16. Time dependence in Black Scholes

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References for this Lecture:

Marco Avellaneda and Peter Laurence,
*Quantitative Modelling of Derivative Securities*,
Plan of Lecture 16

(16a) Term Structure of Interest rates

(16b) Term Structure of Volatilities

(16c) A detailed Example
16a. Term Structure of Interest rates\(^1\)

A first simple extension in Black and Scholes model is to incorporate time dependence.

For example, the following table gives the interest rates of Treasury Bills (zero coupon bonds issued by the US Treasury Department), for June 8, 2006:

<table>
<thead>
<tr>
<th></th>
<th>4-week</th>
<th>3-month</th>
<th>6-month</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield</td>
<td>4.70</td>
<td>4.74</td>
<td>4.87</td>
</tr>
</tbody>
</table>

Treasury bills can be considered to be the most risk free investment, and this is why they are widely used in Black-Scholes formula\(^2\).

The given yields are in percent per annum on a **360-days basis**.

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\(^1\)This section expands section 14d.

\(^2\)It must be taken into account that these rates apply for USD.
360 calendar day

The 360 day calendar is a method of measuring durations used in financial markets. It is based on the assumption of a 360 day year, consisting of 12 months of 30 days each. To arrive at such a calendar from the standard 365/366 day Gregorian calendar, certain days are skipped, in our case, we do not count July 31, August 31, and October 31 (as our computations will refer to this time period).
Instantaneous forward rates

In order to model the interest rates with different maturities, we assume:

- Exists and instantaneous deterministic forward rate $r(t)$,
- $r(t)$ is constant between consecutive maturities,
- the observed term rates $r(t, T)$, satisfy:

$$r(t, T) = \frac{1}{T - t} \int_t^T r(s)ds.$$  \hspace{1cm} (1)
To **calibrate** the term structure of interest rates:

- **Given** a set of term rates $r(t, T)$, for different maturities $T$, **find** the instantaneous forward interest rates $r(t)$.

In our example

- Maturities are $T_1 = \frac{28}{360}$, $T_2 = \frac{90}{360}$, $T_3 = \frac{180}{360}$.
- Find $r_1$, $r_2$ and $r_3$:

$$r(t) = \begin{cases} 
  r_1 & \text{for } 0 \leq t \leq T_1 \\
  r_2 & \text{for } T_1 \leq t \leq T_2 \\
  r_3 & \text{for } T_2 \leq t \leq T_3 
\end{cases}$$

such that (1) gives:

$$r(0, T_1) = 0.0470, \quad r(0, T_2) = 0.0474, \quad r(0, T_3) = 0.0487.$$
Step 1. We find $r_1$. As the integrand takes a constant value, we have

$$r(0, T_1) = \frac{1}{T_1} \int_0^{T_1} r(s)ds$$

$$= \frac{1}{28/360}(28/360)r_1 = r_1 = 0.0470$$

So $r_1 = r(0, T_1)^3$

Step 2. For the second period the integrand takes two values:

$$\frac{1}{(90/360)}\left[ \frac{28}{360}r_1 + \frac{62}{360}r_2 \right] = 0.0474$$

Observe that the 360 simplifies, to give

$$\frac{1}{90}(28r_1 + 62r_2) = 0.0474$$

Here $62 = 90 - 28$. This gives $r_2 = 0.0476$.

\(^a\)Always the first forward rate coincides with the first term rate.
Step 3. For the third interest rate $r_3$ we arrive to a similar computation:

$$\frac{1}{180}(28r_1 + 62r_2 + 90r_3) = 0.0487$$

This gives $r_3 = 0.0500$
In conclusion, from the term rates:

<table>
<thead>
<tr>
<th>Duration</th>
<th>4-weeks</th>
<th>3 months</th>
<th>6 months</th>
</tr>
</thead>
<tbody>
<tr>
<td>In days</td>
<td>28</td>
<td>90</td>
<td>180</td>
</tr>
<tr>
<td>$r(t, T)$ %</td>
<td>4.70</td>
<td>4.74</td>
<td>4.87</td>
</tr>
</tbody>
</table>

we find the forward rates:

<table>
<thead>
<tr>
<th>Intervals</th>
<th>Jn 9 - Jl 6</th>
<th>Jl 7 - Sp 8</th>
<th>Sp 9 - Dc 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>In days</td>
<td>28</td>
<td>62</td>
<td>90</td>
</tr>
<tr>
<td>$r(t)$ %</td>
<td>4.70</td>
<td>4.76</td>
<td>5.00</td>
</tr>
</tbody>
</table>
**Example**  Compute the forward interest rate for 60 days. Time interval is June 9 - August 8:

- The first 28 days (June 9 - July 6) apply $r_1$,
- The second days (July 7 - August 8) apply $r_2$:

$$r(\text{June 9, August 8}) = \frac{1}{T-t} \int_t^T r(s)ds$$

$$= \frac{28 \times r_1 + 32 \times r_2}{60} = 0.0473$$
16b. Term Structure of Volatilities

The same procedure applies to the implied volatilities. Here the situation is more complex: for a maturity $T$ we have different implied volatilities, depending on the strike price (volatility smile).

We choose at the money options ($S(0) = K$) to compute the term implied volatilities $\sigma(t, T)$.

Our model assumes

- Exists an instantaneous deterministic forward volatility $\sigma(t)$,
- $\sigma(t)$ is constant between consecutive maturities,
- The term volatilities satisfy

$$\sigma^2(t, T) = \frac{1}{T-t} \int_t^T \sigma^2(s) ds.$$
Merton (1973)\(^4\) proved that Black-Scholes formula holds for time dependent interest rates and volatilities. For an option written today \(t\), expiring at time \(T\) we must use BS formula with:

- \(\sigma(t, T)\) instead of \(\sigma\)
- \(r(t, T)\) instead of \(r\).

The calibration of the Volatility Term Structure consists in:

- **Given** a set of at the money option prices with different maturities, **compute** the implied term volatilities \(\sigma(t, T)\).
- **Compute** the instantaneous forward volatility \(\sigma(t)\),

16c. A detailed Example

We want to price an at the money call option on the Hang Seng Index, written over the counter on June 8 with a duration of 60 calendar days. More precisely:

- Time interval of the option is June 9 - August 8.
- Spot price is closing price on June 8, $S(0) = 15816.5$
- Strike price is 15800,

This time we use risk-free interest rates from zero coupon US Bond corresponding to the maturity (see previous example).

Here we do not take into account

- currency bid-ask spread when converting from USD to HKD

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5 This is an improvement over Example in 14d.
6 I.e. A direct agreement between the buyer and the holder
- Exposure to currency risk (i.e. HKD/USD exchange rate is fixed and risk-free).

In other terms we assume that we have a risk-free instrument in HKD with the same interest rate structure of the US Bond (Treasury Bills in this case).

This is the information that we have:

<table>
<thead>
<tr>
<th>Month</th>
<th>Strike</th>
<th>Price</th>
<th>$r(t, T)$</th>
<th>$\sigma(t, T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>June, 29</td>
<td>15800</td>
<td>335</td>
<td></td>
<td></td>
</tr>
<tr>
<td>July, 28</td>
<td>15800</td>
<td>507</td>
<td></td>
<td></td>
</tr>
<tr>
<td>August, 8</td>
<td>15800</td>
<td>Price?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>August, 30</td>
<td>15800</td>
<td>625</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Step 1. Compute the term interest rates, from the table of last section, using calendar days:

\[ r(\text{June 9, June 29}) = 0.0470, \]
\[ r(\text{Jn 9, Jl 28}) = \frac{0.0470 \times 28 + 0.0476 \times 22}{28 + 22} = 0.04726, \]
\[ r(\text{Jn 9, Ag 30}) = \frac{0.0470 \times 28 + 0.0476 \times 54}{28 + 54} = 0.04740. \]

Step 2. Compute the term volatilities implied by the quoted prices (trading days), with this interest rates (calendar days).

\[ C(S(0); K; r; \sigma) = QP, \]
where \( QP \) is the quoted price (We use Newton Raphson to solve this equation in \( \sigma \)).

