On a conjecture of Darmon–Rotger in the adjoint CM case

Seminario Latinoamericano de Teoría de Números

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Outline

Kato classes

- $E/\mathbb{Q}$ elliptic curve of conductor N.
- $p \nmid 2N$ good ordinary prime for E.
- $V_pE := (\varprojlim_n E[p^n]) \otimes \mathbb{Q}_p$.

Theorem (Kato's ERL)

There is a class $\kappa_p^{\mathrm{Kato}} \in \mathrm{H}^1(\mathbb{Q}, V_pE)$ such that

$$\exp^*_{\mathrm{BK}}(\mathrm{Loc}_p(\kappa_p^{\mathrm{Kato}})) \doteq \frac{L(E,1)}{\Omega_E}$$

where $\mathrm{Loc}_p : \mathrm{H}^1(\mathbb{Q}, V_pE) \to \mathrm{H}^1(\mathbb{Q}_p, V_pE)$ is the restriction maps at p, and

$$\exp^*_{\mathrm{BK}} : \mathrm{H}^1(\mathbb{Q}_p, V_pE) \to \mathbb{Q}_p$$

is Bloch–Kato's dual exponential map.

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Perrin-Riou's conjecture

Let $\mathrm{Sel}(\mathbb{Q}, V_pE) \subset \mathrm{H}^1(\mathbb{Q}, V_pE)$ be the p-adic Selmer group of E:

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p \to \mathrm{Sel}(\mathbb{Q}, V_pE) \to (\varprojlim Ш(E/\mathbb{Q})[p^n]) \otimes \mathbb{Q}_p \to 0.$$

By Kato's ERL, $L(E,1) = 0 \iff \kappa_p^{\mathrm{Kato}} \in \mathrm{Sel}(\mathbb{Q}, V_pE)$.

Conjecture (Perrin-Riou, 1993)

Suppose that $L(E,1) = 0$. *Then the following are equivalent:*

(1) $\kappa_p^{\mathrm{Kato}} \neq 0$.

(2) $\mathrm{ord}_{s=1} L(E,s) = 1$.

(3) $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1$ *and* $\# Ш(E/\mathbb{Q})[p^\infty] < \infty$.

Moreover, in that case $\log_E(\kappa_p^{\mathrm{Kato}}) = \log_E(P)^2 \pmod{\mathbb{Q}^\times}$, *where* $P \in E(\mathbb{Q}) \otimes \mathbb{Q}$ *is any generator.*

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- (2) $\Leftrightarrow$ (3) : proved by Gross–Zagier, Kolyvagin; Skinner, W. Zhang, etc..
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- As a consequence, if $\mathrm{ord}_{s=1} L(E,s) \geqslant 2$ then $\kappa_p^{\mathrm{Kato}} = 0$!

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Darmon–Rotger's explicit reciprocity law

Generalised Kato classes are attached to

- a triple of eigenforms $(f, g, h) \in S_2(N_f) \times S_1(N_g, \epsilon) \times S_1(N_h, \bar{\epsilon})$,
- a choice of roots $(\gamma, \delta) \in \{\alpha_g, \beta_g\} \times \{\alpha_h, \beta_h\}$.

Theorem (Darmon–Rotger's ERL)

There is a generalised Kato class

$$\kappa_{\gamma,\delta}(f, g, h) \in \mathrm{H}^1(\mathbb{Q}, V_{fgh}),$$

where $V_{fgh} = V_f \otimes V_g \otimes V_h$, *such that*

$$\exp^*_{\mathrm{BK}}(\mathrm{Loc}_p(\kappa_{\gamma,\delta}(f, g, h))) \doteq L(1, f \otimes g \otimes h).$$

Note: Roughly speaking, $\kappa_{\gamma,\delta}(f, g, h) = \lim_{k \to 1} \mathrm{AJ}_p(\Delta^{fg_k h_k})$.

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Running hypotheses:

- $\alpha_g \neq \beta_g$ and $\alpha_h \neq \beta_h$, so we have **four** *a priori distinct*

$$\kappa_{\gamma,\delta}(f,g,h) \in \mathrm{H}^1(\mathbb{Q}, V_{fgh}).$$

- $\gcd(N_f, N_g N_h) = 1$, so $\epsilon(f \otimes g \otimes h) = +1$.

Conjecture (Darmon–Rotger)

Suppose that $L(1, f \otimes g \otimes h) = 0$. *Then the following are equivalent:*

(1) *The classes* $\kappa_{\gamma,\delta}(f,g,h)$ *span a non-trivial subspace of* $\mathrm{Sel}(\mathbb{Q}, V_{fgh})$.

(2) $\dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_{fgh}) = 2$.

Note:

- The conjecture does **not** predict that the classes $\kappa_{\gamma,\delta}(f,g,h)$ span the entire $\mathrm{Sel}(\mathbb{Q}, V_{fgh})$.
- The assumptions imply that $\mathrm{ord}_{s=1} L(s, f \otimes g \otimes h) \geq 2$.

