Mixed Hodge numbers and factorial ratios

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Abstract

This note is an extended version of the slides for my talk with the same title at the *Arithmetic, geometry, and modular forms: a conference in honour of Bill Duke* in June 2019 at the ETH in Zürich. The results presented concern three geometric criteria for the integrality of factorial ratios, numbers such as $(30n)!n!/(6n)!(10n)!(15n)!$, which are integral in a non-immediate way for all n. This work is an offshoot of an ongoing project on hypergeometric motives joint with D. Roberts and M. Watkins.

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In his work on the prime number theorem Chebyshev [11] used indirectly that the numbers

$$c_n := \frac{(30n)!n!}{(6n)!(10n)!(15n)!},$$

are integral for $n = 0, 1, 2, \ldots$. The only know proof of this fact seems to be to use that the valuation

$$v_p(c_n) \geq 0, \qquad n \geq 0$$

for all primes p; this in turn relies on the fact that

$$\lfloor 30x \rfloor + \lfloor x \rfloor - \lfloor 6x \rfloor - \lfloor 10x \rfloor - \lfloor 15x \rfloor \geq 0$$

for all real numbers x, which is easy to check. We will encode the data defining c_n by the list $\gamma := (-30, -1, 6, 10, 15)$ of positive integers with zero sum.

I have been fascinated by this fact since pointed out to me by P. Sarnak in the late 90's. It is not entirely clear how to understand the integrality of ratios of factorials such as these

*I would like to thank A. Mellit for several useful discussions on the topic of this note

(see [6], [8], [27], [1] and the bibliography therein for recent developments). In this note I will discuss three different criteria. These involve

(A) Interior lattice points of dilations of an associated polytope Δ

(B) Hodge numbers of a certain hypersurface $Z_t \subseteq \mathbb{T}$ of a torus $\mathbb{T} := (\mathbb{C}^\times)^d$

(C) Effective weight of the corresponding hypergeometric motive $\mathcal{H}(\gamma \,|\, t)$

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Before discussing these criteria, I would like to briefly sketch the connection between the integrality of c_n and the algebraicity of the corresponding power series.

Early on [25] I noticed that the hypergeometric series

$$c(t) := \sum_{n\geq 0} c_n \left(\frac{t}{M}\right)^n = {}_8F_7\left[\begin{matrix} \frac{1}{30} & \frac{7}{30} & \frac{11}{30} & \frac{13}{30} & \frac{17}{30} & \frac{19}{30} & \frac{23}{30} & \frac{29}{30} \\ \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \end{matrix} \,\Big|\, t\right], \qquad M := \frac{30^{30}}{6^6 10^{10} 15^{15}}, \tag{1}$$

is an algebraic function of t. This follows from the work of Beukers and Heckman [5] who classified all algebraic hypergeometric series, extending the classical result of Schwartz on the algebraicity of the ${}_2F_1$ hypergeometric series. Our series c is number 67 in the Beukers-Heckman list.

Concretely, there exists a polynomial $A(x, y) \in \mathbb{Z}[x, y]$, which we may assume is irreducible, such that

$$A(t, c(t)) = 0$$

in the ring of power series in t. What is the connection between the algebraicity of c and the integrality of the coefficients c_n?

A little work using a theorem of Eisenstein shows that in our case the latter is a consequence of the former (see [24][Prop.2] for the details). But algebraicity is a much stronger property of a power series than integrality of its coefficients. Nevertheless, there is some connection and Landau [21] had already exploited it to reprove Schwartz's result.

The Galois group Γ of the normal closure of the extension $\mathbb{Q}(c(t))/\mathbb{Q}(t)$ is the Weyl group $W(E_8)$ of the Lie algebra E_8 of order $696, 729, 600$. It turns out, as we will see shortly, that

$$\deg_y A = 483, 840.$$

So it is not going to be very easy to find A (though D. Roberts [22] has computed a degree 240 polynomial with the same Galois closure.)

