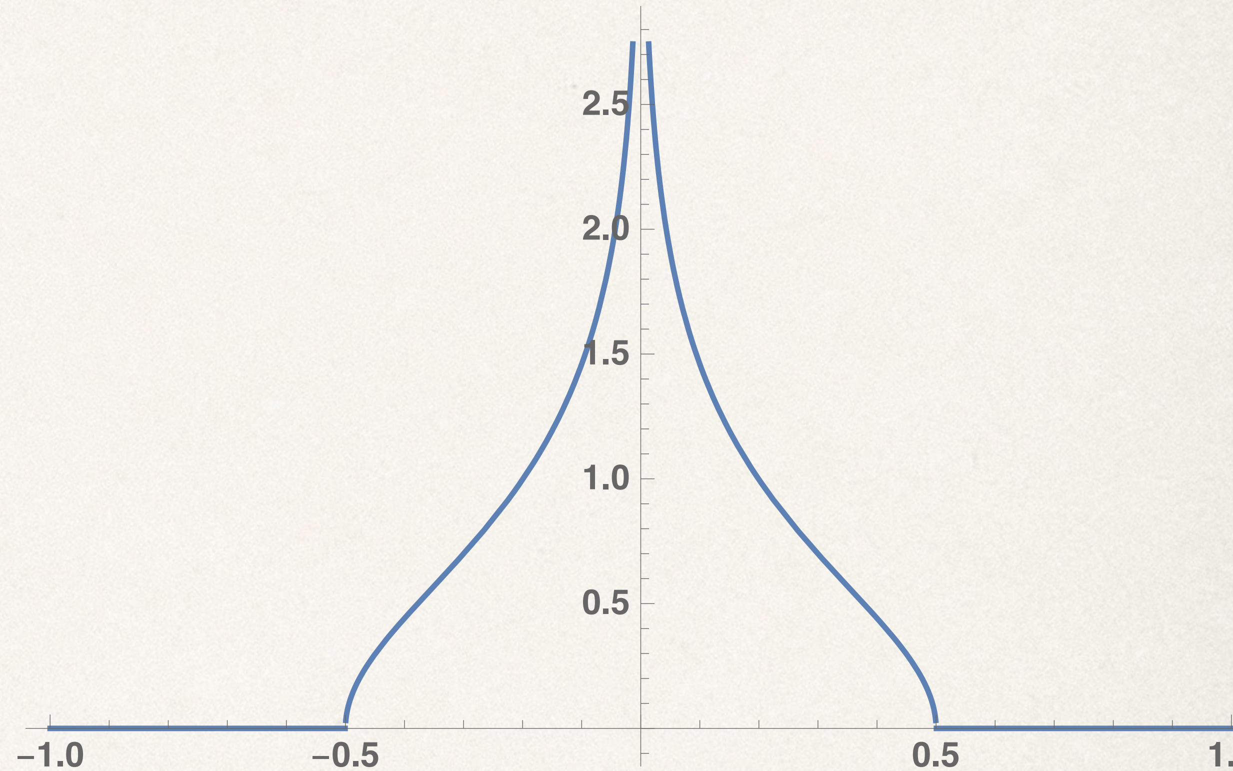


Magic functions and the equidistribution of zeros of polynomials

Emanuel Carneiro (ICTP - Trieste and IMPA - Rio de Janeiro)

May 2021

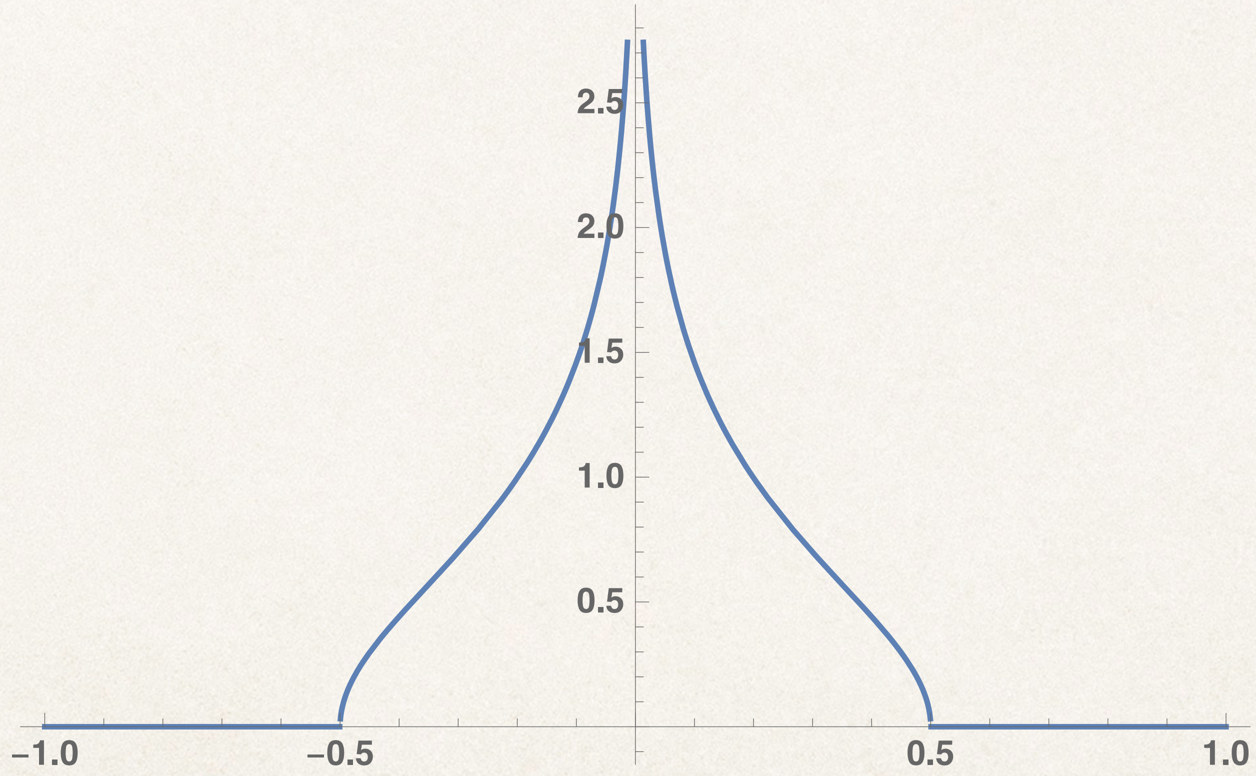
Prologue

Special things…

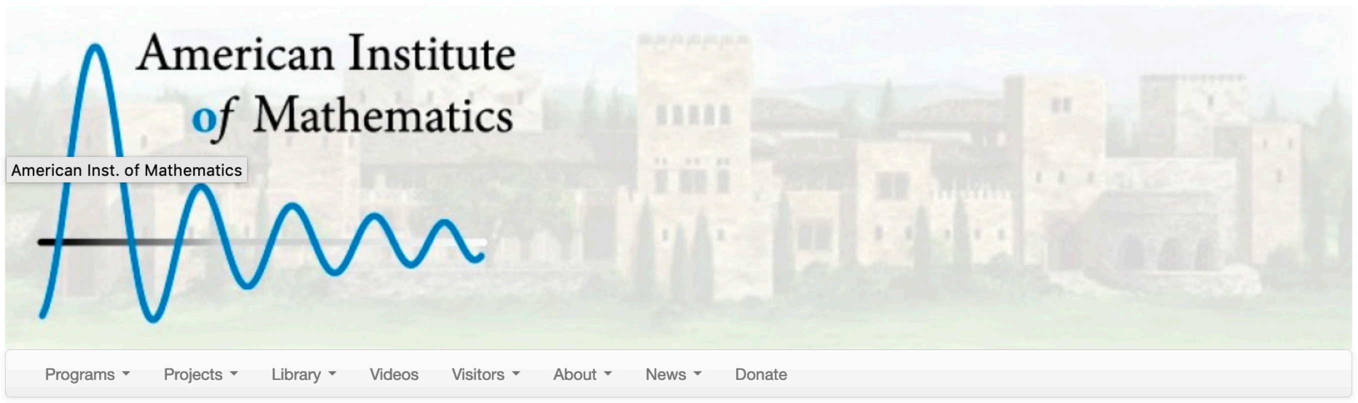
Example: 30/October

My goal today: convince you that something is special…

$$F(x) := \frac{2}{\pi}\log\left(\frac{1+\sqrt{1-4x^2}}{2|x|}\right) \qquad \left(\text{for } -\tfrac{1}{2} \le x \le \tfrac{1}{2}\right)$$



