $\overline{b}$. Since $(\rho_n)^{-1}(x_{i_1}) = \{\hat{x}_{i_1}\}$ and $\deg(\rho_n, \hat{x}_{i_1}) = 2$, any $\rho_n$-lift of $\sigma(x_{i_1})$ starts and ends at different points. Say $\hat{\sigma}_1$ is the $\rho_n$-lift starting at $\hat{\delta}_1(1)$. Hence, $\hat{\delta}_1 * \hat{\sigma}_1 * \overline{\delta}_1^{-1}$ is the $\rho_n$-lift of $\gamma_1$ starting at $\hat{b}$. In the same way, let $\hat{\delta}_2$ and $\overline{\delta}_2$ be respectively the $\rho_n$-lifts of $\delta_2$ starting at $\hat{b}$ and $\overline{b}$, and let $\overline{\sigma}_2$ be the $\rho_n$-lift of $\sigma(x_{i_2})$ starting at $\overline{\delta}_2(1)$. Then, $\overline{\delta}_2 * \overline{\sigma}_2 * \hat{\delta}_2^{-1}$ is the $\rho_n$-lift of $\gamma_2$ starting at $\overline{b}$.

To finish, we have that $\hat{\delta}_1 * \hat{\sigma}_1 * \overline{\delta}_1^{-1} * \overline{\delta}_2 * \overline{\sigma}_2 * \hat{\delta}_2^{-1}$ is the $\rho_n$-lift of $\gamma_1 * \gamma_2$ starting at $\hat{b}$, and the statement follows as $\hat{\delta}_2^{-1}(1) = \hat{\delta}_2(0) = \hat{b}$. □

We now fix some base points $b_0 \in \widehat{\mathbb{C}}^0_\star$, $b_{n+1} \in f^{-1}(b^n)$ and $\hat{b}^n \in \rho_n^{-1}(b^n)$ for $n \geq 0$. Using the lifting criterion, we see that it suffices to show that

$$(f \circ \rho_{n+1})_* \pi_1(M^{n+1}_\star, \hat{b}_{n+1}) \subset (\rho_n)_* \pi_1(M^n_\star, \hat{b}^n), \tag{4.1}$$

in order to prove there is a map $F_{n+1} : M^{n+1}_\star \to M^n_\star$ with $f \circ \rho_{n+1} = \rho_n \circ F_{n+1}$. We will start by giving some generators of the fundamental groups involved.

For a puncture $x$ in a punctured surface $X_\star$ with basepoint $b \in X_\star$, we will choose a curve $\gamma(x) : [0,1] \to X_\star$ of the form $\gamma(x) = \delta(x) * \sigma(x) * \delta(x)^{-1}$, where $\delta(x) : [0,1] \to X_\star$ is a path such that $\delta(x)(0) = b$ and $\delta(x)(1) = \sigma(x)(0)$. This gives an element of $\pi_1(X_\star, b)$.

If $n \geq 1$, then the surface $M^n_\star$ has the punctures $\hat{x}^n_1, \ldots, \hat{x}^n_{2^n+2}$, that are the critical points of $\rho_n$, as well as the punctures $\hat{y}^n_\infty$ and $\overline{y}^n_\infty$, that are the lifts of the point $\infty$, and the punctures $\hat{y}^n_0$ and $\overline{y}^n_0$, that are the lifts of the point 0. On the other hand, $M^0_\star$ has only one lift $\hat{y}^0_\infty = \hat{x}^0_4$ of $\infty$, and the other punctures are $\hat{x}^0_1, \hat{x}^0_2$ and $\hat{x}^0_3$.

For the surfaces $M^n_\star$, also consider the curves $\zeta^n_1, \eta^n_1, \ldots, \zeta^n_{g_n}, \eta^n_{g_n}$ shown in Figure 4.2. As we need elements in $\pi_1(M^n_\star, \hat{b}^n)$, we will again conjugate by a path $\delta^n_\ell : [0,1] \to M^n_\star$ starting at $\hat{b}^n$ and ending at the point that $\zeta^n_\ell$ and $\eta^n_\ell$ share. That is, we consider the loops $\alpha^n_\ell = \delta^n_\ell * \zeta^n_\ell * (\delta^n_\ell)^{-1}$ and $\beta^n_\ell = \delta^n_\ell * \eta^n_\ell * (\delta^n_\ell)^{-1}$.

Hence, for $n \geq 1$, the group $\pi_1(M^n_\star, \hat{b}^n)$ is the free group generated by the following curves:

$$\pi_1(M^n_\star, \hat{b}^n) = \left\langle \alpha^n_1, \beta^n_1, \ldots, \alpha^n_{g_n}, \beta^n_{g_n}, \gamma(\hat{x}^n_1), \ldots, \gamma(\hat{x}^n_{2^n+2}), \gamma(\hat{y}^n_0), \gamma(\hat{y}^n_\infty), \gamma(\overline{y}^n_0) \right\rangle.$$

Of course, above we are considering homotopy classes of loops, and not actual loops. Notice also that $\gamma(\overline{y}^n_\infty)$ is not in the list of generators. The case $n = 0$ is simpler. We have:

$$\pi_1(M^0_\star, \hat{b}_0) = \left\langle \alpha^0_1, \beta^0_1, \gamma(\hat{x}^0_1), \gamma(\hat{x}^0_2), \gamma(\hat{x}^0_3) \right\rangle.$$

To prove the inclusion given in (4.1), it suffices to show the image by $(f \circ \rho_{n+1})_*$ of each generator belongs to $(\rho_n)_* \pi_1(M^n_\star, \hat{b}^n)$.

