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Jean-Marc Azaïs · Mario Wschebor

On the regularity of the distribution of the maximum of one-parameter Gaussian processes

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Abstract. The main result in this paper states that if a one-parameter Gaussian process has C^{2k} paths and satisfies a non-degeneracy condition, then the distribution of its maximum on a compact interval is of class C^k. The methods leading to this theorem permit also to give bounds on the successive derivatives of the distribution of the maximum and to study their asymptotic behaviour as the level tends to infinity.

1. Introduction and main results

Let $X = \{X_t : t \in [0, 1]\}$ be a stochastic process with real values and continuous paths defined on a probability space $(\Omega, \Im, P)$. The aim of this paper is to study the regularity of the distribution function of the random variable $M := max\{X_t : t \in [0, 1]\}$.

X is said to satisfy the hypothesis H_k, k a positive integer, if:

(1) X is Gaussian;

(2) a.s. X has C^k sample paths;

(3) For every integer $n \geq 1$ and any set $t_1, ..., t_n$ of pairwise different parameter values, the distribution of the random vector:

$$X_{t_1}, ..., X_{t_n}, X'_{t_1}, ..., X'_{t_n}, ..., X^{(k)}_{t_1}, ..., X^{(k)}_{t_n}$$

is non degenerate.

We denote $m(t)$ and $r(s, t)$ the mean and covariance functions of X, that is $m(t) := E(X_t)$, $r(s, t) := E\big((X_s - m(s))(X_t - m(t))\big)$ and $r_{ij} := \frac{\partial^{i+j}}{\partial s^i \partial t^j} r$ $(i, j = 0, 1, ..)$ the partial derivatives of r, whenever they exist.

Our main results are the following:

J.-M. Azaïs, M. Wschebor: Laboratoire de Statistique et Probabilités, UMR-CNRS C55830, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cedex 4. France. e-mail: azais@cict.fr

M. Wschebor: Centro de Matemática, Facultad de Ciencias, Universidad de la República, Calle Igua 4225, 11400 Montevideo, Uruguay. e-mail: wscheb@fcien.edu.uy

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Theorem 1.1. *Let $X = \{X_t : t \in [0,1]\}$ be a stochastic process satisfying H_{2k}. Denote by $F(u) = P(M \leq u)$ the distribution function of M.*

Then, F is of class C^k and its succesive derivatives can be computed by repeated application of Lemma 3.3.

Corollary 1.1. *Let X be a stochastic process verifying H_{2k} and assume also that $E(X_t) = 0$ and $Var(X_t) = 1$.*

Then, as $u \to +\infty$, $F^{(k)}(u)$ is equivalent to

$$(-1)^{k-1}\frac{u^k}{2\pi}e^{-u^2/2}\int_0^1 \sqrt{r_{11}(t,t)}dt. \tag{1}$$

The regularity of the distribution of M has been the object of a number of papers. For general results when X is Gaussian, one can mention:Ylvisaker (1968); Tsirelson (1975); Weber (1985); Lifshits (1995); Diebolt and Posse (1996) and references therein.

Theorem 1.1 appears to be a considerable extension, in the context of one-parameter Gaussian processes, of existing results on the regularity of the distribution of the maximum which, as far as the authors know, do not go beyond Lipschitz condition for the first derivative. For example, it implies that if the process is Gaussian with $\mathscr{C}^\infty$ paths and satisfies the non-degeneracy condition for every k = $1, 2, \ldots,$ then the distribution of the maximum is $\mathscr{C}^\infty$. The same methods provide bounds for the successive derivatives as well as their asymptotic behaviour as their argument tends to $+\infty$ (Corollary 1.1).

Except in Theorem 3.1, which contains a first upper bound for the density of M, we will assume X to be Gaussian.

The proof of Theorem 1.1 is based upon the main Lemma 3.3. Before giving the proofs we have stated Theorem 3.2 which presents the result of this Lemma in the special case leading to the first derivative of the distribution function of M. As applications one gets upper and lower bounds for the density of M under conditions that seem to be more clear and more general than in previous work (Diebolt and Posse, 1996). Some extrawork is needed to extend the implicit formula (9) to non-Gaussian processes, but this seems to be feasible.

As for Theorem 1.1 for derivatives of order greater than 1, its statement and its proof rely heavily on the Gaussian character of the process.

The main result of this paper has been exposed in the note by Azaïs and Wschebor (1999).

2. Crossings

Our methods are based on well-known formulae for the moments of crossings of the paths of stochastic processes with fixed levels, that have been obtained by a variety of authors, starting from the fundamental work of S.O.Rice (1944–1945). In this section we review without proofs some of these and related results.

Let $f : I \longrightarrow \mathbb{R}$ be a function defined on the interval I of the real numbers,

$$C_u(f; I) := \{t \in I : f(t) = u\}$$

$$N_u(f;I) = \sharp\big(C_u(f;I)\big)$$

denote respectively the set of roots of the equation $f(t) = u$ on the interval I and the number of these roots, with the convention $N_u(f;I) = +\infty$ if the set C_u is infinite. $N_u(f;I)$ is called the number of "crossings" of f with the "level" u on the interval I.

In the same way, if f is a differentiable function the number of "upcrossings" and "downcrossings" of f are defined by means of

$$U_u(f;I) := \sharp(\{t \in I : f(t) = u,\ f'(t) > 0\})$$

$$D_u(f;I) := \sharp(\{t \in I : f(t) = u,\ f'(t) < 0\}).$$

For a more general definition of these quantities see Cramér and Leadbetter (1967).

In what follows, $\|f\|_p$ is the norm of f in $L^p(I,\lambda),\ 1 \le p \le +\infty$, λ denoting the Lebesgue measure. The joint density of the finite set of real-valued random variables $X_1, \dots X_n$ at the point $(x_1, \dots x_n)$ will be denoted $p_{X_1,\dots,X_n}(x_1, \dots x_n)$ whenever it exists. $\phi(t) := (2\pi)^{-1/2} exp(-t^2/2)$ is the density of the standard normal distribution, $\Phi(t) := \int_{\infty}^{t} \phi(u)du$ its distribution function.

The following proposition (sometimes called Kac's formula) is a common tool to count crossings.

Proposition 2.1. *Let $f : I = [a,b] \longrightarrow \mathbb{R}$ be of class $\mathscr{C}^1$, $f(a), f(b) \neq u$. If f does not have local extrema with value u on the inteval I, then*

$$N_u(f;I) = \lim_{\delta \downarrow 0} 1/(2\delta) \int_I \mathbb{1}_{\{|f(t)-u|<\delta\}} |f'(t)| dt.$$

For m and k, positive integers, $k \le m$, define the factorial kth power of m by

$$m^{[k]} := m(m-1)\cdots(m-k+1).$$

For other real values of m and k we put $m^{[k]} := 0$. If k is an integer $k \ge 1$ and I an interval in the real line, the "diagonal of I^k " is the set:

$$D_k(I) := \{(t_1, \dots, t_k) \in I^k,\ t_j = t_h \text{ for some pair } (j,h),\ j \neq h\}.$$

Finally, assume that $X = \{X_t : t \in \mathbb{R}\}$ is a real valued stochastic process with C^1 paths. We set , for $(t_1, \dots, t_k) \in I^k \backslash D_k(I)$ and $x_j \in \mathbb{R}$ $(j = 1, \dots, k)$:

$$\begin{aligned} A_{t_1,\dots t_k}(x_1, \dots x_k) := \int_{\mathbb{R}^k} \Big[\prod_{j=1}^{k} |x'_j|\Big] p_{X_{t_1},\dots,X_{t_k},X'_{t_1},\dots,X'_{t_k}} \\ \times (x_1, \dots x_k, x'_1, \dots x'_k) dx'_1 \dots dx'_k \end{aligned}$$

and

$$I_k(x_1, \dots x_k) := \int_{I^k} A_{t_1,\dots t_k}(x_1, \dots x_k) dt_1 \dots dt_k,$$

where it is understood that the density in the integrand of the definition of $A_{t_1,\dots t_k}$ $(x_1, \dots x_k)$ exists almost everywhere and that the integrals above can take the value $+\infty$.

Proposition 2.2. *Let k be a positive integer, u a real number and I a bounded interval in the line. With the above notations and conditions, let us assume that the process X also satisfies the following conditions:*

1. *the density*
$$p_{X_{t_1},...,X_{t_k},X'_{s_1},...,X'_{s_k}}(x_1,...x_k,x'_1,...x'_k)$$
exists for $(t_1,...t_k),(s_1,...s_k) \in I^k \backslash D_k(I)$ and is a continuous function of $(t_1,...t_k)$ and of $x_1,...x_k$ at the point $(u,...,u)$.
2. *the function*
$$(t_1,...,t_k,x_1,...x_k) \longrightarrow A_{t_1,...t_k}(x_1,...x_k)$$
is continuous for $(t_1,...,t_k) \in I^k \backslash D_k(I)$ and $x_1,...x_k$ belonging to a neighbourhood of u.
3. *(additional technical condition)*
$$\int_{\mathbb{R}^3} |x'_1|^{k-1}|x'_2-x'_3| p_{X_{t_1},...,X_{t_k},X'_{s_1},X'_{s_2},X'_{t_1}}(x_1,...x_k,x'_1,x'_2,x'_3)dx'_1dx'_2dx'_3 \longrightarrow 0$$
as $|s_2-t_1| \longrightarrow 0$, uniformly as $(t_1,...,t_k)$ varies in a compact subset of $I^k\backslash D_k(I)$ and $x_1,...,x_k$ in a fixed neighbourhood of u.

Then,
$$E((N_u(X,I))^{[k]}) = I_k(u,...,u). \qquad (2)$$
Both members in (2) may be $+\infty$

Remarks. (a) For $k=1$ formula (2) becomes
$$E[N_u(X;I)] = \int_I dt \int_{-\infty}^{+\infty} |x'| p_{X_t,X'_t}(u,x')dx'. \qquad (3)$$
(b) Simple variations of (3), valid under the same hypotheses are:
$$E[U_u(X;I)] = \int_I dt \int_0^{+\infty} x' p_{X_t,X'_t}(u,x')dx' \qquad (4)$$
$$E[D_u(X;I)] = \int_I dt \int_{-\infty}^{0} |x'| p_{X_t,X'_t}(u,x')dx'. \qquad (5)$$
In the same way one can obtain formulae for the factorial moments of “marked crossings”, that is, crossings such that some additional condition holds true. For example, if $Y = \{Y_t : t \in \mathbb{R}\}$ is some other stochastic process with real values such that for every t , (Y_t, X_t, X'_t) admit a joint density, $-\infty \le a < b \le +\infty$ and
$$N_u^{a,b}(X,I) := \sharp\{t : t \in I,\ X_t = u,\ a < Y_t < b\}.$$
Then
$$E[N_u^{a,b}(X;I)] = \int_a^b dy \int_I dt \int_{-\infty}^{+\infty} |x'| p_{Y_t,X_t,X'_t}(y,u,x')dx'. \qquad (6)$$

In particular, if $M^+_{a,b}$ is the number of strict local maxima of $X_{(.)}$ on the interval I such that the value of $X_{(.)}$ lies in the interval (a, b), then $M^+_{a,b} = D^{a,b}_0(X', I)$ and:

$$E[M^+_{a,b}] = \int_a^b dy \int_I dt \int_{-\infty}^0 |x''| p_{X_t, X'_t, X''_t}(x, 0, x'') dx''. \tag{7}$$

Sufficient conditions for the validity of (6) and (7) are similar to those for 3.

(c) Proofs of (2) for Gaussian processes satisfying certain conditions can be found in Belayev (1966) and Cramér-Leadbetter (1967). Marcus (1977) contains various extensions. The present statement of Proposition 2.2 is from Wschebor (1985).

(d) It may be non trivial to verify the hypotheses of Proposition 2.2. However some general criteria are available. For example if X is a Gaussian process with $\mathscr{C}^1$ paths and the densities

$$p_{X_{t_1},\ldots,X_{t_k},X'_{s_1},\ldots,X'_{s_k}}$$

are non-degenerate for $(t_1, \ldots t_k)$, $(s_1, \ldots s_k) \in I^k \backslash D_k$, then conditions 1, 2, 3 of Proposition 2.2 hold true (cf Wschebor, 1985, p.37 for a proof and also for some manageable sufficient conditions in non-Gaussian cases).