How our story begins



Applications are closed for this workshop

Arithmetic statistics, discrete restriction, and Fourier analysis

February 15 to February 19, 2021
at the
American Institute of Mathematics, San Jose, California

organized by

Theresa Anderson, Frank Thorne, and Trevor Wooley

This workshop, sponsored by AIM and the NSF, aims to explore several problems at the interface of harmonic analysis and analytic number theory, with an eye to bringing both groups of researchers together to make progress in discrete restriction, arithmetic statistics, exponential sum estimates and discrete harmonic analysis by using tools from both fields.

Number theory and analysis share many interactions, and there are several emerging areas where input from both fields will likely be quite fruitful. Arithmetic statistics is a subject focused on counting of objects of algebraic interest, has been extensively investigated by Bhargava and collaborators, and seems ripe for Fourier analytic input. Discrete restriction, as pioneered by Bourgain, is rooted in analysis but is sometimes amenable to number theoretic exponential sum estimates inaccessible to such tools as decoupling methods. Discrete analogues in harmonic analysis have been classified in many ways, but are frequently impeded by limited progress on deep number theoretic problems. By bringing together researchers from both analysis and number theory and having them interact on a variety of problems of emerging interest, we hope to make progress on several areas including:

Fourier optimization wonderland

- Tools from Fourier analysis are often useful when treating problems in number theory.

- Ideal situation: to be able to identify an analysis problem, as pure as possible, inside your number problem (and hopefully solve it).

Part I - A brief overview

Our project

Equidistribution of Zeros of Polynomials

K. Soundararajan

Abstract. A classical result of Erdős and Turán states that if a monic polynomial has small size on the unit circle and its constant coefficient is not too small, then its zeros cluster near the unit circle and become equidistributed in angle. Using Fourier analysis we give a short and self-contained proof of this result.

1. INTRODUCTION. Any set of N complex numbers may be viewed as the zero set of a polynomial of degree N. If, however, we start with a polynomial that "arises naturally"—for example, think of polynomials with coefficients ± 1—then the zeros will tend to be "evenly distributed near the unit circle." In [6], Erdős and Turán proved the beautiful result that if the size of a monic polynomial on the unit circle is small, and its constant term is not too small, then its zeros cluster around the unit circle and become equally distributed in sectors. We shall make precise both the hypothesis and conclusion of this statement later, but we hope Figure 1 gives an impression of the

- Mithun Das
- Alexandra Florea
- Angel Kumchev
- Amita Malik
- Micah Milinovich
- Caroline Turnage-Butternaugh
- Jiuya Wang

$$P(z) = z^N + a_{N-1}z^{N-1} + \ldots + a_1 z + a_0$$

- Erdös-Turán (1950): If the size of P(z) in the unit circle is “small” and the constant coefficient is “not too small” then the zeros of the polynomial tend to cluster around the unit circle and the angles of such roots tend to become equidistributed as the degree grows.

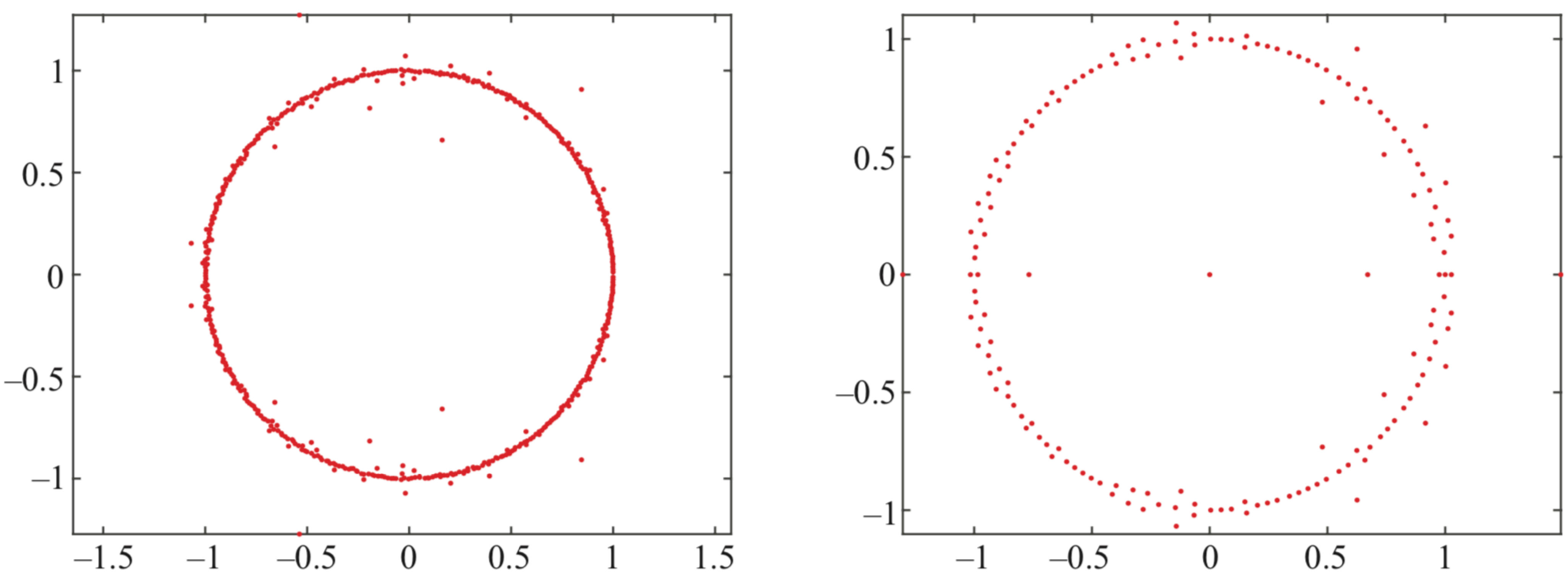


Figure 1. Left: Zeros of a polynomial of degree 500 formed with the decimal digits of π: $3z^{500} + z^{499} + 4z^{498} + \cdots$. Right: Zeros of the Fekete polynomial $\sum_{j=0}^{162}(\frac{j}{163})z^j$ where the coefficients are given by the Legendre symbol $(\frac{j}{163}) = \pm 1$ for $1 \leq j \leq 162$.

Source: K. Soundararajan, Amer. Math. Monthly, 2019.