LEMMA 4.8. Let $n \geq 0$ and let $x$ be a puncture in $M_\star^{n+1}$. Then, the homotopy class of $(f \circ \rho_{n+1}) \circ \gamma(x)$ belongs to $(\rho_n)_* \pi_1(M_\star^n, \hat{b}^n)$.

PROOF. Call $g = f \circ \rho_{n+1}$ and let $x' := g(x)$. Since $\deg(g, x) = 2$ for all punctures $x$ in $M_\star^{n+1}$, we have $g \circ \sigma(x) = \sigma(x')^2$. Note that $\deg(\rho_n, \hat{x}') \in \{1, 2\}$ for all $\hat{x}' \in \rho_n^{-1}(x')$. Hence, any $\rho_n$-lift of $\sigma(x')^2$ is a loop. Namely, it can be either $\sigma(\hat{x}')$ or $\sigma(\hat{x}')^2$, for $\hat{x}' \in \rho_n^{-1}(x')$. This implies any $\rho_n$-lift of $g \circ \gamma(x)$ is also a loop as we have $g \circ \gamma(x) = g \circ \delta(x) * \sigma(x')^2 * g \circ \delta(x)^{-1}$. □

On the other hand, as can be seen in Figure 4.2, we have the following:

CLAIM 4.9. Let $n \geq 0$. Hence, both $\rho_{n+1} \circ \zeta_\ell^{n+1}$ and $\rho_{n+1} \circ \eta_\ell^{n+1}$ bound a disk in $\widehat{\mathbb{C}}$ containing an even number of punctures of type $x_i$.

LEMMA 4.10. Let $n \geq 0$. Then, both homotopy classes of $(f \circ \rho_{n+1}) \circ \alpha_\ell^{n+1}$ and $(f \circ \rho_{n+1}) \circ \beta_\ell^{n+1}$ belong to $(\rho_n)_* \pi_1(M_\star^n, \hat{b}^n)$.

PROOF. We will give a proof only for $\alpha_\ell^{n+1}$ as the same argument holds for $\beta_\ell^{n+1}$. Following Claim 4.9, let $x_{i_1}, \ldots, x_{i_{2k}}$ be the punctures contained in the disk bounded by $\zeta_\ell^{n+1}$. Hence, up to changing the order of the indices $i_1, \ldots, i_{2k}$, there is a homotopy of loops in the disk, fixing basepoint, that takes $\rho_{n+1} \circ \zeta_\ell^{n+1}$ to a loop of the form $\eta_1 * \cdots * \eta_{2k}$, where $\eta_r = \delta_r * \sigma(x_{i_r}) * \delta_r^{-1}$ for some path $\delta_r : [0,1] \to \widehat{\mathbb{C}}_\star^{n+1}$ with $\delta_r(1) = \sigma(x_{i_r})(0)$. Thus, we may assume $\rho_{n+1} \circ \zeta_\ell^{n+1}$ is of the form $\eta_1 * \cdots * \eta_{2k}$, and $\rho_{n+1} \circ \alpha_\ell^{n+1} = \rho_{n+1} \circ \delta_\ell^{n+1} * \eta_1 * \cdots * \eta_{2k} * \rho_{n+1} \circ (\delta_\ell^{n+1})^{-1}$.

Now we study $(f \circ \rho_{n+1}) \circ \alpha_\ell^{n+1}$. Since $\deg(f, x_i) = 1$ for all $i$, we have $f \circ \sigma(x_{i_r}) = \sigma(f(x_{i_r}))$ for all $r$. Moreover, $f(x_{i_r}) = x_{j_r}$ for some $j_r$, and so the loops $\gamma_r := f \circ \eta_r$ are given by $\gamma_r = f \circ \delta_r * \sigma(x_{j_r}) * f \circ \delta_r^{-1}$. Using Lemma 4.7, we see that $\gamma_1 * \cdots * \gamma_{2k}$ is a product of $k$ loops whose $\rho_n$-lifts are loops and, in consequence, any $\rho_n$-lift of this loop is also a loop.

To finish, we have

$$(f \circ \rho_{n+1}) \circ \alpha_\ell^{n+1} = (f \circ \rho_{n+1}) \circ \delta_\ell^{n+1} * \gamma_1 * \cdots * \gamma_{2k} * (f \circ \rho_{n+1}) \circ (\delta_\ell^{n+1})^{-1}.$$

Then, any $\rho_n$-lift of $(f \circ \rho_{n+1}) \circ \alpha_\ell^{n+1}$ is a loop. □

CHAPTER 5

# Problems and related questions

In this chapter, we will discuss the current state of our research, outlining the points where we left off and identifying objects that require further exploration.