(e) Another point related to Rice formulae is the non existence of local extrema at a given level. We mention here two well-known results:

Proposition 2.3 (Bulinskaya, 1961). *Suppose that X has $\mathscr{C}^1$ paths and that for every $t \in I$, X_t has a density $p_{X_t}(x)$ bounded for x in a neighbourhood of u.*

Then, almost surely, X has no tangencies at the level u, in the sense that if

$$T^X_u := \{t \in I, X_t = u, X'_t = 0\},$$

then $P(T^X_u = \emptyset) = 1$.

Proposition 2.4 (Ylvisaker's Theorem, 1968). *Suppose that $\{X_t : t \in T\}$ is a real-valued Gaussian process with continuous paths, defined on a compact separable topological space T and that $Var(X_t) > 0$ for every $t \in T$. Then, for each $u \in \mathbb{R}$, with probability 1, the function $t \to X_t$ does not have any local extrema with value u.*

3. Proofs and related results

Let ξ be a random variable with values in $\mathbb{R}^k$ with a distribution that admits a density with respect to the Lebesgue measure λ. The density will be denoted by $p_\xi(.)$. Further, suppose E is an event. It is clear that the measure

$$\mu_\xi(B; E) := P(\{\xi \in B\} \cap E)$$

defined on the Borel sets B of $\mathbb{R}^k$, is also absolutely continuous with respect to λ . We will denote the "density of ξ related to E" the Radon derivative:

$$p_\xi(x; E) := \frac{d\mu_\xi(.; E)}{d\lambda}(x).$$

It is obvious that $p_\xi(x; E) \leq p_\xi(x)$ for λ-almost every $x \in \mathbb{R}^k$.

Theorem 3.1. *Suppose that X has $\mathscr{C}^2$ paths, that X, X', X'' admit a joint density at every time t, that for every t, X'_t has a bounded density $p_{X'_t}(.)$ and that the function*

$$I(x,z) := \int_0^1 dt \int_{-\infty}^0 |x''| p_{X_t, X'_t, X''_t}(x, z, x'') dx''$$

is uniformly continuous in z for (x,z) in some neighbourhood of $(u,0)$. Then the distribution of M admits a density $p_M(.)$ satisfying a.e.

$$\begin{aligned} p_M(u) \le\ & p_{X_0}(u; X'_0 < 0) + p_{X_1}(u; X'_1 > 0) \\ & + \int_0^1 dt \int_{-\infty}^0 |x''| p_{X_t, X'_t, X''_t}(u, 0, x'') dx''. \end{aligned} \tag{8}$$

Proof . Let $u \in \mathbb{R}$ and $h > 0$. We have

$$\begin{aligned} & P(M \le u) - P(M \le u-h) = P(u-h < M \le u) \\ & \quad \le P(u-h < X_0 \le u, X'_0 < 0) + P(u-h < X_1 \le u, X'_1 > 0) \\ & \qquad + P(M^+_{u-h,u} > 0), \end{aligned}$$

where $M^+_{u-h,u} = M^+_{u-h,u}(0,1)$, since if $u-h < M \le u$, then either the maximum occurs in the interior of the interval $[0,1]$ or at 0 or 1, with the derivative taking the indicated sign. Note that

$$P(M^+_{u-h,u} > 0) \le E(M^+_{u-h,u}).$$

Using Proposition 2.3, with probability 1, $X'(.)$ has no tangencies at the level 0, thus an upper bound for this expectation follows from the Kac's formula:

$$M^+_{u-h,u} = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^1 \mathbb{1}_{\{X(t) \in [u-h,u]\}} \mathbb{1}_{\{X'(t) \in [-\delta,\delta]\}} \mathbb{1}_{\{X''(t) < 0\}} |X''(t)| dt \qquad a.s.$$

which together with Fatou's lemma imply:

$$E(M^+_{u-h,u}) \le \liminf_{\delta \longrightarrow 0} \frac{1}{2\delta} \int_{-\delta}^{\delta} dz \int_{u-h}^{u} I(x,z) dx = \int_{u-h}^{u} I(x,0) dx.$$

Combining this bound with the preceeding one, we get

$$\begin{aligned} & P(M \le u) - P(M \le u-h) \\ & \le \int_{u-h}^{u} \left[p_{X_0}(x; X'_0 < 0) + p_{X_1}(x; X'_1 > 0) + I(x,0) \right] dx, \end{aligned}$$

which gives the result.

In spite of the simplicity of the proof, this theorem provides the best known upper-bound for Gaussian processes. In fact, in this case, formula (8) is a simpler expression of the bound of Diebolt and Posse (1996). More precisely, if we use their parametrization by putting

$$m(t) = 0 \ ; \ r(s,t) = \frac{\rho(s,t)}{\tau(s)\tau(t)},$$

with

$$\rho(t,t)=1,\ \rho_{11}(t,t)=1,\ \rho_{10}(t,t)=0,\ \rho_{12}(t,t)=0,\ \rho_{02}(t,t)=-1,$$

after some calculations, we get exactly their bound $M(u)$ (their formula (9)) for the density of the maximum.

Let us illustrate formula (8) explicitly when the process is Gaussian, centered with unit variance. By means of a deterministic time change, one can also assume that the process has "unit speed" ($Var(X'_t)\equiv 1$). Let L the length of the new time interval. Clearly $\forall\ t,\ m(t)=0,\ r(t,t)=1,\ r_{11}(t,t)=1,\ r_{10}(t,t)=\ 0,$ $r_{12}(t,t)=\ 0, r_{02}(t,t)=-1$. Note that

$$Z\sim N(\mu,\sigma^2)\Rightarrow E(Z^-)=\sigma\phi(\mu/\sigma)-\mu\Phi(-\mu/\sigma).$$

The formulae for regression imply that conditionally on $X_t=u$, $X'_t=0$, X''_t has expectation $-u$ and variance $r_{22}(t,t)-1$. Formula (8) reduces to

$$p_M(u)\le p^+(u):=\phi(u)\left[1+(2\pi)^{-1/2}\int_0^L C_g(t)\phi(u/C_g(t))+u\Phi(u/C_g(t))dt\right],$$

with $C_g(t):=\sqrt{r_{22}(t,t)-1}$

As $x\to+\infty$, $\Phi(x)=1-\frac{\phi(x)}{x}+\frac{\phi(x)}{x^3}+O\left(\frac{\phi(x)}{x^5}\right)$. This implies that

$$\begin{aligned}p^+(u)=\phi(u)&\left[1+Lu(2\pi)^{-1/2}+(2\pi)^{-1/2}u^{-2}\int_0^L C_g^3(t)\phi(u/C_g(t))dt\right]\\&+O\left(u^{-4}\phi(u/C^+)\right),\end{aligned}$$

with $C^+:=\sup_{t\in[0,L]}C_g(t)$.

Furthermore the exact equivalent of $p_M(u)$ when $u\to+\infty$ is

$$(2\pi)^{-1}\ u\ L\ exp(-u^2/2)$$

as we will see in Corollary 1.1.

The following theorem is a special case of Lemma 3.3. We state it separately since we use it below to compare the results that follow from it with known results.

Theorem 3.2. *Suppose that X is a Gaussian process satisfying H_2. Then M has a continuous density p_M given for every u by*

$$\begin{aligned}p_M(u)=&\ p_{X_0}(u^-;M\le u)+p_{X_1}(u^-;M\le u)\\&+\int_0^1 dt\int_{-\infty}^0|x''|p_{X_t,X'_t,X''_t}(u^-,0,x'';M\le u)dx'',\end{aligned}\qquad(9)$$

where $p_{X_0}(u^-;M\le u)=\lim_{x\uparrow u}p_{X_0}(x;M\le u)$ exists and is a continuous function of u , as well as $p_{X_1}(u^-;M\le u)$ and $p_{X_t,X'_t,X''_t}(u^-,0,x'';M\le u)$.

Again, we obtain a simpler version of the expression by Diebolt and Posse (1996).

In fact, the result 3.2 remains true if X is Gaussian with $\mathscr{C}^2$ paths and one requires only that X_s, X_t, X'_t, X''_t admit a joint density for all $s, t, s \neq t \in [0,1]$.

If we replace the event $\{M \leq u\}$ respectively by $\{X'_0 < 0\}$, $\{X'_1 > 0\}$ and Ω in each of the three terms in the right hand member in formula (9) we get the general upper-bound given by (8).

To obtain lower bounds for $p_M(u)$, we use the following immediate inequalities:

$$\begin{aligned} P(M \leq u / X_0 = u) &= P(M \leq u, X'_0 < 0 / X_0 = u) \\ &\geq P\left(X'_0 < 0 / X_0 = u\right) \\ &\quad - E(U_u[0,1]\mathbf{1}_{\{X'_0<0\}} / X_0 = u). \end{aligned}$$

In the same way

$$\begin{aligned} P(M \leq u / X_1 = u) &= P(M \leq u, X'_1 > 0 / X_1 = u) \\ &\geq P\left(X'_1 > 0 / X_1 = u\right) \\ &\quad - E(D_u[0,1]\mathbf{1}_{\{X'_1>0\}} / X_1 = u) \end{aligned}$$

and if $x'' < 0$:

$$\begin{aligned} &P(M \leq u / X_t = u, X'_t = 0, X''_t = x'') \\ &\quad \geq 1 - E([D_u([0,t]) + U_u([t,1])] / X_t = u, X'_t = 0, X''_t = x''). \end{aligned}$$

If we plug these lower bounds into Formula (9) and replace the expectations of upcrossings and downcrossings by means of integral formulae of (4), (5) type, we obtain the lower bound:

$$\begin{aligned} &p_M(u) \geq p_{X_0}(u; X'_0 < 0) + p_{X_1}(u; X'_1 < 0) \\ &+ \int_0^1 dt \int_{-\infty}^0 |x''| p_{X_t, X'_t, X''_t}(u, 0, x'') dx'' \\ &- \int_0^1 ds \int_{-\infty}^0 dx' \int_0^{+\infty} x'_s p_{X_s, X'_s, X_0, X'_0}(u, x'_s, u, x') dx'_s \\ &- \int_0^1 dt \int_{-\infty}^0 |x''| \left[\begin{array}{l} \int_0^t ds \int_{-\infty}^0 |x'| p_{X_s, X'_s, X_t, X'_t, X''_t}(u, x', u, 0, x'') dx' \\ + \int_t^1 ds \int_0^{+\infty} x' p_{X_s, X'_s, X_t, X'_t, X''_t}(u, x', u, 0, x'') dx' \end{array} \right] dx''. \end{aligned} \tag{10}$$

Simpler expressions for (10) also adapted to numerical computations, can be found in Cierco (1996).

Finally, some sharper upperbounds for $p_M(u)$ are obtained when replacing the event $\{M > u\}$ by $\{X_0 + X_1 > 2u\}$, the probability of which can be expressed using the conditionnal expectation and variance of $X_0 + X_1$; we are able only to express these bounds in integral form.

We now turn to the proofs of our main results.

Lemma 3.1. *(a) Let Z be a stochastic process satisfying H_k $(k \geq 2)$ and t a point in $[0,1]$. Define the Gaussian processes $Z^{\vdash}, Z^{\dashv}, Z^t$ by means of the orthogonal decompositions:*

$$Z_s = a^{\vdash}(s)\, Z_0 + s Z_s^{\vdash} \quad s \in (0,1]. \tag{11}$$

$$Z_s = a^{\dashv}(s)\, Z_1 + (1-s)\, Z_s^{\dashv} \quad s \in [0,1). \tag{12}$$

$$Z_s = b^t(s) Z_t + c^t(s)\, Z_t' + \frac{(s-t)^2}{2} Z_s^t \quad s \in [0,1] \;\; s \neq t. \tag{13}$$

Then, the processes $Z^{\vdash}, Z^{\dashv}, Z^t$ can be extended defined at $s=0, s=1, s=t$ respectively so that they become pathwise continuous and satisfy $H_{k-1}, H_{k-1}, H_{k-2}$ respectively.