Height and discrepancy

$$P(z) = \prod_{j=1}^{N} (z - \alpha_j) = z^N + a_{N-1} z^{N-1} + \ldots + a_1 z + a_0 \quad ; \quad a_0 \neq 0 \;\; ; \;\; \alpha_j = \rho_j \, e^{2\pi i \theta_j} .$$

- Two notions of height have been considered:

$$H(P) = \max_{|z|=1} \frac{|P(z)|}{\sqrt{|a_0|}} \quad \text{and} \quad h(P) = \int_0^1 \log^+ \frac{\left|P\left(e^{2\pi i \theta}\right)\right|}{\sqrt{|a_0|}} \, d\theta .$$

- It can be shown that : $h(P) \leq \log H(P)$;

$$\sum_{j=1}^{N} \log \left(\max \left\{ \rho_j \, , \rho_j^{-1} \right\} \right) \leq 2h(P) .$$

- Given an interval $I \subset \mathbb{R}/\mathbb{Z}$ let $N(I;P) = \#\{\alpha_j = \rho_j e^{2\pi i\theta_j} \ : \ \theta_j \in I\}$.

- Define the angular discrepancy by $\mathscr{D}(P) := \sup_I \Big| N(I;P) - |I|N \Big|$.

$\mathscr{D}(P)$	$\leq C\sqrt{N\,h(P)}$	$\leq C\sqrt{N \log H(P)}$
Erdös-Turán (1950)		$C = 16$
Ganelius (1954)		$C = 2.561\ldots$
Mignotte (1992)	$C = 2.561\ldots$	
Soundararajan (2019)	$C = 8/\pi = 2.546\ldots$	

- Theorem 1: $\mathscr{D}(P) \leq \left(4/\sqrt{\pi}\right)\sqrt{N\,h(P)} = (2.256\ldots)\sqrt{N\,h(P)}$.

PS: The example $P(z) = (z-1)^N$ shows that $C \geq 1.75936\ldots$

$$P(z) = (z-1)^N$$

$$\mathscr{D}(P) = N$$

$$h(P) = N\int_0^1 \log^+\left|e^{2\pi i\theta} - 1\right| d\theta = N\,\frac{3\sqrt{3}\,L(2,\chi_3)}{4\pi} = N\,(0.322...),$$

(C. Smyth)

(D. Boyd, Appendix, 1981)

Hence, if $\mathscr{D}(P) \leq C\sqrt{N\,h(P)} \Rightarrow C \geq 1.759..$

Hilbert transforms

Continuous world

For $F : \mathbb{R} \to \mathbb{C}$:

$$\mathscr{H}(F)(x) := \mathrm{p.v.}\ \frac{1}{\pi} \int_{\mathbb{R}} F(x-t) \frac{1}{t}\, dt$$

$$:= \left(-i\, \mathrm{sgn}(t)\, \widehat{F}(t) \right)^{\check{}} \quad (\text{if } F \in L^2(\mathbb{R}))$$

$$\widehat{F}(t) = \int_{-\infty}^{\infty} F(x)\, e^{-2\pi i x t}\, dx$$

$\mathscr{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ $(1 < p < \infty)$ is a bounded operator

Hilbert transforms

Periodic world

For $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$:

$$\mathscr{H}(f)(\theta) = \mathrm{p.v.} \int_{\mathbb{R}/\mathbb{Z}} f(\theta - \alpha)\ \cot(\pi\alpha)\, d\alpha$$

$$:= \Big(-i\, \mathrm{sgn}(k)\, \hat{f}(k) \Big)^{\vee} \quad (\text{if } f \in L^2(\mathbb{R}/\mathbb{Z}))$$

$$\hat{f}(k) = \int_{\mathbb{R}/\mathbb{Z}} f(x)\, e^{-2\pi i x k}\, dx$$

$\mathscr{H} : \ell^p(\mathbb{R}/\mathbb{Z}) \to \ell^p(\mathbb{Z})\ \ (1 < p < \infty)$ is a bounded operator

Class of functions

$$\mathscr{A} = \begin{cases} F : \mathbb{R} \to \mathbb{R} \ \text{ even, continuous and non-negative;} \\ \operatorname{supp}(F) \subseteq [-\frac{1}{2}, \frac{1}{2}]; \\ \widehat{F} \in L^1(\mathbb{R}) . \end{cases}$$

For each $F \in \mathscr{A}$ we consider its periodization $f_F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$

$$f_F(\theta) := \sum_{k \in \mathbb{Z}} F(\theta + k) .$$

Extremal Problem 1 Find the infimum:

$$\mathbf{C} := \inf_{0 \neq F \in \mathscr{A}} \frac{\max \left\{ \|\mathscr{H}(F)\|_{L^\infty(\mathbb{R})}, \|\mathscr{H}(f_F)\|_{L^\infty(\mathbb{R}/\mathbb{Z})} \right\}}{\|F\|_{L^1(\mathbb{R})}} .$$

Theorem 2: $\mathscr{D}(P) \leq \dfrac{4\sqrt{\mathbf{C}}}{\sqrt{\pi}} \sqrt{N\, h(P)}$

Theorem 3: $\mathbf{C} = 1$

Part II - How does the extremal problem appear?

Simplification 1: Schur

Let $P(z) = \prod_{j=1}^{N} \left(z - \rho_j e^{2\pi i \theta_j}\right)$ and define $Q(z) = \prod_{j=1}^{N} \left(z - e^{2\pi i \theta_j}\right)$. Then, for $|z| = 1$

$$\left| \frac{z}{\sqrt{\rho_j}} - \sqrt{\rho_j}\, e^{2\pi i \theta_j} \right|^2 \leq \left| z - e^{2\pi i \theta_j} \right|^2 \Longrightarrow |P(z)|/\sqrt{|a_0|} \geq |Q(z)| \Longrightarrow h(P) \geq h(Q)$$

Simplification 2:

Since

$$N(I; P) - |I| N = |I^c| N - N(I^c; P),$$

it suffices to prove upper bounds for $N(I; P) - |I| N$.

Idea: Given an interval $I \subset \mathbb{R}/\mathbb{Z}$ let $g_\delta \geq \chi_I$

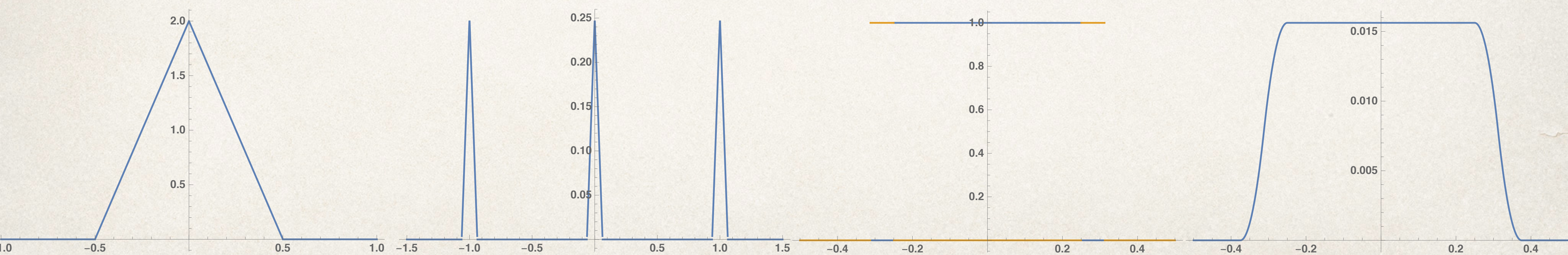
$$N(I;P) - |I|N \leq \sum_{j=1}^{N} g_\delta(\theta_j) - |I|N \lesssim \ldots \lesssim N\delta + \frac{h(P)}{\delta}.$$

Proper implementation: Let $0 \neq F \in \mathscr{A}$ and $0 < \delta \leq 1$. Assume $\|F\|_1 = 1$.

Define $F_\delta(x) := \frac{1}{\delta} F\left(\frac{x}{\delta}\right)$ and $f_\delta(\theta) := \sum_{k \in \mathbb{Z}} F_\delta(\theta + k)$.

