4. Markowitz Diversification Portfolio Investment

Question:
How to invest $\nu$ HKD into assets $A_1, \ldots, A_n$

Central idea: Diversification
“Don’t put all your eggs in one basket”

*MA6622, Ernesto Mordecki, CityU, HK, 2006.
4a. Formulation of the Problem.

Suppose that you have $v$ HKD to invest in several assets $A_1, \ldots, A_n$.

Prices today ($t = 0$) are $S_1(0), \ldots, S_n(0)$ (known).

Prices tomorrow ($t = 1$) will be

$$S_k(1) = S_k(0)(1 + X_k), \quad k = 1, \ldots, n,$$

where $X_k$ is the random interest rate of each asset.
The random interest rates $X_k$ have **positive** expectation

$$r_k = E X_k \geq 0, \quad k = 1, \ldots, n,$$

**variances**

$$\sigma^2_k = \text{var} X_k, \quad k = 1, \ldots, n,$$

**covariances**

$$\text{cov}_{jk} = E(X_j - r_j)(X_k - r_k), \quad j \neq k = 1, \ldots, n,$$

and **correlations**

$$\rho_{jk} = \frac{\text{cov}_{jk}}{\sigma_j \sigma_k}, \quad j \neq k = 1, \ldots, n.$$
So you will buy a portfolio $\pi = (\alpha_1, \ldots, \alpha_n)$ such that

$$v = \alpha_1 S_1(0) + \cdots + \alpha_n S_n(0).$$

At time $t = 1$ your capital will be

$$V_\pi(1) = \alpha_1 S_1(1) + \cdots + \alpha_n S_n(1).$$

Problem: choose best possible $\pi = (\alpha_1, \ldots, \alpha_n)$, according to your risk-return preferences.
The main instruments of analysis proposed by Markowitz (1952) are

- \( EV_\pi(1) \) to measure the expected return,

- \( \text{var} V_\pi(1) \) to measure the risk.

Let us compute these numbers.
Consider the proportions

\[ a_1 = \frac{\alpha_1 S_1(0)}{v}, \ldots, a_n = \frac{\alpha_n S_n(0)}{v} \]

such that \( a_1 + \cdots + a_n = 1 \). Now

\[ V_{\pi}(1) = \alpha_1 S_1(1) + \cdots + \alpha_n S_n(1) \]
\[ = \alpha_1 S_1(0)(1 + X_1) + \cdots + \alpha_n S_n(0)(1 + X_n) \]
\[ = va_1(1 + X_1) + \cdots + va_n(1 + X_n) \]
\[ = v(1 + a_1 X_1 + \cdots + a_n X_n) \]
\[ = v(1 + X_{\pi}). \]

where we define the return of portfolio \( \pi \) as

\[ X_{\pi} = a_1 X_1 + \cdots + a_n X_n. \]
We have

\[ E V_{\pi}(1) = \nu(1 + E X_{\pi}), \quad \text{var } V_{\pi}(1) = \nu^2 \text{var } X_{\pi} \]

and we can write

\[ E X_{\pi} = a_1 X_1 + \cdots + a_n X_n, \]

\[ \text{var } X_{\pi} = \sum_{k=1}^{n} (a_k \sigma_k)^2 + 2 \sum_{1 \leq j < k \leq n} a_j a_k \text{cov}_{j,k}. \]
4b. Re-formulation of the Problem.

Given \( r_k \geq 0, \sigma_k > 0, -1 \leq \rho_{jk} \leq 1 \), find \((a_1, \ldots, a_n)\) such that \(a_1 + \cdots + a_n = 1\):

- expected return: \(a_1 r_1 + \cdots + a_n r_n\) as higher as possible,

- risk: \(\sum_{k=1}^{n} (a_k \sigma_k)^2 + 2 \sum_{1 \leq j < k \leq N} a_i a_j \sigma_j \sigma_k \rho_{jk}\) as lower as possible,

This is the risk-return tradeoff.
Let us examine the case $n = 2$. Results are

$$E X_\pi = a_1 r_1 + a_2 r_2,$$

and, as $\text{cov}_{12} = \sigma_1 \sigma_2 \rho_{12}$, we have

$$\text{var } X_\pi = (a_1 \sigma_1)^2 + (a_2 \sigma_2)^2 + 2a_1 a_2 \sigma_1 \sigma_2 \rho_{12}
= (a_1 \sigma_1 - a_2 \sigma_2)^2 + 2a_1 a_2 \sigma_1 \sigma_2 (1 + \rho_{12})$$
Example: Assume that $r_1 < r_2$, $\sigma = \sigma_1 = \sigma_2$ and $\sigma_{12} = 0$

- $a_1 = 0, a_2 = 1$ gives the maximum possible return $r_2$, with variance $\sigma^2$.