(b) Let f be any function of class C^k. When there is no ambiguity on the process Z, we will define $f^{\vdash}, f^{\dashv}, f^t$ in the same manner, putting f instead of Z in (11), (12), (13), but still keeping the regression coefficients corresponding to Z. Then $f^{\vdash}, f^{\dashv}, f^t$ can be extended by continuity in the same way to functions in $C^{k-1}, C^{k-1}, C^{k-2}$ respectively.

(c) Let m be a positive integer, suppose Z satisfies H_{2m+1} and $t_1, ..., t_m$ belong to $[0,1] \cup \{\vdash, \dashv\}$. Denote by $Z^{t_1,...,t_m}$ the process obtained by repeated application of the operation of part (a) of this Lemma, that is

$$Z_s^{t_1,...,t_m} = \left(Z^{t_1,...,t_{m-1}}\right)_s^{t_m}.$$

Denote by $s_1, ..., s_p$ $(p \leq m)$ the ordered p-tuple of the elements of $t_1, ..., t_m$ that belong to $[0,1]$ (i.e. they are not "$\vdash$" or "$\dashv$"). Then, a.s. for fixed values of the symbols "$\vdash$", "$\dashv$" the application:

$$\left(s_1, ..., s_p, s\right) \to \left(Z_s^{t_1,...,t_m}, \left(Z^{t_1,...,t_m}\right)_s'\right)$$

is continuous.

Proof. (a) and (b) follow in a direct way, computing the regression coefficients $a^{\vdash}(s)$, $a^{\dashv}(s)$, $b^t(s)$, $c^t(s)$ and substituting into formulae (11), (12), (13). Note that (b) also follows from (a) by applying it to $Z+f$ and to Z. We prove now (c) which is a consequence of the following:
Suppose $Z(t_1, ..., t_k)$ is a Gaussian field with C^p sample paths $(p \geq 2)$ defined on $[0,1]^k$ with no degeneracy in the same sense that in the definition of hypothesis H_k (3) for one-parameter processes. Then the Gaussian fields defined by means of:

$$Z^{\vdash}(t_1, ..., t_k) = (t_k)^{-1}\left(Z(t_1, ..., t_{k-1}, t_k) - a^{\vdash}(t_1, ..., t_k) Z(t_1, ..., t_{k-1}, 0)\right) \quad \text{for } t_k \neq 0,$$

$$Z^{\dashv}(t_1, ..., t_k) = (1-t_k)^{-1}\left(Z(t_1, ..., t_{k-1}, t_k) - a^{\dashv}(t_1, ..., t_k) Z(t_1, ..., t_{k-1}, 1)\right) \quad \text{for } t_k \neq 1,$$

$$\begin{aligned}\widetilde{Z}(t_1, ..., t_k, t_{k+1}) = 2\,(t_{k+1} - t_k)^{-2}\,(&Z(t_1, ..., t_{k-1}, t_{k+1}) \\ &- b(t_1, ..., t_k, t_{k+1}) Z(t_1, ..., t_k) \\ &- c(t_1, ..., t_k, t_{k+1}) \frac{\partial Z}{\partial t_k}(t_1, ..., t_k)) \qquad \text{for } t_{k+1} \neq t_k\end{aligned}$$

can be extended to $[0,1]^k$ (respectively $[0,1]^k$, $[0,1]^{k+1}$) into fields with paths in C^{p-1} (respectively C^{p-1}, C^{p-2}). In the above formulae,

- $a^{\vdash}(t_1, ..., t_k)$ is the regression coefficient of $Z(t_1, ..., t_k)$ on $Z(t_1, ..., t_{k-1}, 0)$,
- $a^{\dashv}(t_1, ..., t_k)$ is the regression coefficient of $Z(t_1, ..., t_k)$ on $Z(t_1, ..., t_{k-1}, 1)$,
- $b(t_1, ..., t_k, t_{k+1})$, $c(t_1, ..., t_k, t_{k+1})$ are the regression coefficients of $Z(t_1, ..., t_{k-1}, t_{k+1})$ on the pair $\left(Z(t_1, ..., t_k), \frac{\partial Z}{\partial t_k}(t_1, ..., t_k)\right)$.

Let us prove the statement on $\widetilde{Z}$. The other two are simpler. Denote by V the subspace of $L^2(\Omega, \Im, P)$ generated by the pair $\left(Z(t_1, ..., t_k), \frac{\partial Z}{\partial t_k}(t_1, ..., t_k)\right)$. Denote by $\Pi_{V^\perp}$ the version of the orthogonal projection of $L^2(\Omega, \Im, P)$ on the orthogonal complement of V, defined by means of.

$$\Pi_{V^\perp}(Y) := Y - \left[bZ(t_1, ..., t_k) + c\frac{\partial Z}{\partial t_k}(t_1, ..., t_k)\right],$$

where b and c are the regression coefficients of Y on the pair

$$Z(t_1, ..., t_k), \frac{\partial Z}{\partial t_k}(t_1, ..., t_k).$$

Note that if $\{Y_\theta : \theta \in \Theta\}$ is a random field with continuous paths and such that $\theta \to Y_\theta$ is continuous in $L^2(\Omega, \Im, P)$, then a.s.

$$\left(\theta, t_1, ..., t_k\right) \to \Pi_{V^\perp}(Y_\theta)$$

is continuous.

From the definition:

$$\widetilde{Z}(t_1, ..., t_k, t_{k+1}) = 2\,(t_{k+1} - t_k)^{-2}\,\Pi_{V^\perp}\left(Z(t_1, ..., t_{k-1}, t_{k+1})\right).$$

On the other hand, by Taylor's formula:

$$Z(t_1, ..., t_{k-1}, t_{k+1}) = Z(t_1, ..., t_k) + (t_{k+1} - t_k)\,\frac{\partial Z}{\partial t_k}(t_1, ..., t_k) + R_2\,(t_1, ..., t_k, t_{k+1})$$

with

$$R_2\,(t_1, ..., t_k, t_{k+1}) = \int_{t_k}^{t_{k+1}} \frac{\partial^2 Z}{\partial t_k^2}(t_1, ..., t_{k-1}, \tau)\,(t_{k+1} - \tau)\,d\tau$$

so that

$$\widetilde{Z}(t_1, ..., t_k, t_{k+1}) = \Pi_{V^\perp}\left[2\,(t_{k+1} - t_k)^{-2}\,R_2\,(t_1, ..., t_k, t_{k+1})\right]. \qquad (14)$$

It is clear that the paths of the random field $\widetilde{Z}$ are $p-1$ times continuously differentiable for $t_{k+1} \neq t_k$. Relation (14) shows that they have a continuous extension to $[0,1]^{k+1}$ with $\widetilde{Z}(t_1, ..., t_k, t_k) = \Pi_{V^\perp}\left(\frac{\partial^2 Z}{\partial t_k^2}(t_1, ..., t_k)\right)$. In fact,

$$\Pi_{V^\perp}\left(2\,(s_{k+1} - s_k)^{-2}\,R_2\,(s_1, ..., s_k, s_{k+1})\right)$$

$$= 2\,(s_{k+1} - s_k)^{-2} \int_{s_k}^{s_{k+1}} \Pi_{V^\perp}\left(\frac{\partial^2 Z}{\partial t_k^2}(s_1, ..., s_{k-1}, \tau)\right)(s_{k+1} - \tau)\, d\tau.$$

According to our choice of the version of the orthogonal projection $\Pi_{V^\perp}$, a.s. the integrand is a continuous function of the parameters therein so that, a.s.:

$$\widetilde{Z}\,(s_1, ..., s_k, s_{k+1}) \to \Pi_{V^\perp}\left(\frac{\partial^2 Z}{\partial t_k^2}(t_1, ..., t_k)\right) \text{when } (s_1, ..., s_k, s_{k+1}) \to (t_1, ..., t_k, t_k).$$

This proves (c). In the same way, when $p \geq 3$, we obtain the continuity of the partial derivatives of $\widetilde{Z}$ up to the order p$-$2.

The following lemma has its own interest besides being required in our proof of Lemma 3.3. It is a slight improvement of Lemma 4.3, p. 76 in Piterbarg (1996) in the case of one-parameter processes.

Lemma 3.2. *Suppose that X is a Gaussian process with $\mathcal{C}^3$ paths and that for all $s \neq t$, the distributions of X_s, X'_s, X_t, X'_t and of $X_t, X'_t, X_t^{(2)}, X_t^{(3)}$ do not degenerate. Then, there exists a constant K (depending on the process) such that*

$$p_{X_s, X_t, X'_s, X'_t}(x_1, x_2, x'_1, x'_2) \leq K(t-s)^{-4}$$

for all $x_1, x_2, x'_1, x'_2 \in \mathbb{R}$ and all $s, t, s \neq t \in [0, 1]$.

Proof .

$$p_{X_s, X_t, X'_s, X'_t}(x_1, x_2, x'_1, x'_2) \leq (2\pi)^{-2}\left[DetVar(X_s, X_t, X'_s, X'_t)\right]^{-1/2},$$

where $DetVar$ stands for the determinant of the variance matrix. Since by hypothesis the distribution does not degenerate outside the diagonal $s = t$, the conclusion of the lemma is trivially true on a set of the form $\{|s-t| > \delta\}$, $\delta > 0$. By a compactness argument it is sufficient to prove it for s, t in a neighbourhood of (t_0, t_0) for each $t_0 \in [0, 1]$. For this last purpose we use a generalization of a technique employed by Belyaev (1966). Since the determinant is invariant by adding linear combination of rows (resp. columns) to another row (resp. column),

$$DetVar(X_s, X_t, X'_s, X'_t) = DetVar(X_s, X'_s, \tilde{X}_s^{(2)}, \tilde{X}_s^{(3)}),$$

with

$$\tilde{X}_s^{(2)} = X_t - X_s - (t-s)X'_s \simeq \frac{(t-s)^2}{2} X_{t_0}^{(2)}$$

$$\tilde{X}_s^{(3)} = X'_t - X'_s - \frac{2}{(t-s)}\tilde{X}_s^{(2)} \simeq \frac{(t-s)^2}{6} X_{t_0}^{(3)},$$

The equivalence refers to $(s, t) \to (t_0, t_0)$. Since the paths of X are of class $\mathcal{C}^3$, $\left(X_s, X'_s, (2(t-s)^{-2})\tilde{X}_s^{(2)}, (6(t-s)^{-2})\tilde{X}_s^{(3)}\right)$ tends almost surely to

$\left(X_{t_0}, X'_{t_0}, X^{(2)}_{t_0}, X^{(3)}_{t_0}\right)$ as $(s,t) \to (t_0,t_0)$. This implies the convergence of the variance matrices. Hence

$$DetVar(X_s, X_t, X'_s, X'_t) \simeq \frac{(t-s)^8}{144} DetVar(X_{t_0}, X'_{t_0}, X^{(2)}_{t_0}, X^{(3)}_{t_0}),$$

which ends the proof.

Remark. the proof of Lemma 3.2 shows that the density of X_s, X'_s, X_t, X'_t exists for $|s-t|$ sufficiently small as soon as the process has $\mathscr{C}^3$ paths and for every t the distribution of $X_t, X'_t, X''_t, X^{(3)}_t$ does not degenerate. Hence, under this only hypothesis, the conclusion of the lemma holds true for $0 < |s-t| < \eta$ and some $\eta > 0$.