(note that $\operatorname{supp}(F_\delta) \subset [-\frac{\delta}{2}, \frac{\delta}{2}]$). Define I_δ as the interval “enlarged” by $\frac{\delta}{2}$ on each side and

$$g_\delta(\theta) := \int_{\mathbb{R}/\mathbb{Z}} \chi_{I_\delta}(\alpha) f_\delta(\theta - \alpha)\, d\alpha$$



$$g_\delta(\theta) := \int_{\mathbb{R}/\mathbb{Z}} \chi_{I_\delta}(\alpha) f_\delta(\theta - \alpha)\, d\alpha \Longrightarrow \int_{\mathbb{R}/\mathbb{Z}} g_\delta(\theta)\, d\theta = |I_\delta| \le |I| + \delta$$

$$\begin{aligned}
N(I;P) - |I|N &\le \sum_{j=1}^{N} g_\delta(\theta_j) - |I|N \\
&= \sum_{j=1}^{N} \left(\sum_{k=-\infty}^{\infty} \widehat{g_\delta}(k)\, e^{2\pi i \theta_j k} \right) - |I|N \\
&= \left(N\widehat{g_\delta}(0) - |I|N \right) + \sum_{j=1}^{N} \left(\sum_{k\neq 0} \widehat{g_\delta}(k)\, e^{2\pi i \theta_j k} \right) \\
&\le N\delta + \sum_{k \neq 0} \widehat{g_\delta}(k) \left(\sum_{j=1}^{N} e^{2\pi i \theta_j k} \right)
\end{aligned}$$

Use the identity:

$$\int_{\mathbb{R}/\mathbb{Z}} -2|k|\, e^{2\pi i k\theta} \log \left| P\left(e^{2\pi i \theta}\right) \right| d\theta = \sum_{j=1}^{N} e^{2\pi i k \theta_j}.$$

$$\sum_{k\neq 0} \widehat{g_\delta}(k) \left(\sum_{j=1}^{N} e^{2\pi i \theta_j k} \right) = \int_{\mathbb{R}/\mathbb{Z}} \sum_{k\neq 0} -2\,|\,k\,|\,\widehat{g_\delta}(k)\, e^{2\pi i k\theta} \log \left| P\left(e^{2\pi i\theta}\right) \right| d\theta$$

$$\leq \mathcal{G}_\delta \int_{\mathbb{R}/\mathbb{Z}} \left| \log \left| P\left(e^{2\pi i\theta}\right) \right| \right| d\theta$$

$$= \mathcal{G}_\delta \left(\int_{\mathbb{R}/\mathbb{Z}} 2\log^+ \left| P\left(e^{2\pi i\theta}\right) \right| d\theta - \int_{\mathbb{R}/\mathbb{Z}} \log \left| P\left(e^{2\pi i\theta}\right) \right| d\theta \right)$$

$$= 2\,\mathcal{G}_\delta\, h(P)$$

where:

$$\mathcal{G}_\delta := \max_\theta \left| \sum_{k\neq 0} 2\,|\,k\,|\,\widehat{g_\delta}(k) e^{2\pi i k\theta} \right|$$

Assume that $I_\delta = [\alpha, \beta]$. Since $g_\delta = \chi_{I_\delta} * f_\delta$

$$\widehat{g_\delta}(k) = \widehat{\chi_{I_\delta}}(k)\,\widehat{f_\delta}(k) = \left(\frac{e^{-2\pi i k\alpha} - e^{-2\pi i k\beta}}{2\pi i k}\right)\widehat{f_\delta}(k).$$

Hence

$$\sum_{k\neq 0} 2\,|\,k\,|\,\widehat{g_\delta}(k)\,e^{2\pi i k\theta} = \frac{1}{\pi}\sum_{k\neq 0} -\,i\,\mathrm{sgn}(k)\,\widehat{f_\delta}(k)\left(e^{2\pi i k(\theta-\alpha)} - e^{2\pi i k(\theta-\beta)}\right),$$

and therefore

$$\begin{aligned}\left|\sum_{k\neq 0} 2\,|\,k\,|\,\widehat{g_\delta}(k)\,e^{2\pi i k\theta}\right| &\leq \frac{1}{\pi}\left|\sum_{k\neq 0} -\,i\,\mathrm{sgn}(k)\,\widehat{f_\delta}(k)\,e^{2\pi i k(\theta-\alpha)}\right| + \frac{1}{\pi}\left|\sum_{k\neq 0} -\,i\,\mathrm{sgn}(k)\,\widehat{f_\delta}(k)\,e^{2\pi i k(\theta-\beta)}\right| \\ &= \frac{1}{\pi}\left|\mathscr{H}(f_\delta)(\theta-\alpha)\right| + \frac{1}{\pi}\left|\mathscr{H}(f_\delta)(\theta-\beta)\right| \\ &\leq \frac{2}{\pi}\|\mathscr{H}(f_\delta)\|_{L^\infty(\mathbb{R}/\mathbb{Z})}.\end{aligned}$$

We ended up proving that

$$N(I;P) - |I|\,N \le N\delta + \frac{4}{\pi}\|\mathscr{H}(f_\delta)\|_{L^\infty(\mathbb{R}/\mathbb{Z})}.$$

Hence we need to understand how big $\delta\,\|\mathscr{H}(f_\delta)\|_{L^\infty(\mathbb{R}/\mathbb{Z})}$ can be.

Lemma: $\displaystyle\sup_{0<\delta\le 1} \delta\,\|\mathscr{H}(f_\delta)\|_{L^\infty(\mathbb{R}/\mathbb{Z})} = \max\left\{\|\mathscr{H}(F)\|_{L^\infty(\mathbb{R})},\, \|\mathscr{H}(f_F)\|_{L^\infty(\mathbb{R}/\mathbb{Z})}\right\}.$

Calling $\mathscr{C}(F) := \max\left\{\|\mathscr{H}(F)\|_{L^\infty(\mathbb{R})},\, \|\mathscr{H}(f_F)\|_{L^\infty(\mathbb{R}/\mathbb{Z})}\right\}.$

$$N(I;P) - |I|\,N \le N\delta + \frac{4\mathscr{C}(F)}{\pi\,\delta}.$$

Choosing $\delta = \sqrt{\dfrac{4\,\mathscr{C}(F)\,h(P)}{\pi N}} \Longrightarrow N(I;P) - |I|\,N \le \dfrac{4\sqrt{\mathscr{C}(F)}}{\sqrt{\pi}}\sqrt{N\,h(P)}.$

