# Distribución $p$-ádica de puntos CM y aplicaciones diofantinas, parte 2

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# Notation

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$j : \mathbb{H} \to \mathbb{C}$ modular function with

$$j(z) = q^{-1} + 744 + \sum_{n=1}^{\infty} c_n q^n, \quad \text{where } q := e^{2\pi i z}.$$

**Facts**:

1. $j$ is a Hauptmodul for $\Gamma$.
2. $j(z)$ is the $j$-invariant of the elliptic curve $E_z \simeq \mathbb{C}/(\mathbb{Z} + z\mathbb{Z})$.

# CM points and singular moduli

$z$ in $\mathbb{H}$ is a CM point if $\mathbb{Q}(z)$ is quadratic (imaginary) over $\mathbb{Q}$.

**Theorem (From CM theory)**

*If $z$ is CM then $j(z)$ is an algebraic integer.*

If $z$ is CM, we call $j(z)$ a singular modulus (following Kronecker).

# A question of Masser

Are there *only finitely many* singular moduli that are algebraic units?

**Motivation:** In

*An effective "Theorem of André" for CM-points on a plane curve* (2013)

Bilu, Masser and Zannier proved that there are no pairs $(j_1, j_2)$ of singular moduli on $X_1 \cdot X_2 = 1$.

# Habegger's theorem

In

*Singular moduli that are algebraic units* (2015)

Habegger proved the there are at most finitely many singular moduli that are algebraic units.

Habegger's proof does not give a numerical bound for the number of singular moduli that are algebraic units.

**Natural question:** Is there any such *singular unit*?

# Refinements

1. In

   *No singular modulus is a unit* (2018)

   Bilu, Habegger and Kühne proved that there are no *singular units*.

2. Let $\Phi_m(X, Y)$ denote the $m$-th modular polynomial ($m \geq 1$ integer). In

   *Singular units and isogenies between* $\mathrm{CM}$ *elliptic curves* (2019)

   Y. Li proved that $\Phi_m(j_1, j_2)$ is never an algebraic unit for $j_1, j_2$ singular moduli.

# Differences of singular moduli

Habegger's work (2015) implies the following result: given an algebraic integer $\alpha$ there are at most finitely many singular moduli $j$ such that $j - \alpha$ is an algebraic unit.

**Example:** If $\alpha = 1$ then

$$j\left(\frac{1+i\sqrt{3}}{2}\right) - \alpha = 0 - 1 = -1$$

is an algebraic unit.

In the case $\alpha = j_2$ is a singular modulus we have, by Y. Li's theorem with $m = 1$, that $j_1 - j_2$ is never an algebraic unit.

**Fact**: Differences of singular moduli are very special.

# More on CM points

For $z$ CM, define

$$D = \mathrm{Disc}(z) = \text{discriminant of min. poly. of } z \text{ over } \mathbb{Z}.$$

Then $z$ is of the form

$$\frac{-b+\sqrt{D}}{2a}$$

with $a, b, c \in \mathbb{Z}$ coprime, $a > 0$, $D = b^2 - 4ac < 0$.
$\Gamma$ acts on the set $\mathrm{CM}_D$ of CM points of discriminant $D$ and we define

$$\Lambda_D = \Gamma \backslash \mathrm{CM}_D.$$

**Theorem (CM theory)**

*$\Lambda_D$ is finite of cardinality $h(D)$ (class number) and $j(\Lambda_D)$ is a full Galois orbit.*

# Norms of differences

Given $\alpha$ in $\overline{\mathbb{Q}}$ define

$$\mathrm{Nm}(\alpha) = \prod_{\sigma:\mathbb{Q}(\alpha)\hookrightarrow\overline{\mathbb{Q}}} \sigma(\alpha).$$

Then $j_1 - j_2$ is an algebraic unit if and only if $\mathrm{Nm}(j_1 - j_2) = \pm 1$.

In

*On singular moduli* (1985)

Gross and Zagier gave an *arithmetic formula* for $\mathrm{Nm}(j_1 - j_2)$ under certain hypotheses.
It is not clear how to use Gross and Zagier's formula (or extensions of it) to prove *directly* that $j_1 - j_2$ is never an algebraic unit.