For June 29 we have 15 trading days and \( r = 0.047 \):

\[ C(15816.5; 15088, 15/247, 0.047, \sigma) = 335. \]
gives \( \sigma = 0.195 \).

- For July 28 we have 36 trading days and \( r = 0.04726 \):
  \[
  C(15816.5; 15088, 36/247, 0.04726, \sigma) = 507.
  \]
gives \( \sigma = 0.184 \).

- For August 30 we have 59 trading days and \( r = 0.0474 \):
  \[
  C(15816.5; 15088, 59/247, 0.0474, \sigma) = 625.
  \]
gives \( \sigma = 0.170 \).

So our table is now:

<table>
<thead>
<tr>
<th>Month</th>
<th>Strike</th>
<th>Price</th>
<th>( r(t, T) ) %</th>
<th>( \sigma(t, T) ) %</th>
</tr>
</thead>
<tbody>
<tr>
<td>June, 29</td>
<td>15800</td>
<td>335</td>
<td>4.70</td>
<td>19.5</td>
</tr>
<tr>
<td>July, 28</td>
<td>15800</td>
<td>507</td>
<td>4.724</td>
<td>18.4</td>
</tr>
<tr>
<td>August, 8</td>
<td>15800</td>
<td>Price?</td>
<td>4.73</td>
<td></td>
</tr>
<tr>
<td>August, 30</td>
<td>15800</td>
<td>625</td>
<td>4.740</td>
<td>17.0</td>
</tr>
</tbody>
</table>
Step 3. Compute the instantaneous volatility for each time interval, to fit the implied term volatilities just computed.

- The first volatility, between June 9 and June 29, $\sigma_1 = 0.195$.

- The second value $\sigma_2$, between June 30 and July 28 satisfies:
  \[
  \frac{15 \times \sigma_1^2 + 21 \times \sigma_2^2}{15 + 21} = 0.184^2,
  \]
  gives $\sigma_2 = 0.175722$.

- For $\sigma_3$, we solve
  \[
  \frac{15 \times \sigma_1^2 + 21 \times \sigma_2^2 + 23\sigma_3^2}{15 + 21 + 23} = 0.17^2,
  \]
  that gives $\sigma_3 = 0.145406$.
Step 4. Use the instantaneous forward volatilities to compute our term volatility:

$$
\sigma(\text{Jn 9, Ag 8}) = \sqrt{\frac{15\sigma_1^2 + 21\sigma_2^2 + 7\sigma_3^2}{15 + 21 + 7}} = 0.178287
$$

Step 5. Use this volatility to compute the price of the option, having 43 trading days:

$$
C(15816.5; 15088, 43/247, 0.0473, 0.1783) = 543.768.
$$

The final table is

<table>
<thead>
<tr>
<th>Month</th>
<th>Strike</th>
<th>Price</th>
<th>$r(t, T)$ %</th>
<th>$\sigma(t, T)$ %</th>
</tr>
</thead>
<tbody>
<tr>
<td>June, 29</td>
<td>15800</td>
<td>335</td>
<td>4.70</td>
<td>19.5</td>
</tr>
<tr>
<td>July, 28</td>
<td>15800</td>
<td>507</td>
<td>4.724</td>
<td>18.4</td>
</tr>
<tr>
<td>August, 8</td>
<td>15800</td>
<td>543.8</td>
<td>4.73</td>
<td>0.1783</td>
</tr>
<tr>
<td>August, 30</td>
<td>15800</td>
<td>625</td>
<td>4.740</td>
<td>17.0</td>
</tr>
</tbody>
</table>
Remember

• $r(t, T)$ are the term interest rate, observed.

• $r(t)$ is the forward interest rate, computed.

• We use a 360 calendar day.

• $\sigma(t, T)$ are the term volatilities, implied by $QP$ and computed through Newton Raphson.

• $\sigma(t)$ is the forward volatility, computed.

• We use trading days.