$$\mathscr{D}(P) \le \frac{4\sqrt{\mathbf{C}}}{\sqrt{\pi}}\sqrt{N\,h(P)}$$

Part III - Magic functions

Competition is fair…

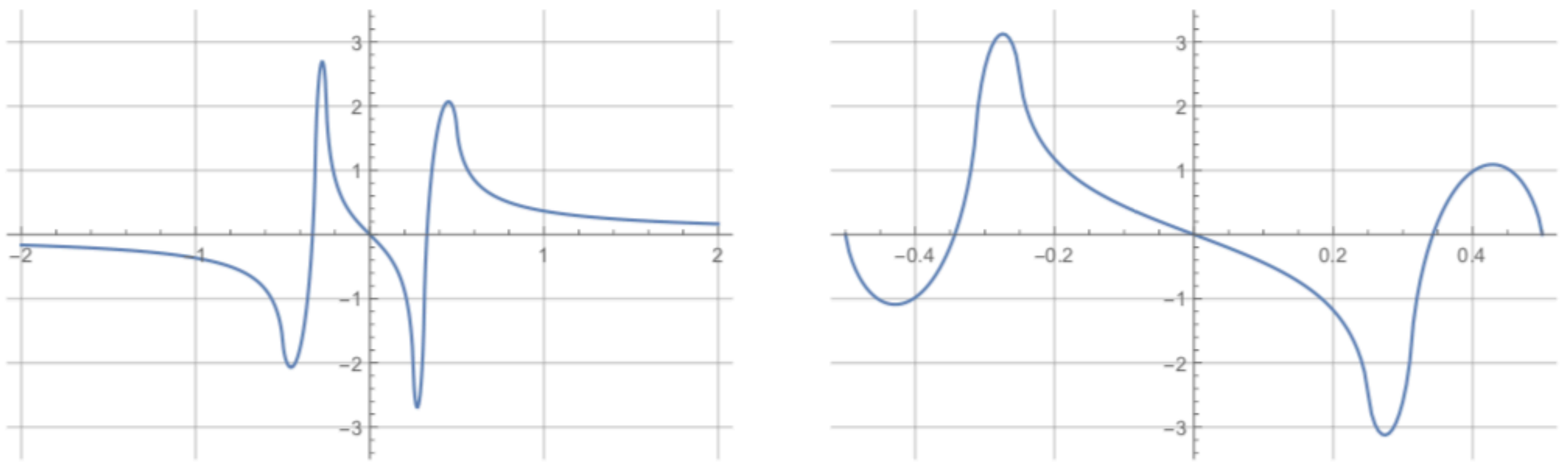
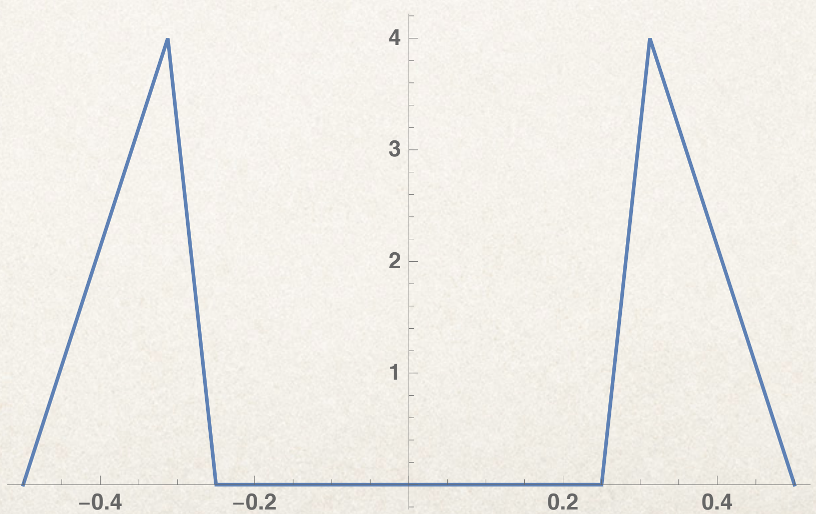
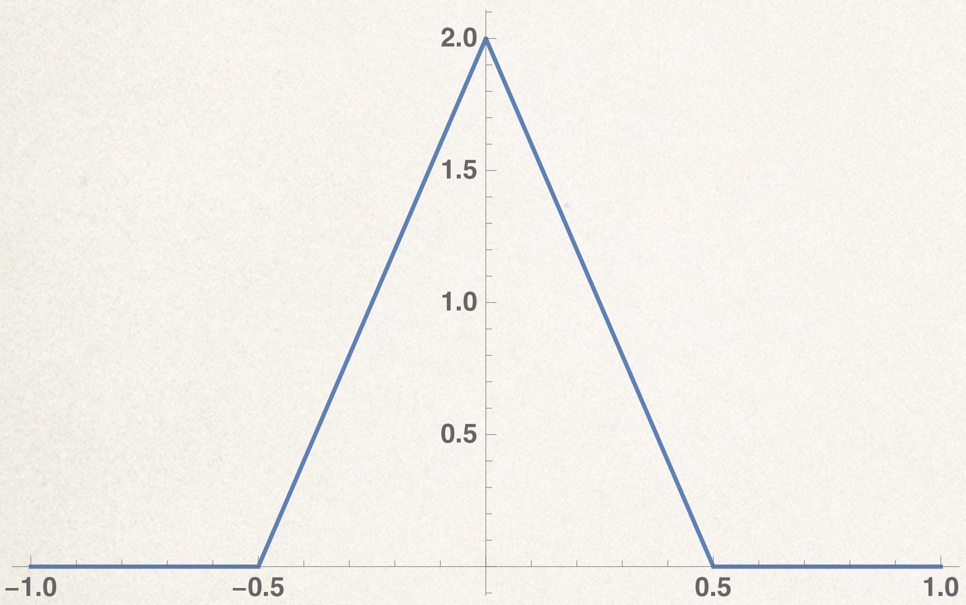


FIGURE 1. For the function F defined in (4.3), on the left we have the graph of $\mathcal{H}(F)$ and on the right we have the graph of $\mathcal{H}(f_F)$.

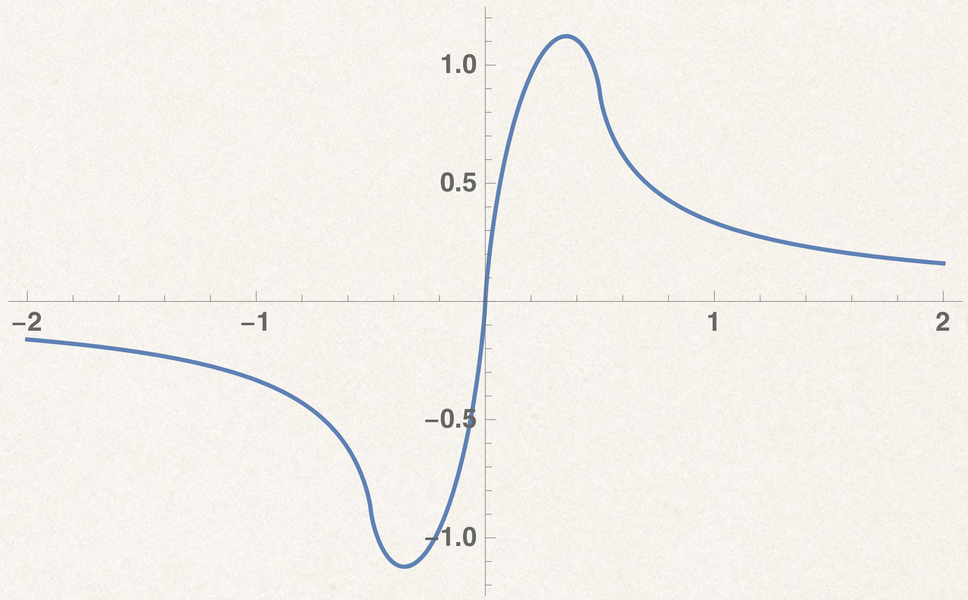
$$F(x) = \begin{cases} 0, \text{ if } 0 \leq |x| \leq \frac{1}{4}; \\ 64|x| - 16, \text{ if } \frac{1}{4} \leq |x| \leq \frac{5}{16}; \\ \frac{1}{3}(32 - 64|x|), \text{ if } \frac{5}{16} \leq |x| \leq \frac{1}{2}. \end{cases}$$