# Singular $S$-units

Fix $S$ a finite set of prime numbers.

An algebraic integer is an $S$-unit if no primes outside $S$ divide $\mathrm{Nm}(\alpha)$.

**Theorem (H–Menares–Rivera-Letelier, 2021)**

*There are at most finitely many singular moduli that are $S$-units.*

Note that every non-zero singular modulus is an $S$-unit for some finite set $S$.

# Numerics: A. Sutherland's table[1]

| $D$ | $\prod_{z\in\Lambda_D} j(z)$ | $D$ | $\prod_{z\in\Lambda_D} j(z)$ | $D$ | $\prod_{z\in\Lambda_D} j(z)$ |
|---|---|---|---|---|---|
| $-3$ | $0$ | $-32$ | $2^6 5^6 23^3$ | $-63$ | $-3^6 5^{12} 17^3 41^3 47^3$ |
| $-4$ | $2^6 3^3$ | $-35$ | $-2^{30} 5^3$ | $-64$ | $-2^3 3^6 23^3 47^3$ |
| $-7$ | $-3^3 5^3$ | $-36$ | $-2^{12} 3^3 11^3 23^3$ | $-67$ | $-2^{15} 3^3 5^3 11^3$ |
| $-8$ | $2^6 5^3$ | $-39$ | $3^{15} 17^3 23^3 29^3$ | $-68$ | $-2^{24} 5^{12} 17^3 47^3$ |
| $-11$ | $-2^{15}$ | $-40$ | $2^{12} 3^6 5^3 29^3$ | $-71$ | $-11^9 17^6 23^3 41^3 47^3 53^3$ |
| $-12$ | $2^4 3^3 5^3$ | $-43$ | $-2^{18} 3^3 5^3$ | $-72$ | $2^{12} 5^6 29^3 53^3$ |
| $-15$ | $-3^6 5^3 11^3$ | $-44$ | $2^{12} 11^3 17^3 29^3$ | $-75$ | $2^{30} 3^6 5^1 11^3$ |
| $-16$ | $2^3 3^3 11^3$ | $-47$ | $-5^{15} 11^6 23^3 29^3$ | $-76$ | $2^{12} 3^9 41^3 53^3$ |
| $-19$ | $-2^{15} 3^3$ | $-48$ | $2^4 3^9 5^6 11^3$ | $-79$ | $-3^{15} 17^3 29^3 47^3 53^3 59^3$ |
| $-20$ | $-2^{12} 5^3 11^3$ | $-51$ | $2^{33} 3^6$ | $-80$ | $2^{12} 5^6 11^3 17^6 59^3$ |
| $-23$ | $-5^9 11^3 17^3$ | $-52$ | $-2^{12} 3^6 5^6 23^3$ | $-83$ | $-2^{48} 5^9$ |
| $-24$ | $2^{12} 3^6 17^3$ | $-55$ | $-3^{12} 5^6 11^3 29^3 41^3$ | $-84$ | $-2^{24} 3^{15} 47^3 59^3$ |
| $-27$ | $-2^{15} 3^1 5^3$ | $-56$ | $2^{24} 11^6 17^3 41^3$ | $-87$ | $3^{18} 5^{18} 23^3 53^3 59^3$ |
| $-28$ | $3^3 5^3 17^3$ | $-59$ | $-2^{48} 11^3$ | $-88$ | $2^{12} 3^6 5^6 17^3 41^3$ |
| $-31$ | $-3^9 11^3 17^3 23^3$ | $-60$ | $3^6 5^3 29^3 41^3$ | $-91$ | $-2^{30} 3^6 17^3$ |

[1] https://math.mit.edu/ drew/NormsOfSingularModuli2000.pdf

# Question

It seems like $j\left(\frac{1+\sqrt{-11}}{2}\right) = -2^{15}$ is the only singular modulus that is an $S$-unit for $S$ a singleton. Is this the case?

A. Sutherland checked this *conjecture* for discriminants $D$ in $]-10^5, -3]$ (private communication).

# Difference of singular moduli

Fix $S$ a finite set of prime numbers.