Lemma 3.3. *Suppose $Z = \{Z_t : t \in [0,1]\}$ is a stochastic process that verifies H_2. Define:*

$$F_v(u) = E\left(\xi_v . \mathbb{1}_{A_u}\right)$$

where

- $A_u = A_u(Z,\beta) = \{Z_t \le \beta(t)\, u \ \ for\ all\ t \in [0,1]\}$,
- *$\beta(.)$ is a real valued C^2 function defined on $[0,1]$,*
- *$\xi_v = G(Z_{t_1} - \beta(t_1)v, \dots, Z_{t_m} - \beta(t_m)v)$ for some positive integer m, $t_1, \dots, t_m \in [0,1]$, $v \in \mathbb{R}$ and some C^∞ function $G : \mathbb{R}^m \to \mathbb{R}$ having at most polynomial growth at ∞, that is, $|G(x)| \le C(1+\|x\|^p)$ for some positive constants C, p and all $x \in \mathbb{R}^m$ ($\|\ .\ \|$ stands for Euclidean norm).*

Then,

For each $v \in \mathbb{R}$, F_v is of class C^1 and its derivative is a continuous function of the pair (u,v) that can be written in the form:

$$\begin{aligned} F'_v(u) = {} & \beta(0) E\left(\xi^{\vdash}_{v,u} . \mathbb{1}_{A_u(Z^{\vdash},\beta^{\vdash})}\right) . p_{Z_0}(\beta(0).u) \\ & + \beta(1) E\left(\xi^{\dashv}_{v,u} . \mathbb{1}_{A_u(Z^{\dashv},\beta^{\dashv})}\right) p_{Z_1}(\beta(1).u) \\ & - \int_0^1 \beta(t) E\left(\xi^t_{v,u}\left(Z^t_t - \beta^t(t).u\right) \mathbb{1}_{A_u(Z^t,\beta^t)}\right) \\ & \times p_{Z_t, Z'_t}\left(\beta(t).u, \beta'(t).u\right) dt, \end{aligned} \tag{15}$$

where the processes $Z^{\vdash}$, $Z^{\dashv}$, Z^t and the functions $\beta^{\vdash}$, $\beta^{\dashv}$, β^t are as in Lemma 3.1 and the random variables $\xi^{\vdash}_{v,u}$, $\xi^{\dashv}_{v,u}$, $\xi^t_{v,u}$ are given by:

$$\begin{aligned} \xi^{\vdash}_{v,u} = G\big[& t_1\left(Z^{\vdash}_{t_1} - \beta^{\vdash}(t_1)\,u\right) + \beta(t_1)(u-v), \dots \\ & \dots, t_m\left(Z^{\vdash}_{t_m} - \beta^{\vdash}(t_m)\,u\right) + \beta(t_m)(u-v)\big] \end{aligned}$$

$$\begin{aligned} \xi^{\dashv}_{v,u} = G\big[& (1-t_1)\left(Z^{\dashv}_{t_1} - \beta^{\dashv}(t_1)\,u\right) + \beta(t_1)(u-v), \dots \\ & \dots, (1-t_m)\left(Z^{\dashv}_{t_m} - \beta^{\dashv}(t_m)\,u\right) + \beta(t_m)(u-v)\big] \end{aligned}$$

$$\xi^t_{v,u} = G\Big[\frac{(t_1-t)^2}{2}\left(Z^t_{t_1} - \beta^t\,(t_1)\,u\right) + \beta\,(t_1)\,(u-v), \ldots$$
$$\ldots, \frac{(t_m-t)^2}{2}\left(Z^t_{t_m} - \beta^t\,(t_m)\,u\right) + \beta\,(t_m)\,(u-v)\Big].$$

Proof. We start by showing that the arguments of Theorem 3.1 can be extended to our present case to establish that F_v is absolutely continuous. This proof already contains a first approximation to the main ideas leading to the proof of the lemma.

Step 1 Assume - with no loss of generality - that $u \geq 0$ and write for $h > 0$:

$$F_v(u) - F_v(u-h) = E\left(\xi_v.\mathbf{1}_{A_u \setminus A_{u-h}}\right\} - E\left(\xi_v.\mathbf{1}_{A_{u-h} \setminus A_u}\right\} \tag{16}$$

Note that:

$$A_u \setminus A_{u-h} \subset \{\beta(0)(u-h) < Z_0 \leq \beta(0)u,\, \beta(0) > 0\}$$
$$\cup\{\beta(1)(u-h) < Z_1 \leq \beta(1)u,\, \beta(1) > 0\} \cup \left\{M^{(1)}_{u-h,u} \geq 1\right\} \tag{17}$$

where:

$$M^{(1)}_{u-h,u} = \sharp\{t : t \in (0,1),\ \beta(t) \geq 0,\ \text{the function } Z_{(.)} - \beta(.)(u-h)$$

has a local maximum at t with value falling in the interval $[0, \beta(t)h]\}$.

Using the Markov inequality

$$P(M^{(1)}_{u-h,u} \geq 1) \leq E\left(M^{(1)}_{u-h,u}\right),$$

and the formula for the expectation of the number of local maxima applied to the process $t \to Z_t - \beta(t)(u-h)$ imply

$$\begin{aligned}
&|E\left(\xi_v.\mathbf{1}_{A_u \setminus A_{u-h}}\right)| \\
&\leq \mathbf{1}_{\{\beta(0)>0\}} \int_{\beta(0)(u-h)}^{\beta(0)u} E\left(|\xi_v|/Z_0 = x\right) p_{Z_o}(x)dx \\
&\quad + \mathbf{1}_{\{\beta(1)>0\}} \int_{\beta(1)(u-h)}^{\beta(1)u} E\left(|\ \xi_v\ |\ /Z_1 = x\right) p_{Z_1}(x)dx \\
&\quad + \int_0^1 \mathbf{1}_{\{\beta(t)>0\}} dt \int_0^{\beta(t)h} E\left(|\xi_v|(Z''_t - \beta''(t)(u-h))^- / V_2 = (x,0)\right) \\
&\quad .p_{V_2}(x,0)dx,
\end{aligned} \tag{18}$$

where V_2 is the random vector

$$\left(Z_t - \beta(t)(u-h),\, Z'_t - \beta'(t)(u-h)\right).$$

Now, the usual regression formulae and the form of ξ_v imply that

$$|E\left(\xi_v.\mathbf{1}_{A_u \setminus A_{u-h}}\right)| \leq (const).h$$

where the constant may depend on u but is locally bounded as a function of u.

An analogous computation replacing $M^{(1)}_{u-h,u}$ by

$$M^{(2)}_{u-h,u} = \sharp\{t : t \in (0,1),\ \beta(t) \le 0,\ \text{the function } Z_{(.)} - \beta(.)u \text{ has a local maximum at } t,\ Z_t - \beta(t)u \in [0, -\beta(t)h]\}$$

leads to a similar bound for the second term in (16). It follows that

$$|F_v(u) - F_v(u-h)| \le (const).h$$

where the constant is locally bounded as a function of u. This shows that F_v is absolutely continuous.

The proof of the Lemma is in fact a refinement of this type of argument. We will replace the rough inclusion (17) and its consequence (18) by an equality.

In the two following steps we will assume the additional hypothesis that Z verifies H_k for every k and $\beta(.)$ is a C^∞ function.

Step 2.

Notice that:

$$\begin{aligned} A_u \setminus A_{u-h} \ = \ A_u \ \cap \Big[&\{\beta(0)(u-h) < Z_0 \le \beta(0)u, \beta(0) > 0\} \\ &\cup \{\beta(1)(u-h) < Z_1 \le \beta(1)u, \beta(1) > 0\} \cup \{M^{(1)}_{u-h,u} \ge 1\}\Big]. \end{aligned} \tag{19}$$

We use the obvious inequality, valid for any three events F_1, F_2, F_3:

$$\sum_1^3 \mathbf{1}_{F_j} - \mathbf{1}_{\cup_1^3 F_j} \ \le \ \mathbf{1}_{F_1\cap F_2} + \mathbf{1}_{F_2\cap F_3} + \mathbf{1}_{F_3\cap F_1}$$

to write the first term in (16) as:

$$\begin{aligned} E\left(\xi_v.\mathbf{1}_{A_u\setminus A_{u-h}}\right) = \ & E\left(\xi_v.\mathbf{1}_{A_u}\mathbf{1}_{\{\beta(0)(u-h)<Z_0\le\beta(0)u\}}\right)\mathbf{1}_{\{\beta(0)>0\}} \\ & + E\left(\xi_v.\mathbf{1}_{A_u}\mathbf{1}_{\{\beta(1)(u-h)<Z_1\le\beta(1)u\}}\right)\mathbf{1}_{\{\beta(1)>0\}} \\ & + E\left(\xi_v.\mathbf{1}_{A_u} M^{(1)}_{u-h,u}\right) + R_1(h) \end{aligned} \tag{20}$$

where

$$\begin{aligned} |R_1(h)| \le \ & E\left(|\xi_v|\mathbf{1}_{\{\beta(0)(u-h)<Z_0\le\beta(0)u,\beta(1)(u-h)<Z_1\le\beta(1)u\}}\right)\mathbf{1}_{\{\beta(0)>0,\beta(1)>0\}} \\ & + E\left(|\xi_v|\mathbf{1}_{\left\{\beta(0)(u-h)<Z_0\le\beta(0)u, M^{(1)}_{u-h,u}\ge 1\right\}}\right)\mathbf{1}_{\{\beta(0)>0\}} \\ & + E\left(|\xi_v|\mathbf{1}_{\left\{\beta(1)(u-h)<Z_1\le\beta(1)u, M^{(1)}_{u-h,u}\ge 1\right\}}\right)\mathbf{1}_{\{\beta(1)>0\}} \\ & + E\left(|\xi_v|\left(M^{(1)}_{u-h,u} - \mathbf{1}_{M^{(1)}_{u-h,u}\ge 1}\right)\right) = T_1(h) + T_2(h) + T_3(h) + T_4(h) \end{aligned}$$

Our first aim is to prove that $R_1(h) = o(h)$ as $h \downarrow 0$.

It is clear that $T_1(h) = O(h^2)$.

Let us consider $T_2(h)$. Using the integral formula for the expectation of the number of local maxima:

$$T_2(h) \leq \mathbf{1}_{\{\beta(0)>0\}} \int_0^1 \mathbf{1}_{\{\beta(t)\geq 0\}} dt \int_0^{\beta(0)h} dz_0 \int_0^{\beta(t)h} dz.$$
$$.E\left(|\xi_v| (Z_t'' - \beta''(t)(u-h))^- / V_3 = v_3\right) p_{V_3}(v_3),$$

where V_3 is the random vector

$$\Big(Z_0 - \beta(0)(u-h), Z_t - \beta(t)(u-h), Z_t' - \beta'(t)(u-h)\Big),$$

and $v_3 = (z_0, z, 0)$.

We divide the integral in the right-hand member into two terms, respectively the integrals on $[0, \delta]$ and $[\delta, 1]$ in the t-variable, where $0 < \delta < 1$. The first integral can be bounded by

$$\int_0^\delta \mathbf{1}_{\{\beta(t)\geq 0\}} dt \int_0^{\beta(t)h} dz \ E\left(|\xi_v| (Z_t'' - \beta''(t)(u-h))^- / V_2 = (z, 0)\right) p_{V_2}(z, 0).$$

where the random vector V_2 is the same as in (18). Since the conditional expectation as well as the density are bounded for u in a bounded set and $0 < h < 1$, this expression is bounded by $(const)\delta h$.

As for the second integral, when t is between δ and 1 the Gaussian vector

$$\left(Z_0 - \beta(0)(u-h), Z_t - \beta(t)(u-h), Z_t' - \beta'(t)(u-h)\right)$$

has a bounded density so that the integral is bounded by $C_\delta h^2$, where C_δ is a constant depending on δ.

Since $\delta > 0$ is arbitrarily small, this proves that $T_2(h) = o(h)$. $T_3(h)$ is similar to $T_2(h)$.

We now consider $T_4(h)$. Put:

$$E_h = \left\{\| \ Z_{(.)}^{(4)} - \beta^{(4)}(.)(u-h) \ \|_\infty \leq h^{-1/4}\right\} \cap \left\{|\xi_v| \leq h^{-1/4}\right\}$$

where $\| \ . \ \|_\infty$stands for the sup-norm in $[0, 1]$. So,

$$T_4(h) \leq E\left(|\xi_v| \mathbf{1}_{E_h} M_{u-h,u}^{(1)} (M_{u-h,u}^{(1)} - 1)\right) + E\left(|\xi_v| \mathbf{1}_{E_h^C} M_{u-h,u}^{(1)}\right) \tag{21}$$

(E^C denotes the complement of the event E).