On the other hand, the series c satisfies a linear differential equation with polynomial coefficients of order 8. This equation has only regular singularities at $t = 0, 1, \infty$. The corresponding *monodromy representation*, obtained by analytic continuation of local solutions at some point $t_0 \neq 0, 1, \infty$,

$$\rho : \quad \pi_1\left(\mathbb{P}^1 \setminus \{0, 1, \infty\}\right) \longrightarrow \mathrm{GL}(V),$$

has the following properties:

$$q_\infty := \mathrm{char}(T_\infty) = \Phi_{30}, \qquad q_0 := \mathrm{char}(T_0^{-1}) = \Phi_1\Phi_2\Phi_3\Phi_5$$

and T_1 fixes a codimension one subspace of V. Here Φ_n denotes the nth cyclotomic polynomial, V is the space of local solutions to the differential equation around t_0 and T_s, for $s = 0, 1, \infty$, are the local monodromies; i.e., the images by ρ of small loops around s. These loops are chosen oriented in a consistent manner so that $T_0T_1T_\infty = \mathrm{id}_V$. The image of ρ is called the *monodromy group*.

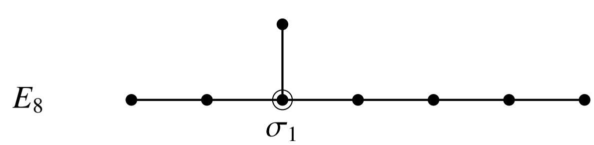
Note that

$$\frac{\Phi_{30}}{\Phi_1\Phi_2\Phi_3\Phi_5} = \frac{(T^{30}-1)(T-1)}{(T^6-1)(T^{10}-1)(T^{15}-1)}$$

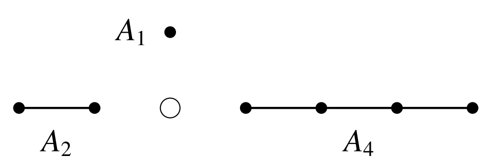
connecting the list γ determining c and the hypergeometric parameters in the series (1).

In general, the monodromy representation of a hypergeometric differential equations is uniquely determined by the analogue of the polynomials q_∞, q_0 (when irreducible or equivalently when q_0 and q_∞ have no common roots). In other words, they give rise to a rigid local system (in the sense of N. Katz [20]): the local monodromies uniquely determine the monodromy representation. Clasically, this is known as not having *accessory parameters.*

We can build this representation in our case by starting directly with the group $W := W(E_8)$ (see [10] for details on Coxeter groups). A Coxeter element $\sigma \in W$ is the product of all simple reflections. Its conjugacy class is uniquely determined independent of the order in which we perform the product. The order of σ is the Coxeter number, which equals 30 for E_8.



Take $\tau = \sigma_1 \cdots \sigma_8$. Then $\sigma_1\tau = \sigma_2 \cdots \sigma_8$ is the product of the Coxeter elements of the diagram obtained by removing the circled dot and all its attached edges in the Dynkin diagram of E_8. Its characteristic polynomial is then q_0.



It follows that if we choose

$$T_\infty := \tau, \qquad T_1 := \sigma_1, \qquad T_0^{-1} := \sigma_1\tau$$

we obtain a representation isomorphic to ρ by rigidity. It is straightforward to check using a computer that the group generated by T_0, T_1, T_∞, the monodromy group, is all of $W(E_8)$.

In our case the Galois group Γ coincides with the monodromy group. The degree of A in y can now be computed by Galois theory. Indeed, using basic properties of reflection groups we find that it equals

$$483,840 = [W(E_8) : W(A_1) \times W(A_2) \times W(A_4)]$$

as promised. (It suffices to find the simple roots orthogonal to that corresponding to σ_1.)