**Theorem (H–Menares–Rivera-Letelier, 2021)**

*Given a singular modulus $j_2$, there are at most finitely many singular moduli $j_1$ such that $j_1 - j_2$ is an $S$-unit.*

We use Habegger's original strategy together with the new ingredient that for every prime number $p$, singular moduli are $p$-adically disperse.

# Habegger's strategy (for singular units)

Habegger considered the absolute logarithmic Weil height

$$h(a) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{1, |a|_v\}$$

for $a$ in $K$ a number field, where

- $M_K$ is the set of places of $K$,
- $|\cdot|_v$ is a representative absolute value extending $|\cdot|_p$ with $p$ prime or $\infty$ (the usual field norms on $\mathbb{Q}$),
- $d_v = [K_v : \mathbb{Q}_p]$.

**First ingredient:** For $j$ a singular modulus of discriminant $D$ we have

$$h(j) \geq A \log |D| + B,$$

with $A, B$ absolute constants, $A > 0$.
This follows from results of Colmez (1989), and Nakkajima and Taguchi (1991).

**Second ingredient:** A density estimate for the number of singular moduli around 0. Given $\varepsilon > 0$ find $r > 0$ small such that

$$\frac{1}{h(D)} \left( j(\Lambda_D) \cap B(0, r) \right) \leq \varepsilon \text{ for } D \to -\infty.$$

This follows from the following equidistribution theorem for CM points.

**Theorem (Duke (1988) + Clozel and Ullmo (2004))**

*When $D \to -\infty$ we have*

$$\frac{1}{h(D)} \sum_{z \in \Lambda_D} \delta_z \to \frac{3}{\pi} \frac{dxdy}{y^2}$$

*weakly on $\Gamma \backslash \mathbb{H}$.*

This step is not effective.

**Third ingredient:** An estimate for the Archimedean distance between a singular modulus and 0. For $j$ a nonzero singular modulus of discriminant $D$ we have

$$-\log|j| \le c_\infty \log|D|,$$

with $c_\infty > 0$ absolute constant.

In the "$(j-\alpha)$ version" of Habegger's theorem ($\alpha$ algebraic integer) one needs David and Hirata-Kohno's deep lower bound for linear forms on $n=2$ elliptic logarithms (2009).

# Putting everything together

If $j$ is a singular unit of discriminant $D$, then

$$\begin{aligned} A\log|D|+B &\leq h(j) \\ &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K} d_v \log\max\{1,|j|_v\} \\ &= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K^\infty} d_v \log\max\{1,|j|_v\} \\ &= -\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K^\infty,|j|_v<1} d_v \log|j|_v, \end{aligned}$$

by the product formula.For $\varepsilon > 0$ convenient we get

$$-\frac{1}{[K:\mathbb{Q}]}\sum_{v\in M_K^\infty,|j|_v<1} d_v \log|j|_v \leq A_\varepsilon \log|D| + B_\varepsilon$$

with $A_\varepsilon < A$.Hence $|D|$ is bounded and the result follows.

# Our proof (for singular $S$-units)

We use Habegger's strategy. For $p$ prime, fix an extension of $|\cdot|_p$ to $\overline{\mathbb{Q}}$.

**First ingredient:** For $j$ a singular modulus of discriminant $D$ we have

$$h(j) \geq A \log |D| + B,$$

with $A, B$ absolute constants, $A > 0$.

**Second ingredient:** A $p$-adic density estimate for the number of singular moduli around 0. Given $\varepsilon > 0$ find $r > 0$ small such that

$$\frac{1}{h(D)} \left(j(\Lambda_D) \cap B_p(0, r)\right) \leq \varepsilon \text{ for } D \to -\infty.$$

**Third ingredient:** An estimate for the $p$-adic distance between a singular modulus and 0. For $j$ a nonzero singular modulus of discriminant $D$ we have

$$-\log |j|_p \leq c_p \log |D|,$$

with $c_p > 0$ absolute constant.