The second term in (21) is bounded as follows:

$$E\left(|\xi_v| \mathbf{1}_{E_h^C} M_{u-h,u}^{(1)}\right) \leq \left[E\left(|\xi_v|^4\right) E\left(\left(M_{u-h,u}^{(1)}\right)^4\right)\right]^{1/4} \left(P(E_h^C)\right)^{1/2}.$$

The polynomial bound on G, plus the fact that $\|Z\|_\infty$ has finite moments of all orders, imply that $E\left(|\xi_v|^4\right)$ is uniformly bounded.

Also, $M_{u-h,u}^{(1)} \leq D_0(Z_{(.)}' - \beta'(.)(u-h), [0, 1]) = D$ (recall that $D_0(g; I)$ denotes the number of downcrossings of level 0 by function g). A bound for $E\left(D^4\right)$

can be obtained on applying Lemma 1.2 in Nualart-Wschebor (1991). In fact, the Gaussian process $Z'_{(.)} - \beta'(.)(u-h)$ has uniformly bounded one-dimensional marginal densities and for every positive integer p the maximum over [0, 1] of its p-th derivative has finite moments of all orders. From that Lemma it follows that $E\left(D^4\right)$ is bounded independently of h, $0 < h < 1$.

Hence,

$$
\begin{aligned}
&E\left(|\xi_v| \mathbf{1}_{E_h^C} M^{(1)}_{u-h,u}\right) \\
&\quad \leq (const)\left[P(\| Z^{(4)}_{(.)} - \beta^{(4)}(.)(u-h) \|_\infty > h^{-1/4}) + P(|\xi_v| > h^{-1/4})\right]^{1/2} \\
&\quad \leq (const)\left[C_1 e^{-C_2 h^{-1/2}} + h^{q/4} E\left(|\xi_v|^q\right)\right]^{1/2},
\end{aligned}
$$

where C_1, C_2 are positive constants and q any positive number. The bound on the first term follows from the Landau-Shepp (1971) inequality (see also Fernique, 1974) since even though the process depends on h it is easy to see that the bound is uniform on h, $0 < h < 1$. The bound on the second term is simply the Markov inequality. Choosing $q > 8$ we see that the second term in (21) is $o(h)$.

For the first term in (21) one can use the formula for the second factorial moment of $M^{(1)}_{u-h,u}$ to write it in the form:

$$
\begin{aligned}
&\int_0^1 \int_0^1 \mathbf{1}_{\{\beta(s)\geq 0, \beta(t)\geq 0\}} ds dt \int_0^{\beta(s)h} dz_1 \int_0^{\beta(t)h} dz_2 \\
&E(|\xi_v| \mathbf{1}_{E_h} (Z''_s - \beta''(s)(u-h))^- (Z''_t - \beta''(t)(u-h))^- / V_4 = v_4).p_{V_4}(v_4),
\end{aligned}
\tag{22}
$$

where V_4 is the random vector

$$
\left(Z_s - \beta(s)(u-h), Z_t - \beta(t)(u-h), Z'_s - \beta'(s)(u-h), Z'_t - \beta'(t)(u-h)\right)
$$

and $v_4 = (z_1, z_2, 0, 0)$.

Let $s \neq t$ and Q be the - unique - polynomial of degree 3 such that
$Q(s) = z_1$, $Q(t) = z_2$, $Q'(s) = 0$, $Q'(t) = 0$. Check that

$$
\begin{aligned}
Q(y) &= z_1 + (z_2 - z_1)(y-s)^2(3t - 2y - s)(t-s)^{-3} \\
Q''(t) &= 6(z_1 - z_2)(t-s)^{-2} \\
Q''(s) &= -6(z_1 - z_2)(t-s)^{-2}.
\end{aligned}
$$

Denote, for each positive h,

$$
\zeta(y) := Z_y - \beta(y)(u-h) - Q(y).
$$

Under the conditioning $V_4 = v_4$ in the integrand of (22), the C^∞ function $\zeta(.)$ verifies $\zeta(s) = \zeta(t) = \zeta'(s) = \zeta'(t) = 0$. So, there exist $t_1, t_2 \in (s, t)$ such that $\zeta''(t_1) = \zeta'''(t_2) = 0$ and for $y \in [s, t]$:

$$
|\zeta''(y)| = \left| \int_{t_1}^{y} \zeta'''(\tau) d\tau \right| = \left| \int_{t_1}^{y} d\tau \int_{t_2}^{\tau} \zeta^{(4)}(\sigma) d\sigma \right| \leq \frac{(t-s)^2}{2} \| \zeta^{(4)} \|_\infty .
$$

Noting that $a^- b^- \le \left(\frac{a+b}{2}\right)^2$ for any pair of real numbers a, b, it follows that the conditional expectation in the integrand of (22) is bounded by:

$$\begin{aligned}
&E(|\xi_v|.\mathbf{1}_{E_h}.(t-s)^4(\| Z^{(4)}_{(.)} - \beta^{(4)}(.)(u-h) \|_\infty)^2 / V_4 = v_4) \\
&\quad \le (t-s)^4.h^{-1/2}.h^{-1/4} = (t-s)^4.h^{-3/4}. \qquad (23)
\end{aligned}$$

On the other hand, applying Lemma 3.2 we have the inequality

$$p_{V_4}(z_1, z_2, 0, 0) \le p_{Z_s, Z_t, Z'_s, Z'_t}(0,0,0,0) \le (const)(t-s)^{-4}$$

the constant depending on the process but not on s, t.

Summing up, the expression in (22) is bounded by

$$(const).h^2.h^{-3/4} = o(h).$$

Replacing now in (20) the expectation $E\left(\xi_v.\mathbf{1}_{A_u} M^{(1)}_{u-h,u}\right)$ by the corresponding integral formula:

$$\begin{aligned}
&E\left(\xi_v.\mathbf{1}_{A_u \setminus A_{u-h}}\right\} \\
&\quad = \mathbf{1}_{\{\beta(0)>0\}}\beta(0) \int_{u-h}^{u} E\left(\xi_v.\mathbf{1}_{A_u}/Z_0 = \beta(0)x\right).p_{Z_0}(\beta(0)x)dx \\
&\qquad + \mathbf{1}_{\{\beta(1)>0\}}\beta(1) \int_{u-h}^{u} E\left(\xi_v.\mathbf{1}_{A_u}/Z_1 = \beta(1)x\right).p_{Z_1}(\beta(1)x)dx \\
&\qquad + \int_0^1 \mathbf{1}_{\{\beta(t)\ge 0\}} dt \int_0^{\beta(t)h} dz E\left(\xi_v.\mathbf{1}_{A_u}(Z''_t - \beta''(t)(u-h))^- / V_2 = (z,0)\right) \\
&\qquad \times p_{V_2}(z,0) + o(h) \\
&\quad = \int_{u-h}^{u} H_1(x,h)dx + o(h) \qquad (24)
\end{aligned}$$

where:

$$\begin{aligned}
H_1(x,h) = {} & \mathbf{1}_{\{\beta(0)>0\}}\beta(0) E\left(\xi_v.\mathbf{1}_{A_u}/Z_0 = \beta(0)x\right).p_{Z_0}(\beta(0)x) \\
& + \mathbf{1}_{\{\beta(1)>0\}}\beta(1) E\left(\xi_v.\mathbf{1}_{A_u}/Z_1 = \beta(1)x\right).p_{Z_1}(\beta(1)x) \\
& + \int_0^1 \mathbf{1}_{\{\beta(t)\ge 0\}} \\
& E(\xi_v.\mathbf{1}_{A_u}(Z''_t - \beta''(t)(u-h))^- / Z_t = \beta(t)x, Z'_t = \beta'(t)(u-h)) \\
& .p_{Z_t, Z'_t}(\beta(t)x, \beta'(t)(u-h))\beta(t)dt. \qquad (25)
\end{aligned}$$

Step 3. Our next aim is to prove that for each u the limit

$$\lim_{h \downarrow 0} \frac{F_v(u) - F_v(u-h)}{h}$$

exists and admits the representation (15) in the statement of the Lemma. For that purpose, we will prove the existence of the limit

$$\lim_{h \downarrow 0} \frac{1}{h} E\left(\xi_v.\mathbf{1}_{A_u \setminus A_{u-h}}\right). \qquad (26)$$

This will follow from the existence of the limit

$$\lim_{h\downarrow 0, u-h<x<u} H_1(x,h).$$

Consider the first term in expression (25). We apply Lemma 3.1(a) and with the same notations therein:

$$Z_t = a^{\vdash}(t)\, Z_0 + t Z_t^{\vdash}, \quad \beta_t = a^{\vdash}(t)\, \beta(0) + t\beta_t^{\vdash} \quad t \in [0,1].$$

For $u - h < x < u$ replacing in (25) we have:

$$\begin{aligned} & E\left(\xi_v.\mathbf{1}_{A_u}/Z_0 = \beta(0)x\right) \\ & \quad = E\Bigg(G\Big(t_1(Z_{t_1}^{\vdash} - \beta^{\vdash}(t_1)x) + \beta(t_1)(x-v), \dots, t_m(Z_{t_m}^{\vdash} - \beta^{\vdash}(t_m)x) \\ & \qquad +\beta(t_m)(x-v)\Big)\mathbf{1}_{B(u,x)}\Bigg) \\ & \quad = E\left(\xi_{v,x}^{\vdash}.\mathbf{1}_{B(u,x)}\right) \end{aligned} \tag{27}$$

where $\xi_{v,x}^{\vdash}$ is defined in the statement and

$$B(u,x) = \left\{ t Z_t^{\vdash} \le \beta(t)u - a^{\vdash}(t)\, \beta(0)x \text{ for all } t \in [0,1] \right\}.$$

For each δ such that $0 < \delta \le 1$ and $a^{\vdash}(s) > 0$ if $0 \le s \le \delta$, we define:

$$\begin{aligned} B_\delta(u,x) &= \left\{ t Z_t^{\vdash} \le \beta(t)u - a^{\vdash}(t)\, \beta(0)x \text{ for all } t \in [\delta,1] \right\} \\ &= \left\{ Z_t^{\vdash} \le \beta^{\vdash}(t)u + \frac{a^{\vdash}(t)\, \beta(0)(u-x)}{t} \text{for all } t \in [\delta,1] \right\}. \end{aligned}$$

It is clear that since we consider the case $\beta(0) > 0$, then

$$B(u,x) = B_{0^+}(u,x) := \lim_{\delta \downarrow 0} B_\delta(u,x).$$

Introduce also the notations:

$$M_{[s,t]} = \sup\left\{ Z_\tau^{\vdash} - \beta^{\vdash}(\tau)u : \tau \in [s,t] \right\},$$

$$\eta_\delta(x) = |u-x| \sup\left\{ \frac{|a^{\vdash}(t)\, \beta(0)|}{t} : t \in [\delta,1] \right\}.$$

We prove that as $x \uparrow u$,

$$E\left(\xi_{v,x}^{\vdash}.\mathbf{1}_{B(u,x)}\right) \to E\left(\xi_{v,u}^{\vdash}.\mathbf{1}_{B(u,u)}\right) \tag{28}$$

We have,

$$\begin{aligned} & |E\left(\xi_{v,x}^{\vdash}.\mathbf{1}_{B(u,x)}\right) - E\left(\xi_{v,u}^{\vdash}.\mathbf{1}_{B(u,u)}\right)| \\ & \quad \le E\left(|\xi_{v,x}^{\vdash} - \xi_{v,u}^{\vdash}|\right) + |E\left(\xi_{v,u}^{\vdash}(\mathbf{1}_{B(u,x)} - \mathbf{1}_{B(u,u)})\right)|. \end{aligned} \tag{29}$$

From the definition of $\xi^{\vdash}_{v,x}$ it is immediate that the first term tends to 0 as $x \uparrow u$. For the second term it suffices to prove that

$$P(B(u,x)\Delta B(u,u)) \to 0 \text{ as } x \uparrow u. \tag{30}$$

Check the inclusion:

$$B(u,x)\Delta B_\delta(u,u) \subset \left\{-\eta_\delta(x) \le M_{[\delta,1]} \le \eta_\delta(x)\right\} \cup \left\{M_{[\delta,1]} \le 0,\, M_{[0,\delta]} > 0\right\}$$

which implies that

$$\begin{aligned} P(B(u,x)\Delta B(u,u)) &\le P(B(u,x)\Delta B_\delta(u,u)) + P(B_\delta(u,u)\Delta B(u,u)) \\ &\le P(|M_{[\delta,1]}| \le \eta_\delta(x)) + 2.P(M_{[\delta,1]} \le 0,\, M_{[0,\delta]} > 0). \end{aligned}$$

Let $x \uparrow u$ for fixed δ. Since $\eta_\delta(x) \downarrow 0$, we get:

$$\limsup_{x \uparrow u} P(B(u,x)\Delta B(u,u)) \le P(M_{[\delta,1]} = 0) + 2.P(M_{[\delta,1]} \le 0,\, M_{[0,\delta]} > 0).$$

The first term is equal to zero because of Proposition 2.4. The second term decreases to zero as $\delta \downarrow 0$ since $\left\{M_{[\delta,1]} \le 0,\, M_{[0,\delta]} > 0\right\}$ decreases to the empty set.