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We now turn to criterion (A) and the associated polytope. Let $\gamma = (\gamma_1, \ldots, \gamma_l)$ be a non-empty list of non-zero integers with zero sum, no pair of entries satisfying $\gamma_i + \gamma_j = 0$ and with no positive integer dividing every entry. Let r be the number of negative γ_i and s the number of positive ones so that $r + s = l$. Assume further that $s \geq r$. We will call such a list a ***gamma list*** for short. Since the order of the γ_i's is irrelevant we will typically choose $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_l$. The numbers we would like to study are then

$$c_n := \frac{\prod_{\gamma_i<0}(-\gamma_i n)!}{\prod_{\gamma_i>0}(\gamma_i n)!}, \qquad n = 0, 1, 2, \ldots$$

Given a gamma list consider $m_1, \ldots, m_l \in \mathbb{Z}^d$ with $d := l - 2$ such that γ spans their affine relations:

$$\gamma_1 m_1 + \cdots \gamma_l m_l = 0.$$

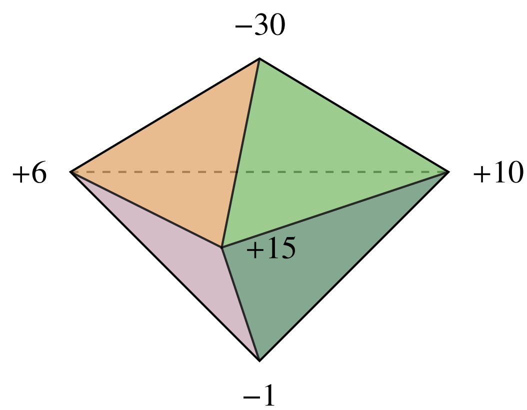
We will choose the m_i's so that their affine span is primitive. In practice, to find m_i we can simply drop say γ_1 from the list and find generators $m_2, \ldots, m_l$ over $\mathbb{Z}$ of the kernel of the $1 \times d$ matrix $(\gamma_2, \ldots, \gamma_l)$ and then set $m_1 = 0$. Finally, we let $\Delta \subseteq \mathbb{Z}^d$ be the convex hull of $m_1, \ldots, m_l$. Primitivity guarantees that Δ is uniquely determined up to invertible affine linear transformation over $\mathbb{Z}$. The normalized volume of Δ equals

$$\mathrm{vol}(\Delta) = \mathrm{vol}(\gamma) := -\sum_{\gamma_i<0} \gamma_i = \sum_{\gamma_i>0} \gamma_i.$$

A possible choice of m_i in the Chebyshev case are the columns of the following matrix

$$\begin{pmatrix} 1 & 0 & 5 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Here is a schematic (and not to scale) picture of the polytope with each m_i labeled by the corresponding γ_i.



In general, for a d-dimensional polytope $\Delta \in \mathbb{Z}^d$ we have, by results of Ehrhart,

$$\sum_{k\geq 0} \#(k\Delta)\, T^k = \frac{\delta(\Delta, T)}{(1-T)^{d+1}}, \tag{2}$$

with $\delta(\Delta, T)$ a polynomial of degree at most d. We define the *codegree* of Δ as $\operatorname{codeg}(\Delta) := d + 1 - \deg(\delta) \geq 1$. Equivalently, the codegree is the smallest positive integer k such that $k\Delta$ has an interior lattice point (see [3] for a recent survey of Ehrhart theory).

So for the Chebyshev example we have

$$\sum_{k\geq 0} \#(k\Delta)\, T^k = \frac{1 + 15T + 15T^2}{(1-T)^4} = 1 + 19T + 85T^2 + 230T^3 + O(T^4)$$

and hence $\operatorname{codeg}(\Delta) = 2$.

We can now formulate our first criterion for integrality (see also [1]).

Criterion A:

Theorem 3.1. *The numbers $c_n \in \mathbb{Z}$ for all n if and only if $s > r$ and*

$$\operatorname{codeg}(\Delta) \geq r; \tag{3}$$

equivalently, $k\Delta$ has no interior lattice points for $k = 1, 2, \ldots, r-1$.

Note that for $r = 1$ the condition (3) is vacuous. Indeed, in this case c_n is a multinomial number and integrality is immediate.

4

We now define the algebraic varieties Z_t appearing in criterion (B) (see the MAGMA manual's *canonical scheme* [9] and [4]). Consider the Laurent polynomial, with $m_1, \ldots, m_l$ as before,

$$f := \sum_{i=1}^{l} u_i x^{m_i}, \qquad x = (x_1, \ldots, x_d),$$

where $x^m := x_1^{m^1} \cdots x_l^{m^l}$ for $m = (m^1, \ldots, m^l) \in \mathbb{Z}^d$ and $u_1, \ldots, u_l$ are parameters in $\mathbb{C}^\times$. Following Watkins and Beukers-Cohen-Mellit we consider the hypersurface $Z \subseteq \mathbb{T} := (\mathbb{C}^\times)^d$ for a given u defined by the vanishing of f. The dimension of Z is $\kappa := l - 3$.

