5. Portfolio diversification (II) and CAPM

MA6622, Ernesto Mordecki, CityU, HK, 2006.

References for Lecture 5:


5a. Computing efficient portfolios

**Purpose**: construct an efficient portfolio in practice, departing from historical prices of the assets.

**Suppose**: we have three assets $A, B, C$ (in the same currency) to construct our portfolio.

Let $S_i(0), \ldots, S_i(n)$ be the historical prices of asset $i = A, B, C$. (in practice $n = 90$ for daily data, of $n = 12$ for monthly data, etc).
STEP 1: Calculate the return of $A$, by the equations

$$x_A(1) = \frac{S_A(1)}{S_A(0)} - 1, \ldots, x_A(n) = \frac{S_A(n)}{S_A(n-1)} - 1,$$

and similar for $B$ and $C$.

STEP 2: Estimate the mean returns of $A$ by

$$\bar{x}_A = \frac{1}{n} \sum_{k=1}^{n} x_A(k).$$

and the same for $B, C$. 
STEP 3: Estimate the variance-covariance matrix:

\[
\begin{bmatrix}
\sigma^2_A & \text{cov}_{AB} & \text{cov}_{AC} \\
\text{cov}_{AB} & \sigma^2_B & \text{cov}_{BC} \\
\text{cov}_{AC} & \text{cov}_{BC} & \sigma^2_C
\end{bmatrix}
\]

As this matrix is symmetric we must estimate only 6 values.
For $A$, the variance is

$$\bar{\sigma}_A^2 = \frac{1}{n - 1} \sum_{k=1}^{n} \left( x_A(k) - \bar{x}_A \right)^2.$$  

and similarly for $B$ and $C$.

The covariance between returns of $A$ and $B$ is

$$\bar{c}_{AB} = \frac{1}{n} \sum_{k=1}^{n} x_A(k)x_B(k) - \bar{x}_A\bar{x}_B,$$

and similarly for $A, C$ and $B, C$. 
Now, we can estimate the expected return of a portfolio $\pi = (\alpha, \beta, \gamma)$, with proportions $(a, b, c)$, $a + b + c = 1$. The expected return is

$$E X_\pi = a \bar{x}_A + b \bar{x}_B + c \bar{x}_C,$$

while the variance of the return is

$$\text{var} X_\pi = a^2 \bar{\sigma}_A^2 + b^2 \bar{\sigma}_B^2 + c^2 \bar{\sigma}_C^2 + 2(ab \bar{c}_A \bar{c}_B + ac \bar{c}_A \bar{c}_C + bc \bar{c}_B \bar{c}_C).$$
Consider for example the values (in %)

\[
\bar{x}_A = 10 \quad \bar{x}_B = 5 \quad \bar{x}_C = 3 \\
\bar{\sigma}_A = 8 \quad \bar{\sigma}_B = 10 \quad \bar{\sigma}_C = 12 \\
\bar{c}_{AB} = 0 \quad \bar{c}_{AC} = 0 \quad \bar{\sigma}_{BC} = 0
\]

Consider the following two portfolios:

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(\bar{x})</th>
<th>(\bar{\sigma})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pi)</td>
<td>0.333</td>
<td>0.333</td>
<td>0.333</td>
<td>6</td>
<td>5.85</td>
</tr>
<tr>
<td>(\pi_{eff})</td>
<td>0.322</td>
<td>0.372</td>
<td>0.305</td>
<td>6</td>
<td>5.81</td>
</tr>
</tbody>
</table>

The second portfolio is efficient, both have the same expected return.
5b. Introduction to CAPM

Consider

- The risk-free interest rate \( r \)

- A global market index with (stochastic) return \( \rho \)

- An asset \( A \) in the market with (stochastic) return \( \rho_A \)
Problem: Quantify the risk and return of $A$ in the market

Example: $r$ is the interest rate of a zero coupon US-bond, $\rho$ is the return of the S&P 500, and $A$ is a share of Google.
W. Sharpe (1964) established that there exists a quantity denoted $\beta$ such that

$$E \rho_A - r = \beta (E \rho - r),$$  \hspace{1cm} (1)

where

$$\beta = \frac{\text{cov}(\rho_A, \rho)}{\text{var} \rho} = \frac{E(\rho \rho_A) - E \rho E \rho_A}{E(\rho^2) - (E \rho)^2}.$$

The amount $E \rho_A - r$ is the risk premium of asset $A$. 
5c. $\beta$ and the expected return

Based on (1) we see that

- If $\beta = 0$ then $E\rho_A = r$.