It is easy to prove that the function

$$(u,v) \to E\left(\xi^{\vdash}_{v,u}.\mathbf{1}_{A_u(Z^{\vdash},\beta^{\vdash})}\right)$$

is continuous. The only difficulty comes from the indicator function $\mathbf{1}_{A_u(Z^{\vdash},\beta^{\vdash})}$ although again the fact that the distribution function of the maximum of the process $Z^{\vdash}_{(.)} - \beta^{\vdash}(.)u$ has no atoms implies the continuity in u in much the same way as above.

So, the first term in the right-hand member of (25) has the continuous limit:

$$\mathbf{1}_{\{\beta(0)>0\}}\beta(0)E\left(\xi^{\vdash}_{v,u}.\mathbf{1}_{A_u(Z^{\vdash},\beta^{\vdash})}\right).p_{Z_0}(\beta(0).u).$$

With minor changes, we obtain for the second term the limit:

$$\mathbf{1}_{\{\beta(1)>0\}}\beta(1)E\left(\xi^{\dashv}_{v,u}.\mathbf{1}_{A_u(Z^{\dashv},\beta^{\dashv})}\right).p_{Z_1}(\beta(1).u),$$

where $Z^{\dashv}$, $\beta^{\dashv}$ are as in Lemma 3.1 and $\xi^{\dashv}_{v,u}$ as in the statement of Lemma 3.3.

The third term can be treated in a similar way. The only difference is that the regression must be performed on the pair (Z_t, Z'_t) for each $t \in [0,1]$, applying again Lemma 3.1 (a),(b),(c). The passage to the limit presents no further difficulties, even if the integrand depends on h.

Finally, note that conditionally on $Z_t = \beta(t)u$, $Z'_t = \beta'(t)u$ one has

$$Z''_t - \beta''(t)u = Z^t_t - \beta^t(t)u$$

and

$$(Z_t - \beta(t)u)^- \mathbf{1}_{A_u(Z,\beta)} = -(Z_t - \beta(t)u)\mathbf{1}_{A_u(Z,\beta)}.$$

Adding up the various parts, we get:

$$\lim_{h\downarrow 0}\frac{1}{h}E\left(\xi_v.\mathbf{1}_{A_u\setminus A_{u-h}}\right)=\mathbf{1}_{\{\beta(0)>0\}}\beta(0)E\left(\xi^{\vdash}_{v,u}.\mathbf{1}_{A_u(Z^{\vdash},\beta^{\vdash})}\right).p_{Z_0}(\beta(0).u)$$
$$+\mathbf{1}_{\{\beta(1)>0\}}\beta(1)E\left(\xi^{\dashv}_{v,u}.\mathbf{1}_{A_u(Z^{\dashv},\beta^{\dashv})}\right).p_{Z_1}(\beta(1).u)$$
$$-\int_0^1\beta(t)\mathbf{1}_{\{\beta(t)>0\}}dt\,E\left(\xi^t_{v,u}(Z^t_t-\beta^t(t).u)\mathbf{1}_{A_u(Z^t,\beta^t)}\right)$$
$$\times\, p_{Z_t,Z'_t}(\beta(t)u,\beta'(t)u).$$

Similar computations – that we will not perform here – show an analogous result for

$$\lim_{h\downarrow 0}\frac{1}{h}E\left(\xi_v.\mathbf{1}_{A_{u-h}\setminus A_u}\right)$$

and replacing into (16) we have the result for processes Z with C^∞ paths.

Step 4. Suppose now that Z and $\beta(.)$ satisfy the hypotheses of the Lemma and define:

$$Z^\epsilon(t)=(\psi_\epsilon*Z)(t)+\epsilon Y(t)\qquad\text{and}\qquad\beta^\epsilon(t)=(\psi_\epsilon*\beta)(t)$$

where $\epsilon>0$, $\psi_\epsilon(t)=\epsilon^{-1}\psi(\epsilon^{-1}t)$, ψ a non-negative C^∞ function with compact support,$\int_{-\infty}^{+\infty}\psi(t)dt=1$ and Y is a Gaussian centered stationary process with C^∞ paths and non-purely atomic spectrum, independent of Z. Proceeding as in Sec. 10.6 of Cramer-Leadbetter (1967), one can see that Y verifies H_k for every k. The definition of Z^ϵ implies that Z^ϵ inherites this property. Thus for each positive ϵ, Z^ϵ meets the conditions for the validity of Steps 2 and 3, so that the function

$$F^\epsilon_v(u)=E\left(\xi^\epsilon_v\mathbf{1}_{A_u(Z^\epsilon,\beta^\epsilon)}\right)$$

where $\xi^\epsilon_v=G(Z^\epsilon_{t_1}-\beta^\epsilon(t_1)v,\ldots,Z^\epsilon_{t_m}-\beta^\epsilon(t_m)v)$ is continuoustly differentiable and its derivative verifies (15) with the obvious changes, that is:

$$\left(F^\epsilon_v\right)'(u)=\beta^\epsilon(0)E\left(\left(\xi^\epsilon_{v,u}\right)^{\vdash}.\mathbf{1}_{A_u\left((Z^\epsilon)^{\vdash},(\beta^\epsilon)^{\vdash}\right)}\right).p_{Z^\epsilon_0}\left(\beta^\epsilon(0).u\right)$$
$$+\beta^\epsilon(1)E\left(\left(\xi^\epsilon_{v,u}\right)^{\dashv}.\mathbf{1}_{A_u\left((Z^\epsilon)^{\dashv},(\beta^\epsilon)^{\dashv}\right)}\right).p_{Z^\epsilon_1}\left(\beta^\epsilon(1).u\right)$$
$$-\int_0^1\beta^\epsilon(t)E\left(\left(\xi^\epsilon_{v,u}\right)^t\left(\left(Z^\epsilon\right)^t_t-\left(\beta^\epsilon\right)^t(t).u\right)\mathbf{1}_{A_u\left((Z^\epsilon)^t,(\beta^\epsilon)^t\right)}\right)$$
$$\times\, p_{Z^\epsilon_t,(Z^\epsilon)'_t}\left(\beta^\epsilon(t).u,\left(\beta^\epsilon\right)'(t).u\right)dt.\tag{31}$$

Let $\epsilon\downarrow 0$. We prove next that $(F^\epsilon_v)'(u)$ converges for fixed (u,v) to a limit function $F^*_v(u)$ that is continuous in (u,v). On the other hand, it is easy to see that for fixed (u,v) $F^\epsilon_v(u)\to F_v(u)$. Also, from (31) it is clear that for each v, there exists $\epsilon_0>0$ such that if $\epsilon\in(0,\epsilon_0)$, $(F^\epsilon_v)'(u)$ is bounded by a fixed constant when u varies in a bounded set because of the hypothesis on the functions G and β and the non-degeneracy of the one and two-dimensional distribution of the process Z.

So, it follows that $F_v^*(u) = F_v'(u)$ and the same computation implies that $F_v'(u)$ satisfies (15).

Let us show how to proceed with the first term in the right-hand member of (31). The remaining terms are similar.

Clearly, almost surely, as $\epsilon \downarrow 0$ one has $Z_t^\epsilon \to Z_t$, $(Z^\epsilon)_t' \to Z_t'$, $(Z^\epsilon)_t'' \to Z_t''$ uniformly for $t \in [0,1]$, so that the definition of $Z^\vdash$ in (11) implies that $(Z^\epsilon)_t^\vdash \to Z_t^\vdash$ uniformly for $t \in [0,1]$, since the regression coefficient $(a^\epsilon)^\vdash(t)$ converges to $a^\vdash(t)$ uniformly for $t \in [0,1]$ (with the obvious notation).

Similarly, for fixed (u,v):

$$(\beta^\epsilon)_t^\vdash \to \beta_t^\vdash,\ (\xi_{v,u}^\epsilon)^\vdash \to \xi_{v,u}^\vdash$$

uniformly for $t \in [0,1]$.

Let us prove that

$$E\left((\xi_{v,u}^\epsilon)^\vdash \mathbf{1}_{A_u\left((Z^\epsilon)^\vdash,(\beta^\epsilon)^\vdash\right)}\right) \to E\left(\xi_{v,u}^\vdash \mathbf{1}_{A_u\left(Z^\vdash,\beta^\vdash\right)}\right).$$

This is implied by

$$P\left(A_u\left(\left(Z^\epsilon\right)^\vdash, \left(\beta^\epsilon\right)^\vdash\right) \Delta\ A_u\left(Z^\vdash,\beta^\vdash\right)\right) \to 0. \qquad (32)$$

as $\epsilon \downarrow 0$. Denote, for $\epsilon > 0$, $\eta \geq 0$:

$$C_{u,\epsilon} = A_u\left(\left(Z^\epsilon\right)^\vdash, \left(\beta^\epsilon\right)^\vdash\right) = \left\{\left(Z^\epsilon\right)_t^\vdash \leq \left(\beta^\epsilon\right)^\vdash(t).u \text{ for every } t \in [0,1]\right\}$$
$$E_{u,\eta} = \left(Z_t^\vdash \leq \beta^\vdash(t)u + \eta \text{ for all } t \in [0,1]\right).$$

One has:

$$P(C_{u,\epsilon}\Delta E_{u,0}) \leq P(C_{u,\epsilon} \setminus E_{u,\eta}) + P(E_{u,\eta} \setminus C_{u,\epsilon}) + P(E_{u,\eta} \setminus E_{u,0}).$$

Let K be a compact subset of the real line and suppose $u \in K$. We denote:

$$D_{\epsilon,\eta} = \left\{\sup_{u\in K, t\in[0,1]} \mid \left[\left(Z^\epsilon\right)_t^\vdash - \left(\beta^\epsilon\right)^\vdash(t).u\right] - \left[Z_t^\vdash - \beta^\vdash(t).u\right] \mid > \eta\right\}$$

and

$$F_{u,\eta} = \left\{-\eta \leq \sup_{t\in[0,1]}\left(Z_t^\vdash - \beta^\vdash(t)u\right) \leq \eta\right\}.$$

Fix $\eta > 0$ and choose ϵ small enough so that:

$$P\left(D_{\epsilon,\eta}\right) < \eta.$$

Check the following inclusions:

$$C_{u,\epsilon} \setminus E_{u,\eta} \subset D_{\epsilon,\eta}, \quad \left(E_{u,\eta} \setminus C_{u,\epsilon}\right) \cap D_{\epsilon,\eta}^c \subset F_{u,\eta}, \quad E_{u,\eta} \setminus E_{u,0} \subset F_{u,\eta}$$

which imply that if ϵ is small enough:

$$P(C_{u,\epsilon}\Delta E_{u,0}) \leq 2.\eta + 2.P\left(F_{u,\eta}\right).$$

For each u, as $\eta \downarrow 0$ one has

$$P\left(F_{u,\eta}\right) \to P\left(\sup_{t\in[0,1]}\left(Z_t^{\vdash} - \beta^{\vdash}(t)u\right) = 0\right) = 0.$$

where the second equality follows again on applying Proposition 2.4.