The triangle function



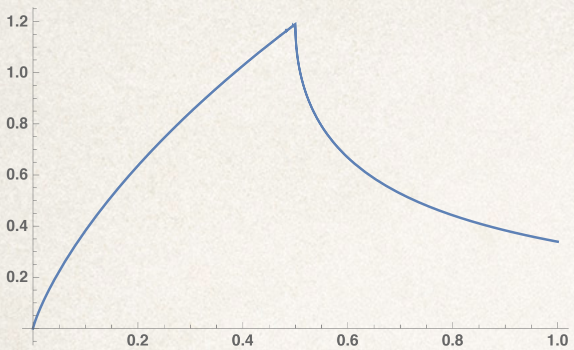
$F_{\blacktriangle}(x) = 2(1 - 2|x|)_+$

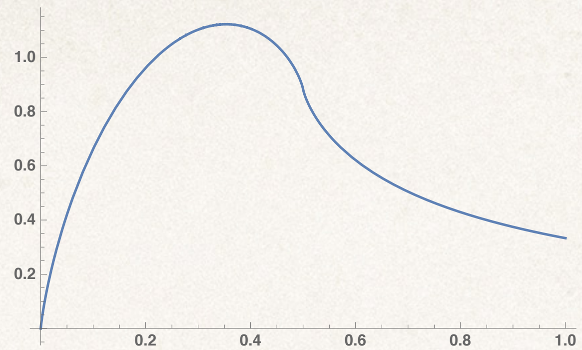


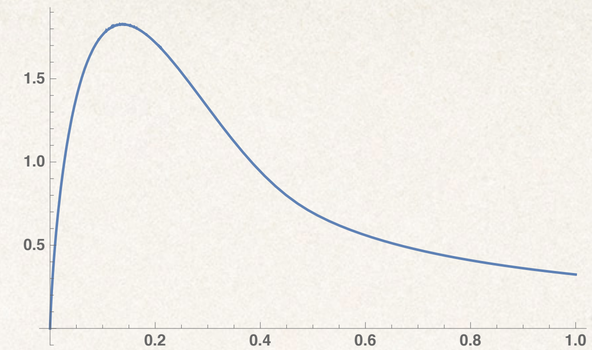
$\mathscr{H}(F_{\blacktriangle})$

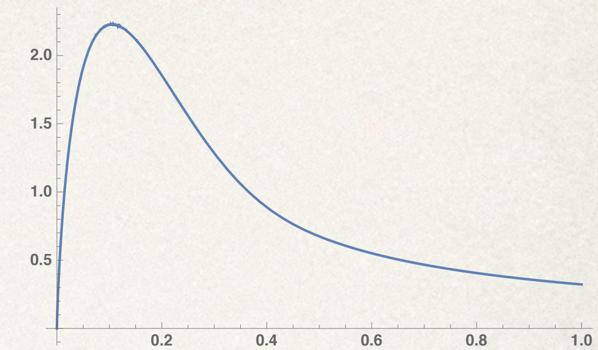
$$\|\mathscr{H}(F_{\blacktriangle})\|_{L^\infty(\mathbb{R})} = \mathscr{H}(F_{\blacktriangle})\left(\frac{1}{2\sqrt{2}}\right) = \frac{4}{\pi}\log(1+\sqrt{2}) = 1.12...$$