- If $\beta = 0$ then $E\rho_A = \rho$.

$E\rho_A$ is a linear function of $\beta$ under the equation

$$E\rho_A = r + \beta(\rho - r).$$
5d. $\beta$ and the risk

As we are interested in risk, we compute the variance of $\rho_A$. Define the random variable $\varepsilon$ by the relation

$$\varepsilon = \rho_A - E\rho_A - \beta(\rho - E\rho).$$

Taking expectations we verify $E\varepsilon = 0$.

$$\text{cov}(\rho, \varepsilon) = E\rho\varepsilon = E\rho\left(\rho_A - E\rho_A - \beta(\rho - E\rho)\right) = E(\rho\rho_A) - E\rho E\rho_A - \beta[E\left(\rho^2\right) - (E\rho)^2] = 0$$
We obtained a decomposition of $\rho_A$ of the form

$$\rho_A - \mathbf{E} \rho_A = \beta (\rho - \mathbf{E} \rho) + \varepsilon$$

where the two terms in the sum are uncorrelated.
This gives

\[ \text{var } \rho_A = \beta^2 \text{var } \rho + \text{var } \varepsilon. \]

Here

- \( \beta^2 \text{var } \rho \) is the systematic (unavoidable) risk of \( A \)
- \( \text{var } \varepsilon \) is the unsystematic (diversifiable) risk of \( A \).

Then, \( \beta \) measures the systematic or market risk of \( A \).
It is important to distinguish between this two risks:

- the first is the **systematic**, intrinsic of the market, can not be reduced.

- The second, called **unsystematic**, can be diversified, for instance, buying other assets.
In practice $\beta$ ranges, approximately from $1/2$ to 2, and, for instance

- If $\beta = 1/2$, the asset $A$ has half of the expected return of the market, but $1/4$ of the systematic risk, we have a **defensive** asset.

- If $\beta = 2$, the asset doubles the expected return of the market, but the systematic risk is raised four times. We have an **aggressive** asset.
5d. Computation of $\beta$

The parameter $\beta$ results as the slope in a simple linear regression model of the form

$$\rho_A - r = \beta (\rho - r) + \varepsilon,$$

where $\varepsilon$ is a statistical error.
In order to estimate $\beta$, you need:

- Prices $(S_A(0), S_A(1), \ldots, S_A(n))$ of the asset $A$,

- corresponding prices $(S(0), S(1), \ldots, S(n))$ of the market index.
**STEP 1:** Compute the corresponding returns by the formula

\[
x(k) = \frac{S(k)}{S(k-1)} - 1, \quad y(k) = \frac{S_A(k)}{S_A(k-1)} - 1.
\]

**STEP 2:** Find the mean returns

\[
\bar{x} = \frac{1}{n} \sum_{k=1}^{n} x(k), \quad \bar{y} = \frac{1}{n} \sum_{k=1}^{n} y(k),
\]
STEP 3: Estimate the variance of $\rho$ by

$$\bar{\sigma}_x^2 = \frac{1}{n} \sum_{k=1}^{n} (x(k) - \bar{x})^2,$$

STEP 4: Estimate the covariance

$$\bar{c}_{xy} = \frac{1}{n} \sum_{k=1}^{n} (x(k)y(k)) - \bar{x}\bar{y}.$$

FINAL STEP: The estimation of $\beta$ is

$$\beta = \frac{\bar{c}_{xy}}{\bar{\sigma}_x^2}.$$