The questions we pose primarily concern the concept of QOTC, introduced in Chapter 4. The correct definition for generalizing a QOTE remains unclear. For now, we say a branched covering map $f : S^2 \to S^2$ is a *quotient of towers of coverings (QOTC)* if there is a family of positive-genus surfaces $M^n$, a family of covering maps $F_{n+1} : M^{n+1} \to M^n$, and a family of finite branched covering maps $\rho_n : M^n \to S^2$ such that $f \circ \rho_{n+1} = \rho_n \circ F_{n+1}$. This gives the following commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & M^{n+1} & \xrightarrow{F_{n+1}} & M^n & \xrightarrow{F_n} & \cdots & \xrightarrow{F_2} & M^1 & \xrightarrow{F_1} & M^0 \\
 & & \Big\downarrow{\scriptstyle \rho_{n+1}} & & \Big\downarrow{\scriptstyle \rho_n} & & & & \Big\downarrow{\scriptstyle \rho_1} & & \Big\downarrow{\scriptstyle \rho_0} \\
\cdots & \longrightarrow & S^2 & \xrightarrow[f]{} & S^2 & \xrightarrow[f]{} & \cdots & \xrightarrow[f]{} & S^2 & \xrightarrow[f]{} & S^2
\end{array}$$

According to Theorem 4.5, every Thurston map $f : S^2 \to S^2$ with a hyperbolic orbifold and no periodic critical points is a QOTC. This leads us to question what additional properties might be derived for the associated maps. For instance, can we expect there is some $r \geq 2$ such that $\deg(\rho_n) = r$ for all $n \geq 0$? Compare this with Corollary 4.6. Additionally, can we choose each $\rho_n$ to be a regular branched covering map? Can each $F_n$ be chosen as regular covering maps? These questions concern the index and normality of subgroups involved.

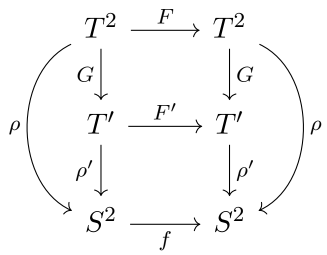
Recall that, for the case of a QOTE $f$, the map $F : T^2 \to T^2$ is always a regular covering map due to the abelian nature of the fundamental group of a torus. Additionally, the associated projections $\rho : T^2 \to S^2$ in Theorem 4.3 were all selected as regular orbifold covering maps.

Suppose now $f : S^2 \to S^2$ is a QOTC. If there exists an associated surface $M_N$ that is a torus, then all $M_n$ must be tori since the coverings $F_n$ are of finite degree. Consequently, $f$ is a QOTE and thus has a parabolic orbifold.

On the other hand, If $f : S^2 \to S^2$ is a QOTC with some $M_N$ such that $\chi(M_N) < 0$, then all $M_n$ also have negative Euler characteristic. An obvious question that arises in this case is whether $f$ has a hyperbolic orbifold. A priori, a QOTE could admit such a tower of coverings in addition to the existing tower with tori.

Perhaps the most ambitious challenge would be attempting to generalize the ideas presented in Chapter 3 to this setting. For example, can we expect $\deg(\rho_0, \cdot)$ to be constant on $\rho_0^{-1}(x)$ for all $x \in S^2$? Compare this with Theorem 3.3. If this holds true, can we infer a statement similar to Theorem 3.4? Specifically, is there some $k \geq 0$ such that $\rho_0 : M_0 \to \mathcal{O}_f^k$ is an orbifold covering map?

On the other hand, in Chapter 3 we saw that, given a QOTE $f : S^2 \to S^2$ with associated maps $F : T^2 \to T^2$ and $\rho : T^2 \to S^2$ not having the injectivity property, one can find "more efficient" associated maps by factoring the existing ones. Specifically, Lemma 3.8 shows how to obtain associated maps $F' : T' \to T'$ and $\rho' : T' \to S^2$ for $f$ and a covering map $G : T^2 \to T'$ with $\deg(G) \geq 2$ such that the diagram below commutes:




We now ask whether it is possible to establish a condition similar to non-$\rho$-injectivity to factor maps associated with a QOTC. Additionally, if the given maps $F_n$ and $\rho_n$ are efficient in the sense that they cannot be factored further, we inquire whether the topology of the surfaces $M_n$ is thereby determined. In other words, we seek to understand if the surfaces associated with a QOTC are unique up to factoring maps.

To conclude, it would also be valuable to understand what additional information about a Thurston map $f : S^2 \to S^2$ can be inferred solely from the tower of coverings $F_n : M_{n+1} \to M_n$, beyond the orbifold structure. More precisely, might this information provide insights into the dynamic behavior of $f$? Exploring this could deepen our understanding of the interplay between topology and dynamics.

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