Leads to $\mathscr{D}(P) \leq (2.39\ldots)\sqrt{N\,h(P)}$





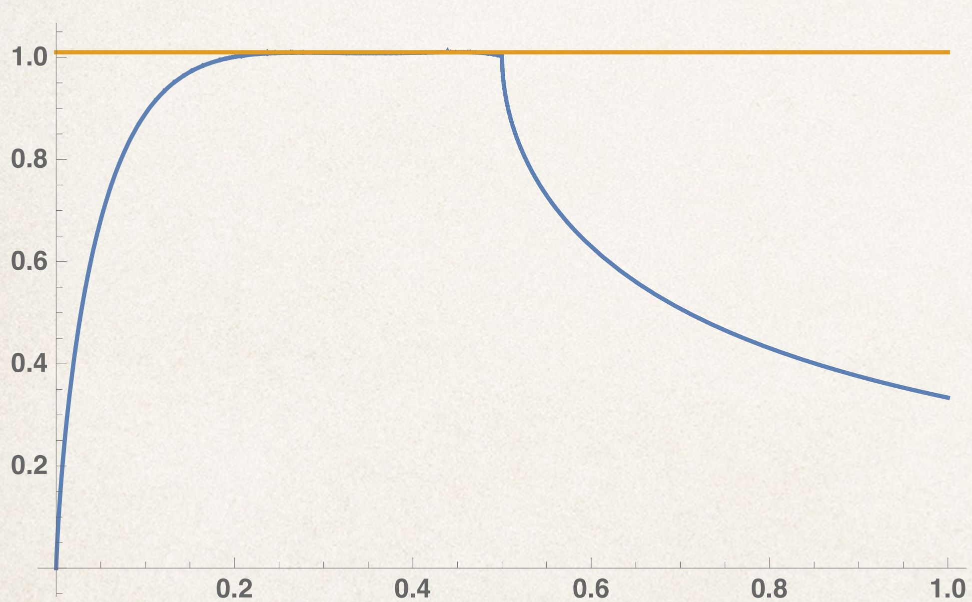


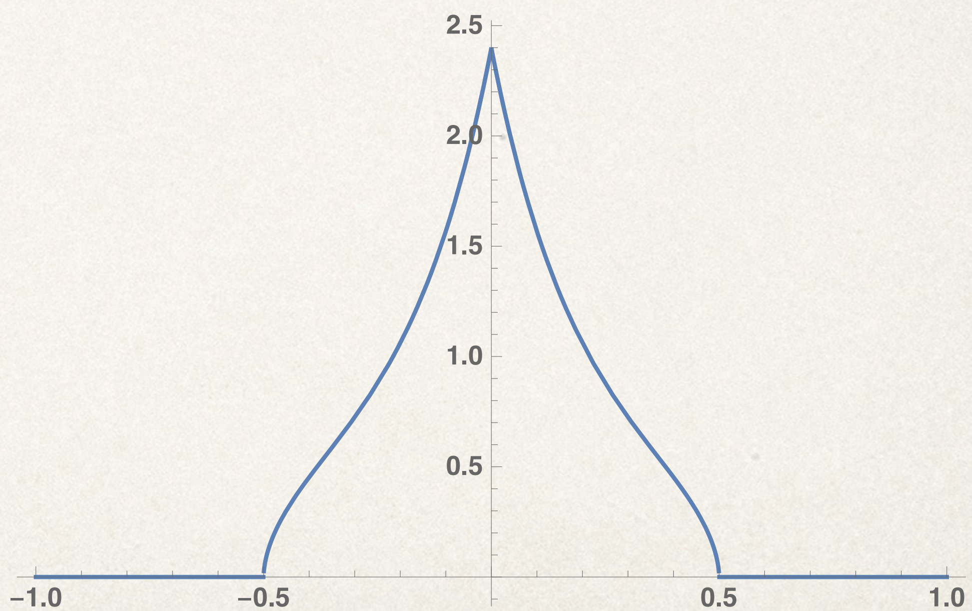


$\nu = 1/2$ $\nu = 1$ $\nu = 3$ $\nu = 4$

$$F_\nu(x) = c_\nu(1 - 2|x|)_+^\nu$$

Lots and lots of convex combinations of convex combinations later…

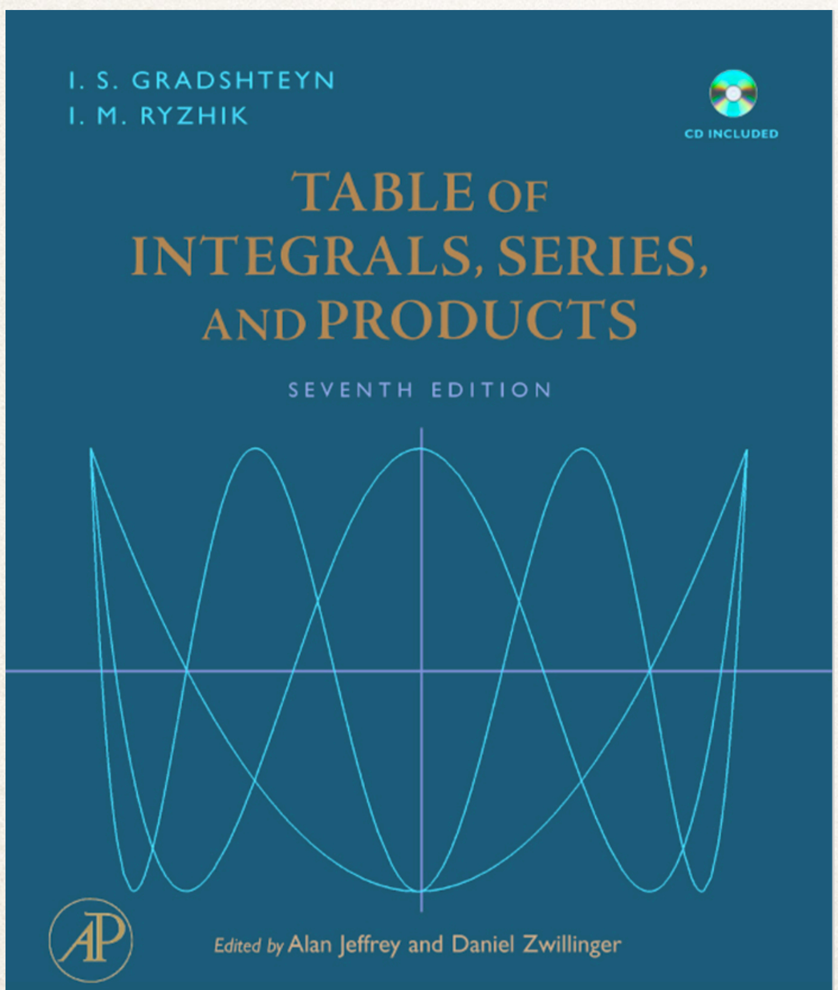




$\mathbf{C} \leq 1.015$

The first miracle

$$\mathscr{H}(F)(x) := \frac{1}{\pi}\int_{\mathbb{R}} F(x-t)\frac{1}{t}\,dt = \frac{1}{\pi}\int_0^{1/2} F'(t)\ \log\left(\frac{|x-t|}{|x+t|}\right)\,dt$$



4.297

1.	$\int_0^1 \ln\frac{ax+b}{bx+a}\frac{dx}{(1+x)^2} = \frac{1}{a-b}\left[(a+b)\ln\frac{a+b}{2} - a\ln a - b\ln b\right]$	$[a>0,\quad b>0]$	BI (115)(16)
2.	$\int_0^\infty \ln\frac{ax+b}{bx+a}\frac{dx}{(1+x)^2} = 0$	$[ab>0]$	BI (139)(23)
3.	$\int_0^1 \ln\frac{1-x}{x}\frac{dx}{1+x^2} = \frac{\pi}{8}\ln 2$		BI (115)(5)
4.	$\int_0^1 \ln\frac{1+x}{1-x}\frac{dx}{1+x^2} = \boldsymbol{G}$		BI (115)(17)
5.[11]	$\int_0^\infty \ln\left(\frac{1+x}{1-x}\right)^2\frac{dx}{x(1+x^2)} = \frac{\pi^2}{2}$		BI (141)(13)
6.	$\int_u^v \ln\frac{v+x}{u+x}\frac{dx}{x} = \frac{1}{2}\left(\ln\frac{v}{u}\right)^2$	$[uv>0]$	BI (145)(33)
7.	$\int_0^\infty \frac{b\ln(1+ax) - a\ln(1+bx)}{x^2}\,dx = ab\ln\frac{b}{a}$	$[a>0,\quad b>0]$	FI II 647
8.	$\int_0^1 \ln\frac{1+ax}{1-ax}\frac{dx}{x\sqrt{1-x^2}} = \pi\arcsin a$	$[\lvert a\rvert \le 1]$	GW (325)(21c), BI (122)(2)
9.	$\int_u^v \ln\left(\frac{1+ax}{1-ax}\right)\frac{dx}{\sqrt{(x^2-u^2)(v^2-x^2)}} = \frac{\pi}{v}F\left(\arcsin av, \frac{u}{v}\right)$	$[\lvert av\rvert < 1]$	BI (145)(35)
10.[8]	$\mathrm{PV}\int_0^1 \ln\left\lvert\frac{a+y}{a-y}\right\rvert\frac{dy}{y\sqrt{1-y^2}} = \frac{\pi^2}{2}$	$[0<a\le 1]$	

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A little bit of reverse engineereing…

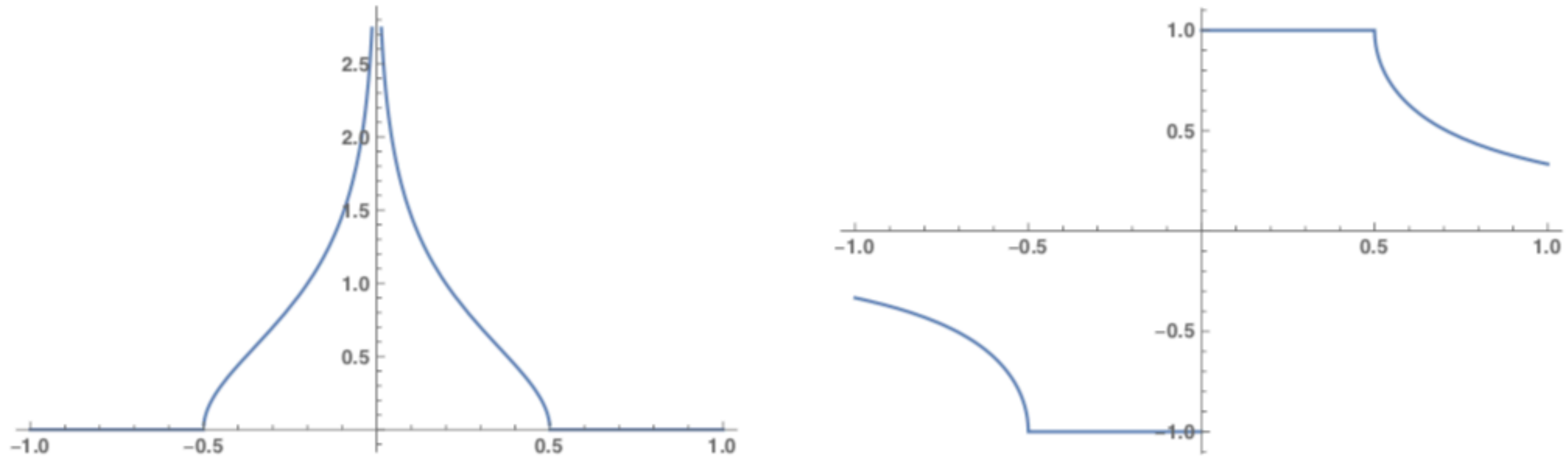

FIGURE 3. On the left, the graph of the magic function $\mathfrak{F}$. On the right, the graph of the Hilbert transform $\mathcal{H}(\mathfrak{F})$.