By scaling the variables by $x_j \mapsto a_j x_j$ and the polynomial itself by $f \mapsto a_0 f$ with $a_0, a_j \in \mathbb{C}^\times$ for $j = 1, \ldots, d$ we obtain isomorphic hypersurfaces. Hence we may take as the natural parameter for the family the quantity $u := u_1^{\gamma_1} \cdots u_l^{\gamma_l} \in \mathbb{C}^\times$. We see that having chosen f to have $l = d + 2$ monomials in d variables is what guarantees that our family really only depends on one parameter.

In fact, it is better to normalize the parameter and use instead

$$t := (-1)^{\mathrm{vol}(\gamma)} u M, \qquad M := \frac{\prod_{\gamma_i < 0} (-\gamma_i)^{-\gamma_i}}{\prod_{\gamma_i > 0} \gamma_i^{\gamma_i}}.$$

Choose a family of hypersurfaces Z_t with given parameter $t \in \mathbb{C}^\times$. Concretely, we may choose integers $k_1, \ldots, k_l$ such that $k_1\gamma_1 + \cdots k_l\gamma_l = 1$ and take $u_i = u^{k_i}$ with $u = (-1)^{\mathrm{vol}(\gamma)} t/M$. Then Z_t is smooth except for $t = 1$ where it has a unique double point (uniqueness follows from the primitivity of $m_1, \ldots, m_l$).

There is a refinement $\delta^\#$ of the polynomial δ in (2) due to Stanley [28] that incorporates the way the faces of Δ are put together. By work of Danilov-Khovanski, Batyrev-Borisov, Katz-Stapledon and others (see [14], [2], [18]) this polynomial gives the weight κ mixed Hodge numbers [15], [16] of the middle cohomology of Z_t for generic $t \in \mathbb{C}^\times$. More precisely,

$$\delta^\#(\Delta, T) = \sum_{j=0}^{\kappa} h_c^{j,\kappa-j,\kappa}(Z_t) T^{j+1}.$$

Since Δ is a simplicial polytope (all proper faces are simplices) it is fairly straightforward to compute this polynomial explicitly and obtain

$$\delta^\#(\Delta, T) = \sum_{\substack{N \geq 1 \\ m_- > m_+}} \frac{T^{m_-} - T^{m_+}}{T - 1} \delta_N^\#(T), \qquad \delta_N^\#(T) := \sum_{\substack{j=1 \\ \gcd(j,N)=1}}^{N-1} T^{\sum_{i=1}^{l} \left\{\frac{j\gamma_i}{N}\right\}}. \tag{4}$$

Here

$$m_\pm := \{i \mid \mathrm{sign}(\gamma_i) = \pm 1, N \mid \gamma_i\}$$

and $\{\cdot\}$ denotes fractional part. One may verify that this formula is equivalent to that giving the Hodge numbers of hypergeometric motives first conjectured by Corti and Golyshev and proved by Fedorov (see [13], [23] and [17]). A completely analogous formula holds where we sum over all N such that $m_- < m_+$. In fact there is also a similar formula for the polynomial $\delta(\Delta, T)$ itself.

$$\delta(\Delta, T) = \sum_{N \geq 1} \frac{T^{m_-} - 1}{T - 1} \delta_N^\#(T) = \sum_{N \geq 1} \frac{T^{m_+} - 1}{T - 1} \delta_N^\#(T).$$

For our running example of Chebyshev with $\gamma = (-30, -1, 6, 10, 15)$, we find that the only contribution to the sum comes from $N = 30$ for which $\delta_{30}^\#(T) = 8T^2$. Hence also

$\delta^\#(\Delta, T) = 8T^2$. We can take

$$Z_t: \quad xyz - \frac{M}{t} + x^5 + y^3 + z^2 = 0, \qquad M := \frac{30^{30}}{6^6 10^{10} 15^{15}},$$

an affine piece of a rational elliptic surface with parameter x; it has twelve bad fibers of type I_1. We have $h_c^{i,j,2}(Z_t) = 0$ for $(i, j) = (2, 0), (0, 2)$ and $h_c^{i,j,2}(Z_t) = 8$ for $(i, j) = (1, 1)$. This exhibits the connection with the E_8 lattice more directly as the Mordell-Weil lattice of the elliptic surface [26].