# Putting everything together

If $j$ is a singular unit of discriminant $D$, then

$$\begin{aligned} A \log |D| + B &\leq h(j) \\ &= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} d_v \log \max\{1, |j|_v\} \\ &= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K^\infty \cup M_K^S} d_v \log \max\{1, |j|_v\} \\ &= -\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K^\infty \cup M_K^S, |j|_v < 1} d_v \log |j|_v, \end{aligned}$$

by the product formula. For $\varepsilon > 0$ convenient we get

$$-\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K^\infty \cup M_K^S, |j|_v < 1} d_v \log |j|_v \leq A_{\varepsilon,S} \log |D| + B_{\varepsilon,S}$$

with $A_{\varepsilon,S} < A$. Hence $|D|$ is bounded and the result follows.

# Singular moduli are $p$-adically disperse

**Theorem (H–Menares–Rivera-Letelier, 2021)**

*Given* $\varepsilon > 0$ *there exists* $r > 0$ *small such that*

$$\frac{1}{h(D)}\left(j(\Lambda_D) \cap B_p(0, r)\right) \leq \varepsilon \text{ for } D \to -\infty.$$

This follows from our identification of all limit measures of CM points in the $p$-adic setting.

# $p$-adic distribution of CM points

For simplicity, restrict to $D < 0$ fundamental discriminant.

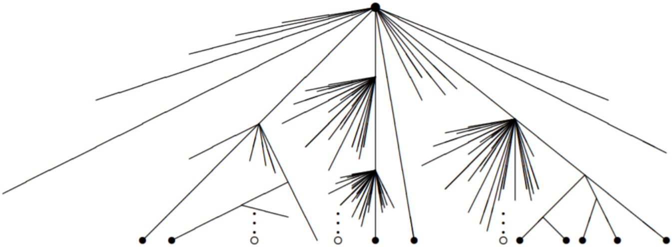
We have $j(\Lambda_D) \subset \overline{\mathbb{Q}} \subset \mathbb{C}_p \subset \mathbb{A}^1_{\mathrm{Berk}}$.

$\mathbb{A}^1_{\mathrm{Berk}}$ can be defined as the set of multiplicative semi-norms on $\mathbb{C}_p[X]$ extending $|\cdot|_p$ on $\mathbb{C}_p$, with a suitable topology (locally compact, arc-connected), and

$$\mathbb{C}_p \hookrightarrow \mathbb{A}^1_{\mathrm{Berk}}, \qquad z \mapsto \iota(z)$$

is defined by $\iota(z)(f) = |f(z)|_p$ for $f$ in $\mathbb{C}_p[X]$.
$\mathbb{C}_p$ is dense in $\mathbb{A}^1_{\mathrm{Berk}}$.

Above the unit disc in $\mathbb{C}_p$ we have the following picture[2]



At the top we have the Gauss point $\zeta$ defined by

$$\zeta(a_0 + a_1X + \ldots + a_nX^n) = \max\{|a_0|_p, |a_1|_p, \ldots, |a_n|_p\}.$$

[2] Illustration of Joe Silverman

# Convergence towards the Gauss point

**Theorem (H–Menares–Rivera-Letelier, 2020)**

1. *For fundamental discriminants* $D < 0$ *with* $\left(\frac{D}{p}\right) = 1$ *we have*

$$\frac{1}{h(D)} \sum_{z \in \Lambda_D} \delta_{j(z)} \to \delta_\zeta$$

   *weakly on* $\mathbb{A}^1_{\mathrm{Berk}}$.

2. *This is not the case for fundamental discriminants* $D < 0$ *with* $\left(\frac{D}{p}\right) \neq 1$.

# The case $\left(\frac{D}{p}\right) \neq 1$

Let $\mathcal{O}_D$ denote the ring of integers of $\mathbb{Q}(\sqrt{D})$. Then

$$\Lambda_D = \{E \text{ ell. curve over } \overline{\mathbb{Q}} \text{ with } \mathrm{End}(E) \simeq \mathcal{O}_D\} \subset Y(\overline{\mathbb{Q}})$$

where $Y(\overline{\mathbb{Q}})$ is the (open) moduli space of elliptic curves over $\overline{\mathbb{Q}}$.