$$F(x) := \frac{2}{\pi} \log\left(\frac{1 + \sqrt{1 - 4x^2}}{2|x|} \right) \qquad \left(\text{for } -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}\right)$$

$$\mathscr{H}(F)(x) = \begin{cases} \operatorname{sgn}(x), \text{ if } |x| \leq \frac{1}{2}; \\ \frac{2}{\pi} \arcsin\left(\frac{1}{2x}\right), \text{ if } |x| > \frac{1}{2}. \end{cases}$$

Conclusion: $\mathbf{C} \leq 1$

How to prove a lower bound? A duality argument

Assume that there exists a function $G : \mathbb{R} \to \mathbb{R}$

$$\|G\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |G(x)|\, dx = 1.$$

$$\mathscr{H}(G)(x) = -1 \quad \text{a.e. on} \quad -\frac{1}{2} < x < \frac{1}{2}$$

Then, for any $F \in \mathscr{A}$ normalized so that $\|F\|_{L^1(\mathbb{R})} = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(x)\, dx = 1$,

$$\|\mathscr{H}(F)\|_{L^\infty(\mathbb{R})} = \|\mathscr{H}(F)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |G(x)|\, dx \geq \left| \int_{\mathbb{R}} \mathscr{H}(F)(x)\, \overline{G(x)}\, dx \right|$$

$$= \left| -\int_{\mathbb{R}} F(x)\, \overline{\mathscr{H}(G)(x)}\, dx \right| = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(x)\, dx = 1$$

The second miracle

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Higher Transcendental Functions, 3 volumes.
Tables of Integral Transforms, 2 volumes.

Elementary functions (cont'd)

	$f(x)$	$\pi^{-1} \fint_{-\infty}^{\infty} f(x)\,(x-y)^{-1}\,dx$
(23)	$-(x^2-a^2)^{-1/2}$ $\quad -\infty<x<-a$ 0 $\quad -a<x<a$ $(x^2-a^2)^{-1/2}$ $\quad a<x<\infty$	0 $\quad -\infty<y<-a$ $(a^2-y^2)^{-1/2}$ $\quad -a<y<a$ 0 $\quad a<y<\infty$
(24)	0 $\quad -\infty<x<0$ $(a-x)^{1/2}(a+x)^{-1/2}$ $\quad 0<x<a$ 0 $\quad a<x<\infty$	$-\frac{1}{2}+\frac{1}{\pi}\left\|\frac{a-y}{a+y}\right\|^{1/2}\cos^{-1}\left(-\frac{a}{y}\right)$ $-\infty<y<-a$ $-\frac{1}{2}+\frac{1}{\pi}\left(\frac{a-y}{a+y}\right)^{1/2}\log\left\|\frac{a+(a^2-y^2)^{1/2}}{-y}\right\|$ $-a<y<a$ $-\frac{1}{2}+\frac{1}{\pi}\left(\frac{y-a}{y+a}\right)^{1/2}\cos^{-1}\left(-\frac{a}{y}\right)$ $a<y<\infty$ $0<\cos^{-1}<\pi$
(25)	0 $\quad -\infty<x<-a$ $(a-x)^{1/2}(a+x)^{-1/2}$ $\quad -a<x<a$ 0 $\quad a<x<\infty$	$-1+(a-y)^{1/2}\,\|y+a\|^{-1/2}$ $-\infty<y<-a$ -1 $\quad -a<y<a$ $-1+(y-a)^{1/2}(y+a)^{-1/2}$ $a<y<\infty$

Again, a little bit of reverse engineereing…

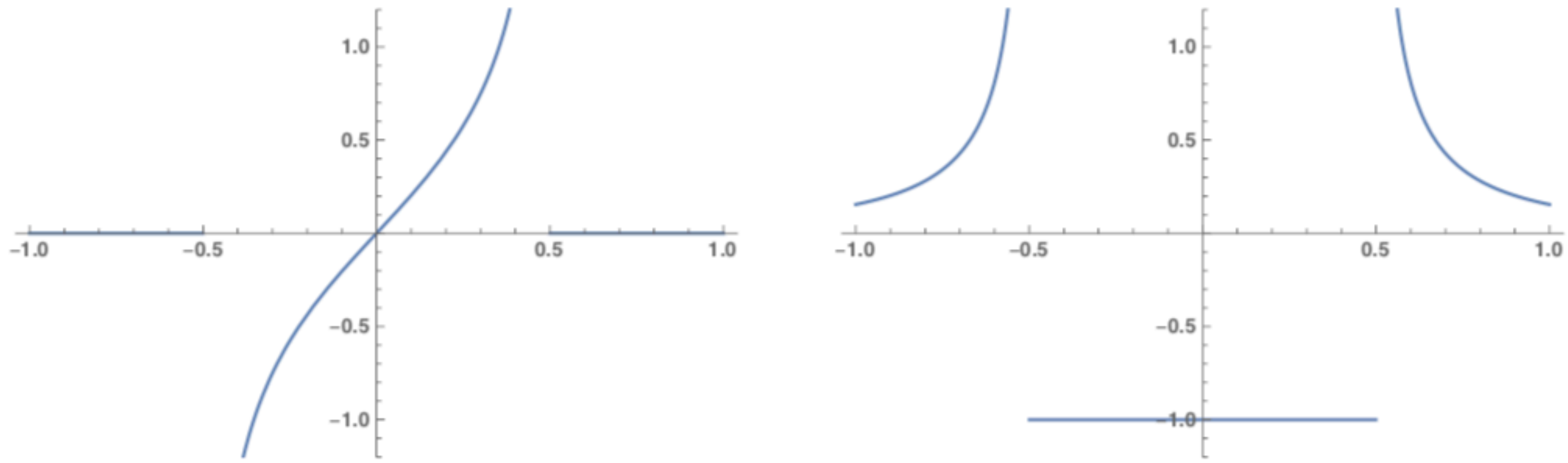

FIGURE 2. On the left, the graph of the magic function $\mathfrak{G}$. On the right, the graph of the Hilbert transform $\mathcal{H}(\mathfrak{G})$.

$$G(x) := \frac{2x}{\sqrt{1-4x^2}} \qquad \left(\text{for } -\tfrac{1}{2} \le x \le \tfrac{1}{2}\right)$$

$$\mathscr{H}(G)(x) = \begin{cases} -1, \text{ if } |x| < \frac{1}{2}; \\ -1 + \frac{2|x|}{\sqrt{4x^2-1}}, \text{ if } |x| > \frac{1}{2}. \end{cases}$$

Conclusion: $\mathbf{C} \ge 1$

An interesting related result

Class of functions

$$\mathscr{A}^* = \begin{cases} F \in L^1(\mathbb{R}) \quad ; \quad F \geq 0 \\ \text{supp}(F) \subseteq [-\tfrac{1}{2}, \tfrac{1}{2}] \, . \end{cases}$$

Extremal Problem 2

Find the infimum: $\mathbf{C}^* := \inf_{0 \neq F \in \mathscr{A}^*} \dfrac{\|\mathscr{H}(F)\|_{L^\infty(\mathbb{R})}}{\|F\|_{L^1(\mathbb{R})}}$.

Theorem 3: $\mathbf{C}^* = 1$ and the unique extremal function is

$$F(x) := \frac{2}{\pi} \log\left(\frac{1 + \sqrt{1 - 4x^2}}{2\,|\,x\,|} \right) \qquad \left(\text{for } -\tfrac{1}{2} \leq x \leq \tfrac{1}{2}\right)$$

Consequence

$$\|\mathscr{H}(F)\|_{L^\infty(\mathbb{R})} \geq |\,I\,|\,\|F\|_{L^1(\mathbb{R})} \quad \text{if} \quad \begin{cases} F \in L^1(\mathbb{R}) \quad ; \quad F \geq 0 \\ \text{supp}(F) \subseteq I \, . \end{cases}$$

Thank you!