In general, $\deg \delta^\# \leq \deg \delta$ but for our polytopes the equality holds. We may therefore reformulate Theorem 3.1 as follows.

Criterion B:

Theorem 4.1. *With the notation of Theorem 3.1 we have that $c_n \in \mathbb{Z}$ for all n if and only if $s > r$ and*

$$h^{\kappa,0}(Z_t) = h^{\kappa-1,1}(Z_t) = \cdots = h^{\kappa-r+1,r-1}(Z_t) = 0, \qquad t \neq 1. \tag{5}$$

For example, for $r = 2$ the condition (5) means that the hypersurface Z_t should have geometric genus $p_g := h^{\kappa,0}$ equal zero.

In fact, we can go further and compute all of the mixed Hodge numbers of Z_t using a formula of Batyrev-Borisov [2][Thm. 3.18] and not just the top degree piece given by (4). It is better to formulate the result in terms of the *primitive* cohomology $PH_c^\kappa(Z_t)$; i.e., the kernel of the Gysin homomorphism [14][Prop. 3.9] $H_c^i(Z_t, \mathbb{C}) \to H_c^{i+2}(\mathbb{T}, \mathbb{C})$. This removes from the cohomology of Z_t the contributions from the ambient torus.

Theorem 4.2. *The E-polynomial of $PH_c^\kappa(Z_t)$ (primitive cohomology) is*

$$E(\Delta; a, b) := \sum_{i,j} h_c^{i,j} a^i b^j = \frac{1}{ab}\left[\delta^\#(\Delta; a, b) + \delta^0(\Delta; a, b) - 1\right],$$

where

$$\delta^\#(\Delta; a, b) = \sum_{\substack{N \geq 1 \\ m_- > m_+}} \frac{(a/b)^{m_-} - (a/b)^{m_+}}{a/b - 1} b^{l-1} \delta_N^\#(a/b)$$

$$\delta^0(\Delta; a, b) = \sum_{N \geq 1} \frac{(ab)^{\min(m_-, m_+)} - 1}{ab - 1} b^{l - m_+ - m_-} \delta_N^\#(a/b)$$

Note that $T\delta^\#(\Delta, T) = T\delta^\#(\Delta; T, 1)$ is indeed the component of (top) degree κ of $E(\Delta; a, b)$. Also

$$\delta(\Delta, T) = TE(\Delta; T, 1) + 1.$$

It is useful to display the two variable polynomial $E(\Delta; a, b)$ as an array with the coefficient of $a^i b^j$ in spot $(i, j) \in \mathbb{Z}^2$. For Chebyshev we have

N	m_-	m_+	$\delta^\#$
1	2	3	1
2	1	2	T
3	1	2	$2T$
5	1	2	$4T$
6	1	1	$T^2 + T$
10	1	1	$2T^2 + 2T$
15	1	1	$4T^2 + 4T$
30	1	0	$8T^2$

and hence

$$E(\Delta; a, b) = \begin{matrix} 7 & 8 \\ 8 & 7 \end{matrix}$$

5

There is also an arithmetic aspect to the story. The number of points of Z_t over finite fields ([4]) involves the finite analogue of a hypergeometric series due to N. Katz [19][(8.2.7)]. More precisely, the trace of the q-th Frobenius (for a good prime p) acting on the weight κ piece of the middle cohomology of Z_t has the form

$$\mathcal{H}(t) := \frac{1}{1-q} \sum_{\chi} \frac{J(\alpha\chi, \beta\chi)}{J(\alpha, \beta)} \chi(t), \tag{6}$$

where α, β are the hypergeometric parameters (as characters of $\mathbb{F}_q^\times$), J are certain Jacobi sums and χ runs over all characters of $\mathbb{F}_q^\times$. We call the weight κ piece of the middle cohomology the ***hypergeometric motive*** $\mathcal{H}(\gamma|t)$ [23] attached to the list γ. It is a pure motive of weight κ.