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$$\kappa_{\gamma,\delta}(f, g, h) \in \mathrm{H}^1(\mathbb{Q}, V_{fgh}).$$

- $\gcd(N_f, N_g N_h) = 1$, so $\epsilon(f \otimes g \otimes h) = +1$.

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Suppose that $L(1, f \otimes g \otimes h) = 0$. *Then the following are equivalent:*

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- The conjecture does **not** predict that the classes $\kappa_{\gamma,\delta}(f, g, h)$ span the entire $\mathrm{Sel}(\mathbb{Q}, V_{fgh})$.
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Darmon–Rotger's conjecture: rank $(2,0)$ adjoint case

- Suppose now that f corresponds to an elliptic curve $E/\mathbb{Q}$, and $h = g^*$. Then
$$V_{fgg^*} \cong V_pE \oplus (V_pE \otimes \mathrm{ad}^0 V_g)$$
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Suppose that $L(E, 1) = 0$ and has sign $+1$, and that $L(E, \mathrm{ad}^0 V_g, 1) \neq 0$. Then the following are equivalent:

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Outline

The rank $(2,0)$ adjoint CM case

We consider the adjoint CM case:

- K imaginary quadratic field of discriminant prime to N in which
$$(p) = \mathfrak{p}\bar{\mathfrak{p}} \text{ splits.}$$
- $g = \theta_\psi = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}}$, with ψ a ray class character of K of conductor prime of Np.

Then we have the four generalised Kato classes

$$\kappa_{\alpha,\alpha^{-1}}, \quad \kappa_{\alpha,\beta^{-1}}, \quad \kappa_{\beta,\alpha^{-1}}, \quad \kappa_{\beta,\beta^{-1}} \in \mathrm{H}^1(\mathbb{Q}, V_p E),$$

where $\alpha = \psi(\bar{\mathfrak{p}})$ and $\beta = \psi(\mathfrak{p})$.

Note: In this case,

$$L(E, \mathrm{ad}^0 V_p(g), s) = L(E^K, s) \cdot L(E/K, \chi, s),$$

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Main result: rank (2,0) adjoint CM case

Let N^- be the largest factor of N divisible only by primes inert in K.

Theorem (C.–Hsieh)

Suppose that $L(E,1)=0$ and has sign $+1$, $L(E^K,1)\cdot L(E/K,\chi,1)\neq 0$, and that:

- *$E[p]$ is absolutely irreducible as $G_{\mathbb{Q}}$-module.*
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Then $\kappa_{\alpha,\beta^{-1}}=\kappa_{\beta,\alpha^{-1}}=0$ and the following hold:

$$\kappa_{\alpha,\alpha^{-1}}\neq 0 \implies \dim_{\mathbb{Q}_p}\mathrm{Sel}(\mathbb{Q},V_pE)=2,$$

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Main result: rank (2,0) adjoint CM case

Let N^- be the largest factor of N divisible only by primes inert in K.

Theorem (C.–Hsieh)

Suppose that $L(E,1)=0$ and has sign $+1$, $L(E^K,1)\cdot L(E/K,\chi,1)\neq 0$, and that:

- *$E[p]$ is absolutely irreducible as $G_{\mathbb{Q}}$-module.*
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- *$E[p]$ is ramified at every prime $\ell | N^-$.*

Then $\kappa_{\alpha,\beta^{-1}} = \kappa_{\beta,\alpha^{-1}} = 0$ and the following hold:

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Remarks

- The condition $\mathrm{Sel}(\mathbb{Q}, V_pE) \neq \ker(\mathrm{Loc}_p)$ is automatic if $\#E(\mathbb{Q}) = \infty$ or $\#Ш(E/\mathbb{Q})[p^\infty] < \infty$.
- Our proof also shows if $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 2$ and $\#Ш(E/\mathbb{Q})[p^\infty] < \infty$, then

$$\kappa_{\alpha,\alpha^{-1}} = \log_E(Q) \cdot P - \log_E(P) \cdot Q \pmod{\mathbb{Q}_p^\times},$$

where (P, Q) is any basis of $E(\mathbb{Q}) \otimes \mathbb{Q}$.

- Thus $\kappa_{\alpha,\alpha^{-1}}$ generates (over $\mathbb{Q}_p$) the image of the line

$$\langle P \wedge Q := P \otimes Q - Q \otimes P \rangle$$

under the map $\bigwedge^2 E(\mathbb{Q}) \otimes \mathbb{Q} \to E(\mathbb{Q}) \otimes \mathbb{Q}_p = \mathrm{Sel}(\mathbb{Q}, V_pE)$ induced by $\log_E$.