Let $\mathfrak{D}$ be the $p$-adic discriminant of the ring of integers $\mathcal{O}_{\mathfrak{D}}$ of $\mathbb{Q}_p(\sqrt{D})$.Then $D \in \mathfrak{D}$, $\mathcal{O}_D \subset \mathcal{O}_{\mathfrak{D}}$ and

$$\Lambda_D \subset \Lambda_{\mathfrak{D}} = \{E \text{ ell. curve over } \overline{\mathbb{Q}}_p \text{ with } \mathrm{End}(\widehat{E}) \simeq \mathcal{O}_{\mathfrak{D}}\} \subset Y(\overline{\mathbb{Q}}_p)$$

where $\widehat{E}$ is the *formal group* of $E$.

# The case $\left(\frac{D}{p}\right) \neq 1$

Every (fundamental) discriminant $D < 0$ with $\left(\frac{D}{p}\right) \neq 1$ belongs to some $p$-adic (fundamental) discriminant $\mathfrak{D}$.

**Theorem (H–Menares–Rivera-Letelier, 2021)**

*For a p-adic discriminant $\mathfrak{D}$ the set $\Lambda_{\mathfrak{D}}$ is compact and there exists a (unique) Borel probability measure $\nu_{\mathfrak{D}}$ with support $\Lambda_{\mathfrak{D}}$ such that for fundamental discriminants $D < 0$ with $D \in \mathfrak{D}$ we have*

$$\frac{1}{h(D)} \sum_{z \in \Lambda_D} \delta_{j(z)} \to \nu_{\mathfrak{D}}$$

*weakly on $Y(\overline{\mathbb{Q}}_p)$.*

There are 3 (for $p > 2$) or 7 (for $p = 2$) $p$-adic fundamental discriminants.

# Singular moduli are $p$-adically disperse

**Theorem (H–Menares–Rivera-Letelier, 2021)**

*None of the limit measures of* $\mathrm{CM}$ *poins in the p-adic topology has an atom in* $\mathbb{C}_p$.

This implies our second ingredient.

**Theorem (H–Menares–Rivera-Letelier, 2021)**

*Given* $\varepsilon > 0$ *there exists* $r > 0$ *small such that*

$$\frac{1}{h(D)} \left( j(\Lambda_D) \cap B_p(0, r) \right) \leq \varepsilon \text{ for } D \to -\infty.$$

# The last ingredient

**Theorem (H–Menares–Rivera-Letelier, 2021)**

*For $j$ a nonzero singular modulus of discriminant $D$ we have*

$$-\log |j|_p \leq c_p \log |D|,$$

*with $c_p > 0$ absolute constant.*

The proof uses ideas of F. Charles (2018) and results of Gross (1986): deformation theory of elliptic curves, formal groups/modules, and canonical/quasi-canonical liftings.

With the three main ingredients, we get the result!
The strategy is essentially the same for differences of singular moduli that are $S$-units.

# Final comments

In

*On singular moduli that are $S$-units* (2020)

F. Campagna shows that $S_0 = \{p \text{ prime }, p \equiv 1 \bmod 3\}$ every singular $S$-unit is a singular unit, hence there are none.

We can use Campagna's result to extend ours to certain classes of infinite sets $S$ of prime numbers (larger than $S_0$).

# Other modular functions

Habegger asked us[3]: What about the $\lambda$-invariants? These are Hauptmoduln for $\Gamma(2)$.

**General question:** What about more general Hauptmoduln?

The method seems to extend without major difficulties to the case of differences of singular moduli that are $S$-units for any Hauptmodul of a genus zero subgroup of $\mathrm{GL}_2^+(\mathbb{Q})$ that is algebraically related to the $j$-function.

[3] (private communication)

# Examples

1. The $\lambda$-invariants: there are six of them, they satisfy

$$2^8(1-\lambda+\lambda^2)^3 - j\lambda^2(1-\lambda)^2 = 0.$$

In

*The lambda invariant at* CM *points* (2018)

Yang, Yin and Yu chose a particular lambda invariant $\lambda_0$ and proved that $\lambda_0(z)$ is an algebraic unit for infinitely many CM points $z$.

2. Weber functions $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2$ are roots of

$$(X^{24} \pm 16)^3 - X^{24} j = 0.$$

Weber's computations show that infinitely many singular moduli for these functions are algebraic units.

**Note that 0 is not a singular modulus for any of these functions.**

# Thanks for your attention!

¡Muchas gracias!