This proves that as $\epsilon \downarrow 0$ the first term in the right-hand member of (31) tends to the limit

$$\beta(0)E\left(\xi_{v,u}^{\vdash}.\mathbf{1}_{A_u(Z^{\vdash},\beta^{\vdash})}\right).p_{Z_0}\left(\beta\left(0\right).u\right).$$

It remains to prove that this is a continuous function of (u, v). It suffices to prove the continuity of the function

$$E\left(\mathbf{1}_{A_u(Z^{\vdash},\beta^{\vdash})}\right) = P\left(A_u\left(Z^{\vdash}, \beta^{\vdash}\right)\right)$$

as a function of u. For that purpose we use inequality:

$$\begin{aligned} &\mid P\left(A_{u+h}\left(Z^{\vdash}, \beta^{\vdash}\right)\right) - P\left(A_u\left(Z^{\vdash}, \beta^{\vdash}\right)\right) \mid \\ &\leq P\left(\mid \sup_{t\in[0,1]}\left(Z_t^{\vdash} - \beta^{\vdash}(t).u\right) \mid \leq \mid h \mid \parallel \beta^{\vdash} \parallel_\infty\right) \end{aligned}$$

and as $h \to 0$ the right-hand member tends to $P\left(\mid \sup_{t\in[0,1]}\left(Z_t^{\vdash} - \beta^{\vdash}(t).u\right) \mid = 0\right)$ which is equal to zero by Propostion 2.4.

Proof of Theorem 1.1 We proceed by induction on k.

We will give some details for the first two derivatives including some implicit formulae that will illustrate the procedure for general k.

We introduce the following additional notations. Put $Y_t := X_t - \beta(t)u$ and define, on the interval $[0, 1]$, the processes $X^{\vdash}$, $X^{\dashv}$, X^t, $Y^{\vdash}$, $Y^{\dashv}$, Y^t, and the functions $\beta^{\vdash}$, $\beta^{\dashv}$, β^t, as in Lemma 3.1. Note that the regression coefficients corresponding to the processes X and Y are the same, so that anyone of them may be used to define the functions $\beta^{\vdash}$, $\beta^{\dashv}$, β^t. One can easily check that

$$Y_s^{\vdash} = X_s^{\vdash} - \beta^{\vdash}(s)u$$

$$Y_s^{\dashv} = X_s^{\dashv} - \beta^{\dashv}(s)u$$

$$Y_s^t = X_s^t - \beta^t(s)u.$$

For $t_1, ..., t_m \in [0, 1] \cup \{\vdash, \dashv\}$, $m \geq 2$, we define by induction the stochastic processes $X^{t_1,...,t_m} = \left(X^{t_1,...,t_{m-1}}\right)^{t_m}$, $Y^{t_1,...,t_m} = \left(Y^{t_1,...,t_{m-1}}\right)^{t_m}$ and the function $\beta^{t_1,...,t_m} = \left(\beta^{t_1,...,t_{m-1}}\right)^{t_m}$, applying Lemma 3.1 for the computations at each stage.

With the aim of somewhat reducing the size of the formulae we will express the successive derivatives in terms of the processes $Y^{t_1,...,t_m}$ instead of $X^{t_1,...,t_m}$. The reader must keep in mind that for each m-tuple $t_1, ..., t_m$ the results depend on u through the expectation of the stochastic process $Y^{t_1,...,t_m}$. Also, for a stochastic process Z we will use the notation

$$A(Z) = A_0(Z, \beta) = \{Z_t \leq 0 : \text{for all } t \in [0, 1]\}.$$

First derivative. Suppose that X satisfies H_2. We apply formula (15) in Lemma 3.3 for $\xi \equiv 1$, $Z = X$ and $\beta(.) \equiv 1$ obtaining for the first derivative:

$$\begin{aligned} F'(u) = {} & E\left(\mathbf{1}_{A(Y^{\vdash})}\right) p_{Y_0}(0) + E\left(\mathbf{1}_{A(Y^{\dashv})}\right) p_{Y_1}(0) \\ & - \int_0^1 E\left(Y_{t_1}^{t_1} \mathbf{1}_{A(Y^{t_1})}\right) p_{Y_{t_1}, Y'_{t_1}}(0,0) dt_1. \end{aligned} \tag{33}$$

This expression is exactly the expression in (9) with the indicated notational changes and after taking profit of the fact that the process is Gaussian, via the regression on the conditionning in each term. Note that according to the definition of the Y-process:

$$\begin{aligned} E\left(\mathbf{1}_{A(Y^{\vdash})}\right) &= E\left(\mathbf{1}_{A_u(X^{\vdash}, \beta^{\vdash})}\right) \\ E\left(\mathbf{1}_{A(Y^{\dashv})}\right) &= E\left(\mathbf{1}_{A_u(X^{\dashv}, \beta^{\dashv})}\right) \\ E\left(Y_{t_1}^{t_1} \mathbf{1}_{A(Y^{t_1})}\right) &= E\left(Y_{t_1}^{t_1} \mathbf{1}_{A_u(X^{t_1}, \beta^{t_1})}\right). \end{aligned}$$

Second derivative. Suppose that X satisfies H_4. Then, $X^{\vdash}$, $X^{\dashv}$, X^{t_1} satisfy H_3, H_3, H_2 respectively. Therefore Lemma 3.3 applied to these processes can be used to show the existence of $F''(u)$ and to compute a similar formula, excepting for the necessity of justifying differentiation under the integral sign in the third term. We get the expression:

$$\begin{aligned} F''(u) = {} & -E\left(\mathbf{1}_{A(Y^{\vdash})}\right) p_{Y_0}^{(1)}(0) - E\left(\mathbf{1}_{A(Y^{\dashv})}\right) p_{Y_1}^{(1)}(0) \\ & + \int_0^1 E\left(Y_{t_1}^{t_1} \mathbf{1}_{A(Y^{t_1})}\right) p_{Y_{t_1}, Y'_{t_1}}^{(1,0)}(0,0) dt_1 \\ & + p_{Y_0}(0) \left[\beta^{\vdash}(0) E\left(\mathbf{1}_{A(Y^{\vdash,\vdash})}\right) p_{Y_0^{\vdash}}(0) + \beta^{\vdash}(1) E\left(\mathbf{1}_{A(Y^{\vdash,\dashv})}\right) p_{Y_1^{\vdash}}(0)\right] \\ & - \int_0^1 \beta^{\vdash}(t_2) E\left(Y_{t_2}^{\vdash, t_2} \mathbf{1}_{A\left(Y^{\vdash, t_2}\right)}\right) p_{Y_{t_2}^{\vdash}, (Y^{\vdash})'_{t_2}}(0,0) dt_2 \\ & + p_{Y_1}(0) \left[\beta^{\dashv}(0) E\left(\mathbf{1}_{A(Y^{\dashv,\vdash})}\right) p_{Y_0^{\dashv}}(0) + \beta^{\dashv}(1) E\left(\mathbf{1}_{A(Y^{\dashv,\dashv})}\right) p_{Y_1^{\dashv}}(0)\right] \\ & - \int_0^1 \beta^{\dashv}(t_2) E\left(Y_{t_2}^{\dashv, t_2} \mathbf{1}_{A\left(Y^{\dashv, t_2}\right)}\right) p_{Y_{t_2}^{\dashv}, (Y^{\dashv})'_{t_2}}(0,0) dt_2 \\ & - \int_0^1 p_{Y_{t_1}, Y'_{t_1}}(0,0) \left[\begin{array}{l} -\beta^{t_1}(t_1) E\left(\mathbf{1}_{A(Y^{t_1})}\right) + \beta^{t_1}(0) E\left(Y_{t_1}^{t_1,\vdash} \mathbf{1}_{A\left(Y^{t_1,\vdash}\right)}\right) p_{Y_0^{t_1}}(0) \\ + \beta^{t_1}(1) E\left(Y_{t_1}^{t_1,\dashv} \mathbf{1}_{A\left(Y^{t_1,\dashv}\right)}\right) p_{Y_1^{t_1}}(1) \\ - \int_0^1 \beta^{t_1}(t_2) E\left(Y_{t_1}^{t_1,t_2} Y_{t_2}^{t_1,t_2} \mathbf{1}_{A(Y^{t_1,t_2})}\right) p_{Y_{t_2}^{t_1}, (Y^{t_1})'_{t_2}}(0,0) dt_2 \end{array}\right] dt_1, \end{aligned} \tag{34}$$

In this formula $p_{Y_{t_0}}^{(1)}$, $p_{Y_{t_1}}^{(1)}$ and $p_{Y_{t_1}, Y'_{t_1}}(0,0)^{(1,0)}$ stand respectively for the derivative of $p_{Y_{t_0}}(.)$, the derivative of $p_{Y_{t_1}}(.)$ and the derivative with respect to the first variable of $(p_{Y_{t_1}, Y'_{t_1}}(.,.))$.

To validate the above formula, note that:

- The first two lines are obtained by differentiating with respect to u, the densities $p_{Y_0}(0) = p_{X_0}(-u)$, $p_{Y_1}(0) = p_{X_1}(-u)$, $p_{Y_{t_1}, Y'_{t_1}}(0,0) = p_{X_{t_1}, X'_{t_1}}(-u, 0)$.
- Lines 3 and 4 come from the application of Lemma 3.3 to differentiate $E(\mathbf{1}_{A(Y^{\vdash})})$. The lemma is applied with $Z = X^{\vdash}$, $\beta = \beta^{\vdash}$, $\xi = 1$.
- Similarly, lines 5 and 6 contain the derivative of $E(\mathbf{1}_{A(Y^{\dashv})})$.
- The remaining corresponds to differentiate the function

$$E\big(Y_{t_1}^{t_1}\mathbf{1}_{A(Y^{t_1})}\big) = E\Big(\big(X_{t_1}^{t_1} - \beta^{t_1}(t_1)u\big)\mathbf{1}_{A_u(X^{t_1},\beta^{t_1})}\Big)$$

in the integrand of the third term in (33). The first term in line 7 comes from the simple derivative

$$\frac{\partial}{\partial v}E\big((X_{t_1}^{t_1} - \beta^{t_1}(t_1)v)\mathbf{1}_{A_u(X^{t_1},\beta^{t_1})}\big) = -\beta^{t_1}(t_1)E(\mathbf{1}_{A(Y^{t_1})}).$$

The other terms are obtained by applying Lemma 3.3 to compute

$$\frac{\partial}{\partial u}E\big((X_{t_1}^{t_1} - \beta^{t_1}(t_1)v)\mathbf{1}_{A_u(X^{t_1},\beta^{t_1})}\big),$$

putting $Z = X^{t_1}$, $\beta = \beta^{t_1}$, $\xi = X_{t_1}^{t_1} - \beta^{t_1}(t_1)v$.
- Finally, differentiation under the integral sign is valid since because of Lemma 3.1, the derivative of the integrand is a continuous function of (t_1, t_2, u) due the regularity and non-degeneracy of the Gaussian distributions involved and Proposition 2.4.

General case. With the above notation, given the $m-$tuple $t_1, ..., t_m$ of elements of $[0,1] \cup \{\vdash, \dashv\}$ we will call the processes $Y, Y^{t_1}, Y^{t_1,t_2}, ..., Y^{t_1,...,t_{m-1}}$ the "ancestors" of $Y^{t_1,...,t_m}$. In the same way we define the ancestors of the function $\beta^{t_1,...,t_m}$.