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Outline

Construction of $\kappa_{\alpha,\alpha^{-1}}$

- Let $\Lambda := \mathbb{Z}_p[[T]]$, $u := (1+p)$, and

$$T \mapsto u^{k-1} - 1, \quad (k \in \mathbb{Z}_{\geqslant 1})$$

  the weight k specialization map $\Lambda \to \overline{\mathbb{Q}}_p$.
- $\underline{g}, \underline{g}^* \in \Lambda[[q]]$ Hida families through the p-stabilizations $g_\alpha, g^*_{\alpha^{-1}}$:

$$\underline{g}|_{T=0} = g_\alpha, \quad \underline{g}^*|_{T=0} = g^*_{\alpha^{-1}}$$

  (i.e., weight 1 specializations).
- By construction,

$$\kappa_{\alpha,\alpha^{-1}} := \kappa_{f\underline{gg}^*}(T)|_{T=0},$$

  where $\kappa_{f\underline{gg}^*}(T) \in \mathrm{H}^1(\mathbb{Q}, V_pE \otimes V_{\underline{gg}^*})$ is such that

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$\kappa_{\alpha,\alpha^{-1}}$ and p-adic L-functions

- Building on Walspurger's work, in 1996 Bertolini–Darmon constructed

$$\theta_p^{\mathrm{BD}}(T) \in \Lambda$$

interpolating square-roots of twists of $L(E/K, 1)$.

Theorem (C.–Hsieh)

There is a generalised Coleman power series map

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Note:

- Here Λ becomes the *anti-cyclotomic variable* for K.
- To prove the main result, we compute the *leading term* of $(\spadesuit)$ at $T = 0$.

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Derived p-adic heights

- By the theory of anticyclotomic *derived* p-adic heights, there is a filtration

$$\mathrm{Sel}(K, V_pE) = S_p^{(1)} \supseteq S_p^{(2)} \supseteq \cdots \supseteq S_p^{(r)} \supseteq \cdots \supseteq S_p^{(\infty)} = 0$$

and a sequence of skew-symmetric (resp. symmetric) pairings for even r (resp. odd r)

$$h_p^{(r)} : S_p^{(r)} \times S_p^{(r)} \longrightarrow \mathbb{Q}_p$$

with $h_p^{(1)}$ = Mazur–Tate pairing, and $\ker(h_p^{(r)}) = S_p^{(r+1)}$.

- The τ-eigenspaces of $\mathrm{Sel}(K, V_pE)$ are isotropic for $h_p^{(1)}$, since $h_p^{(1)}(x^\tau, y^\tau) = h_p^{(1)}(x, y)^\tau = -h_p^{(1)}(x, y)$.
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Building on the formula

$$\mathcal{L}_p(\kappa_{f\underline{gg}^*}(T)) = \theta_p^{\mathrm{BD}}(T) \qquad (\spadesuit)$$

we show:

Theorem (C.–Hsieh)

Let $\mathfrak{r} := \mathrm{ord}_T \theta_p^{\mathrm{BD}}(T)$. *Then* $\kappa_{\alpha,\alpha^{-1}} \in S_p^{(\mathfrak{r})}$, *and*

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Sketch of proof of non-vanishing

- For the proof that $\kappa_{\alpha,\alpha^{-1}} \neq 0$, under our hypotheses we have:

$$\dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_pE) = 2, \quad \dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_pE^K) = 0,$$

and the filtration reduces to

$$\mathrm{Sel}(\mathbb{Q}, V_pE) = S_p^{(1)} = S_p^{(2)} = \cdots = S_p^{(r)} \supset S_p^{(r+1)} = \cdots = S_p^{(\infty)} = 0$$

for some $r \geqslant 2$.

- Skinner–Urban's divisibility in IMC implies $\mathfrak{r} \leqslant r$, so the above gives

$$S_p^{(\mathfrak{r})} = \mathrm{Sel}(\mathbb{Q}, V_pE).$$

- If $\mathrm{Sel}(\mathbb{Q}, V_pE) \neq \ker(\mathrm{Loc}_p)$, the proof that $\kappa_{\alpha,\alpha^{-1}} \neq 0$ then follows from our generalised Rubin formula.
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First instance of non-vanishing: Gross's example

- The elliptic curve $E/\mathbb{Q}$ with $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 2$ of smallest conductor:

$$E = 389a1 : y^2 + y = x^3 + x^2 - 2x.$$

- Take $K = \mathbb{Q}(\sqrt{-2})$ and $p = 11$. Then:
  - p is ordinary for E, and splits in K.
  - $E[p]$ is irreducible, and ramified at $N^- = 389$.
  - $L(E^K, 1) \neq 0$.
  - Can find infinitely many χ with $L(E/K, \chi, 1) \neq 0$ by Vatsal.
- We numerically check

$$\theta_p^{\mathrm{BD}}(T) \equiv -T^2 + 58T^3 + \cdots \pmod{(p^2, T^p)}.$$

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