- $a_1 = a_2 = 1/2$ gives the minimum variance $\sigma^2/4$, with medium return $(r_1 + r_2)/2$.

and these two portfolios are very different.
So:

- high expected return rises with an increase in risk,

- Low levels risk are associated with low expected returns

In other words: high returns requires risky investments (read Financial Times article4.pdf)
Negative correlated assets

If $\rho_{12} = -1$, i.e. $A_1$ and $A_2$ are negatively correlated,

$$\text{var} \ X_\pi = (a_1 \sigma_1 - a_2 \sigma_2)^2$$

we can achieve $\text{var} \ V_\pi(1) = 0$ by choosing $a_1 \sigma_1 = a_2 \sigma_2$, that gives

$$a = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \quad b = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$
In this situation we have no risk, and our return is

\[ V_\pi(1) = v \left( 1 + \frac{r_1 \sigma_2 + r_2 \sigma_1}{\sigma_1 + \sigma_2} \right) > v. \]

**Phenomenon of negative correlation:** When making an investment portfolio, negatively correlated assets reduces the risk.
Diversification

Assume that the assets $A_1, \ldots, A_n$ are uncorrelated:

$$\rho_{jk} = 0, \quad \text{for all } j \neq k,$$

with bounded variances:

$$\sigma^2_1 \leq C, \ldots, \sigma^2_n \leq C.$$

Choose $a_1 = \cdots = a_n = 1/n$.

$$\text{var } X_\pi = a_1^2 \sigma^2_1 + \cdots + a_n^2 \sigma^2_n =$$

$$\frac{1}{n^2}(\sigma^2_1 + \cdots + \sigma^2_n) \leq \frac{C}{n} \to 0 \quad (n \to \infty).$$
Phenomenon of absence of correlation: In absence of correlation the number $n$ of assets should be possible larger to reduce the risk (variance) of the investment.
4c. Systematic and Unsystematic Risk

If we have \( n \) correlated assets, the variance of a portfolio with proportions \( a_1, \ldots, a_n \) and return

\[
X_\pi = a_1 X_1 + \cdots + a_n X_n
\]

is

\[
\text{var } X_\pi = \sum_{k=1}^{n} (a_k \sigma_k)^2 + 2 \sum_{1 \leq j < k \leq n} a_i a_j \text{cov}_{jk}.
\]

If we take \( a_1 = \cdots = a_n = 1/n \), we obtain

\[
\sum_{k=1}^{n} (a_k \sigma_k)^2 = \frac{1}{n} \bar{\sigma}_n,
\]
defining the mean variance as

$$\overline{\sigma}_n = \frac{1}{n} \sum_{k=1}^{n} \text{var} \ X_k.$$ 

The mean covariance is

$$\overline{c}_n = \frac{2}{n^2 - n} \sum_{1 \leq i < j \leq n} \text{cov}(i, j),$$

and, we conclude

$$\text{var} \ X_\pi = \frac{1}{n} \overline{\sigma}_n + \left(1 - \frac{1}{n}\right) \overline{c}_n$$

Assume that $\overline{\sigma}_n \leq C$ and $\overline{c}_n \to \overline{c}$. Then

$$\text{var} \ X_\pi \to \overline{c} \quad (n \to \infty)$$
In conclusion:

- The first (variance) term can be reduced by diversification, it is the **unsystematic** risk.

- The second (covariance) term can **not** be reduced, it is the **systematic** called also the **market** risk.
4d. Efficient Portfolio

In order to compare two portfolios, rational behaviour is:

- With equal expected returns, we prefer lower variance,
- With equal variance, we prefer higher expected returns.
We now plot, for $a_k \geq 0$ with $a_1 + \cdots + a_n = 1$ all the possible values of

$$\left(\sqrt{\text{var} V_\pi(1)}, \mathbb{E} V_\pi(1)\right).$$

It is not difficult to see, that

- Given a constant $\sigma^2_0 = \text{var} V_\pi(1)$ we have an interval of values for $\mathbb{E} V_\pi(1)$,

- Given a constant $m_0 = \mathbb{E} V_\pi(1)$ we have an interval of values of $\text{var} V_\pi(1)$
Conclusion: Given $\sigma^2_0$ we have one portfolio with this variance and maximum expectation, in the efficient curve.

Similarly, given $m_0$ there is only one portfolio with this expected return and minimum variance, in the same curve.

This portfolios are called efficient.

These results are known as the mean-variance analysis of Markowitz.