Assume the following induction hypothesis: If X satisfies H_{2k} then F is k times continuously differentiable and $F^{(k)}$ is the sum of a finite number of terms belonging to the class D_k which consists of all expressions of the form:

$$\int_0^1 .. \int_0^1 ds_1..ds_p Q(s_1,..,s_p)E\left(\xi\mathbf{1}_{A\left(Y^{t_1,...,t_m}\right)}\right)K_1(s_1,..,s_p)K_2(s_1,..,s_p) \quad (35)$$

where:

- $1 \le m \le k$.
- $t_1, ..., t_m \in [0,1] \cup \{\vdash, \dashv\}$, $m \ge 1$.
- $s_1, .., s_p$, $0 \le p \le m$, are the elements in $\{t_1, ..., t_m\}$ that belong to $[0,1]$ (that is, which are neither "$\vdash$" nor "$\dashv$"). When $p = 0$ no integral sign is present.
- $Q(s_1, .., s_p)$ is a polynomial in the variables $s_1, .., s_p$.
- ξ is a product of values of $Y^{t_1,...,t_m}$ at some locations belonging to $\{s_1, .., s_p\}$.
- $K_1(s_1, .., s_p)$ is a product of values of some ancestors of $\beta^{t_1,...,t_m}$ at some locations belonging to the set $\{s_1, .., s_p\} \cup \{0,1\}$.
- $K_2(s_1, .., s_p)$ is a sum of products of densities and derivatives of densities of the random variables Z_τ at the point 0, or the pairs (Z_τ, Z'_τ) at the point $(0,0)$ where $\tau \in \{s_1, .., s_p\} \cup \{0,1\}$ and the process Z is some ancestor of $Y^{t_1,...,t_m}$.

Note that K_1 does not depend on u but K_2 is a function of u.

It is clear that the induction hypothesis is verified for $k = 1$. Assume that it is true up to the integer k and that X satisfies H_{2k+2}. Then $F^{(k)}$ can be written as a sum of terms of the form (35). Consider a term of this form and note that the variable u may appear in three locations:

1. In ξ, where differentiation is simple given its product form, the fact that $\frac{\partial}{\partial u} Y_s^{t_1,\ldots,t_q} = -\beta^{t_1,\ldots,t_q}(s), q \le m, s \in \{s_1, \ldots, s_p\}$ and the boundedness of moments allowing to differentiate under the integral and expectation signs.
2. In $K_2(s_1, .., s_p)$ which is clearly $\mathscr{C}^\infty$ as a function of u. Its derivative with respect to u takes the form of a product of functions of the types $K_1(s_1, .., s_p)$ and $K_2(s_1, .., s_p)$ defined above.
3. In $\mathbf{1}_{A\left(Y^{t_1,\ldots,t_m}\right)}$. Lemma 3.3 shows that differentiation produces 3 terms depending upon the processes $Y^{t_1,\ldots,t_m,t_{m+1}}$ with t_{m+1} belonging to $[0,1] \cup \{\vdash, \dashv\}$. Each term obtained in this way belongs to D_{k+1}.

The proof is achieved by noting that, as in the computation of the second derivative, Lemma 3.1 implies that the derivatives of the integrands are continuous functions of u that are bounded as functions of $(s_1, .., s_p, t_{m+1}, u)$ if u varies in a bounded set.

The statement and proof of Theorem 1.1 can not, of course, be used to obtain explicit expressions for the derivatives of the distribution function F. However, the implicit formula for $F^{(k)}(u)$ as sum of elements of D_k can be transformed into explicit upper-bounds if one replaces everywhere the indicator functions $\mathbf{1}_{A(Y^{t_1,\ldots,t_m})})$ by 1 and the functions $\beta^{t_1,\ldots,t_m}(.)$ by their absolute value.

On the other hand, Theorem 1.1 permits to have the exact asymptotic behaviour of $F^{(k)}(u)$ as $u \to +\infty$ in case $Var(X_t)$ is constant. Even though the number of terms in the formula increases rapidly with k, there is exactly one term that is dominant. It turns out that as $u \to +\infty$, $F^{(k)}(u)$ is equivalent to the k-th derivative of the equivalent of $F(u)$. This is Corollary 1.1.

Proof of Corollary 1.1. To prove the result for $k = 1$ note that under the hypothesis of the Corollary, one has $r(t,t) = 1,\ r_{01}(t,t) = 0,\ r_{02}(t,t) = -r_{11}(t,t)$ and an elementary computation of the regression (13) replacing Z by X, shows that:

$$b^t(s) = r(s,t), \qquad c^t\,(s) = \frac{r_{01}(s,t)}{r_{11}(t,t)}$$

and

$$\beta^t(s) = 2\frac{1 - r(s,t)}{(t-s)^2}$$

since we start with $\beta(t) \equiv 1$.

This shows that for every $t \in [0,1]$ one has $\inf_{s \in [0,1]}(\beta^t(s)) > 0$ because of the non-degeneracy condition and $\beta^t(t) = -r_{02}(t,t) = r_{11}(t,t) > 0$. The expression for F' becomes:

$$F'(u) = \phi(u)L(u), \tag{36}$$

where

$$L(u) = L_1(u) + L_2(u) + L_3(u),$$

$$L_1(u) = P(A_u(X^{\vdash}, \beta^{\vdash}),$$

$$L_2(u) = P(A_u(X^{\dashv}, \beta^{\dashv}),$$

$$L_3(u) = -\int_0^1 E\big((X_t^t - \beta^t(t)u)\mathbf{1}_{A_u(X^t,\beta^t)}\big)\frac{dt}{(2\pi r_{11}(t,t))^{1/2}}.$$

Since for each $t \in [0, 1]$ the process X^t is bounded it follows that

$$a.s.\ \mathbf{1}_{A_u(X^t,\beta^t)} \to 1 \text{ as } u \to +\infty.$$

A dominated convergence argument shows now that $L_3(u)$ is equivalent to

$$-\frac{u}{(2\pi)^{1/2}}\int_0^1 \frac{r_{02}(t,t)}{(r_{11}(t,t))^{1/2}}dt = \frac{u}{(2\pi)^{1/2}}\int_0^1 \sqrt{r_{11}(t,t)}dt.$$

Since $L_1(u), L_2(u)$ are bounded by 1, (1) follows for $k = 1$.

For $k \geq 2$, write

$$F^{(k)}(u) = \phi^{(k-1)}(u)L(u) + \sum_{h=2}^{h=k}\binom{k-1}{k-1}\phi^{(k-h)}(u)L^{(h-1)}(u). \qquad (37)$$

As $u \to +\infty$, for each $j = 0, 1, ..., k-1$, $\phi^{(j)}(u) \simeq (-1)^j\, u^j\phi(u)$ so that the first term in (37) is equivalent to the expression in (1). Hence, to prove the Corollary it suffices to show that the succesive derivatives of the function L are bounded. In fact, we prove the stronger inequality

$$|L^{(j)}(u)| \leq l_j\phi(\frac{u}{a_j}),\ \ j = 1, ..., k-1 \qquad (38)$$

for some positive constants $l_j,\ a_j,\ \ j = 1, ..., k-1$.

We first consider the function L_1. One has:

$$\beta^{\vdash}(s) = \frac{1 - r(s,0)}{s}\ \ for\ 0 < s \leq 1,\ \beta^{\vdash}(0) = 0,$$

$$(\beta^{\vdash})'(s) = \frac{-1 + r(s,0) - s.r_{10}(s,0)}{s^2}\ \ for\ 0 < s \leq 1,\ (\beta^{\vdash})'(0) = \frac{1}{2}r_{11}(0,0).$$

The derivative $L_1'(u)$ becomes

$$\begin{aligned} &L_1'(u) = \beta^{\vdash}(1)E[\mathbf{1}_{A_u(X^{\vdash,\dashv},\beta^{\vdash,\dashv})}]\ p_{X_1^{\vdash}}(\beta^{\vdash}(1)u) \\ &-\int_0^1 \beta^{\vdash}(t)E\left((X_t^{\vdash,t} - \beta^{\vdash,t}(t)u)\mathbf{1}_{A_u(X^{\vdash,t},\beta^{\vdash,t})}\right)\ p_{X_t^{\vdash},(X^{\vdash})_t'}(\beta^{\vdash}(t)u, (\beta^{\vdash})'(t)u)\ dt. \end{aligned}$$

Notice that $\beta^{\vdash}(1)$ is non-zero so that the first term is bounded by a constant times a non-degenerate Gaussian density. Even though $\beta^{\vdash}(0) = 0$, the second term is also bounded by a constant times a non-degenerate Gaussian density because the joint distribution of the pair $(X_t^{\vdash}, (X^{\vdash})_t')$ is non-degenerate and the pair $(\beta^{\vdash}(t), (\beta^{\vdash})'(t)) \neq (0,0)$ for every $t \in [0,1]$.

Applying a similar argument to the succesive derivatives we obtain (38) with L_1 instead of L.

The same follows with no changes for

$$L_2(u) = P(A_u(X^{\dashv}, \beta^{\dashv}).$$

For the third term

$$L_3(u) = -\int_0^1 E\big((X_t^t - \beta^t(t)u)\mathbf{1}_{A_u(X^t,\beta^t)}\big)\frac{dt}{(2\pi r_{11}(t,t))^{1/2}}$$

we proceed similarly, taking into account $\beta^t(s) \neq 0$ for every $s \in [0,1]$. So (38) follows and we are done.

Remark. Suppose that X satisfies the hypotheses of the Corollary with $k \geq 2$. Then, it is possible to refine the result as follows.

For $j = 1, ..., k$:

$$\begin{aligned} F^{(j)}(u) &= (-1)^{j-1}(j-1)!h_{j-1}(u) \\ &\times \left[1 + (2\pi)^{-1/2}.u.\int_0^1 (r_{11}(t,t))^{1/2}dt\right]\phi(u) + \rho_j(u)\phi(u) \quad (39) \end{aligned}$$

where $h_j(u) = \frac{(-1)^j}{j!}(\phi(u))^{-1}\phi^{(j)}(u)$, is the standard j-th Hermite polynomial $(j = 0, 1, 2, ...)$ and

$$|\ \rho_j(u)\ | \leq C_j \exp(-\delta u^2)$$

where $C_1, C_2, ...$ are positive constants and $\delta > 0$ does not depend on j.

The proof of (39) consists of a slight modification of the proof of the Corollary.

Note first that from the above computation of $\beta^{\vdash}(s)$ it follows that 1) if $X_0^{\vdash} < 0$, then if u is large enough $X_s^{\vdash} - \beta^{\vdash}(s).u \leq 0$ for all $s \in [0,1]$ and 2) if $X_0^{\vdash} > 0$, then $X_0^{\vdash} - \beta^{\vdash}(0).u > 0$ so that:

$$L_1(u) = P(X_s^{\vdash} - \beta^{\vdash}(s).u \leq 0) \text{ for all } s \in [0,1]) \uparrow \frac{1}{2} \quad \text{as } u \uparrow +\infty.$$

On account of (38) this implies that if $u \geq 0$:

$$0 \leq \frac{1}{2} - L_1(u) = \int_u^{+\infty} L_1'(v)dv \leq D_1 \exp(-\delta_1 u^2)$$

with D_1, δ_1 positive constants.

$L_2(u)$ is similar. Finally:

$$L_3(u) = -\int_0^1 E\left((X_t^t - \beta^t(t)u)\right)\frac{dt}{(2\pi r_{11}(t,t))^{1/2}}$$

$$-\int_0^1 E\big((X_t^t-\beta^t(t)u)\mathbf{1}_{(A_u(X^t,\beta^t))^C}\big)\frac{dt}{(2\pi r_{11}(t,t))^{1/2}}. \qquad (40)$$

The first term in (40) is equal to:

$$(2\pi)^{-1/2}.u.\int_0^1 (r_{11}(t,t))^{1/2}dt.$$

As for the second term in (40) denote $\beta_\# = \inf_{s,t\in[0,1]} \beta^t(s) > 0$ and let $u > 0$. Then:

$$P\left(\left(A_u(X^t,\beta^t)\right)^C\right) \le P(\exists\, s \in [0,1] \text{ such that } X_s^t > \beta_\#.u) \le D_3\exp(-\delta_3 u^2)$$

with D_3, δ_3 are positive constants, the last inequality being a consequence of the Landau-Shepp-Fernique inequality.

The remainder follows in the same way as the proof of the Corollary.

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