For our running Chebyshev example, we consider the hypersurface of $\mathbb{A}^3$ given by

$$\tilde{Z}_t : \quad -\tfrac{M}{t} + xyz + x^5 + y^3 + z^2 = 0, \qquad M := \frac{30^{30}}{6^6 10^{10} 15^{15}},$$

and find (see [4][Cor. 1.8] for an equivalent statement) that

$$\#\tilde{Z}_t(\mathbb{F}_q) = q^2 + q\mathcal{H}(t) + 1, \qquad t \neq 0, \quad q = p^k, \quad p > 5.$$

A number of features of hypergeometric motives have already been implemented in MAGMA by M. Watkins [9]. For example, we can easily verify a few instances of the above formula as follows.

Define the function

```
function surfcheb(t,q);

        A<x,y,z>:=AffineSpace(GF(q),3);
        M:=30^30/6^6/10^10/15^15;u:=-M/t;
        S:=Surface(A,u+x*y*z+z^5+x^3+y^2);
        return((#Points(S)-q^2-1)/q);

end function;
```

This will output the number $(\#\tilde{Z}_t(\mathbb{F}_q) - q^2 - 1)/q$ for a given value of t and q. Then for example

```
> H:=HypergeometricData([*-30,-1,6,10,15*]);
> [HypergeometricTrace(H, t, 7): t in [1..6]];
[ 0, 0, 1, -1, -1, 0 ]
> [surfcheb(t,7): t in [1..6]];
[ 0, 0, 1, -1, -1, 0 ]
> [HypergeometricTrace(H, t, 23): t in [1..22]];
[ 0, 0, 1, -1, 0, 0, -1, 2, -1, 0, 1, 0, -2, -2, 0, 1, 0, 0, 1, 0, -1, 1 ]
> [surfcheb(t,23): t in [1..22]];
[ 0, 0, 1, -1, 0, 0, -1, 2, -1, 0, 1, 0, -2, -2, 0, 1, 0, 0, 1, 0, -1, 1 ]
```

In fact, the MAGMA package can compute many more things about $\mathcal{H}(\gamma|t)$, notably, a big chunk (and in many cases all) of its L-function. I refer to the MAGMA manual [9] for many worked out examples. Here are for example, some Euler factors for $\mathcal{H}(\gamma\,|\,2)$.

```
> EulerFactor(H,2,7);
x^8 + x^5 + x^3 + 1
> EulerFactor(H,2,23);
x^8 - x^4 + 1
```

We can formulate our last criterion in the following form.

Criterion C:

Theorem 5.1. *With the notation of Theorem 3.1 we have that $c_n \in \mathbb{Z}$ for all n if and only if $s > r$ and the hypergeometric motive $\mathcal{H}(\gamma|t)$ is a Tate twist of a pure effective motive of weight $s - r - 1$.*

For example, if $s - r = 1$ then the integrality of c_n for all n is equivalent to $\mathcal{H}(\gamma|t)$ being a Tate twist of a pure motive of weight zero. It can be shown directly [24][Thm. 1.4] that when $s - r = 1$ the integrality of c_n is equivalent to Beukers and Heckman's criterion for the algebraicity of the power series $c = \sum_{n\geq 0} c_n t^n$. Hence in a sense the above theorem is a generalization of this criterion.

For $s - r = 2$ the integrality of c_n is equivalent to the hypergeometric motive $\mathcal{H}(\gamma|t)$ being a Tate twist of a motive with Hodge numbers (m, m) for some positive integer m. Conjecturally, $\mathcal{H}(\gamma|t)$ should then correspond to an abelian variety.

For example, consider $\gamma = (-11, -2, 1, 3, 4, 5)$. The characteristic polynomials of local monodromies are

$$q_\infty = \Phi_{11}, \qquad q_0 = \Phi_1^2 \Phi_3 \Phi_4 \Phi_5.$$

We can take Δ as the convex hull of the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 & 0 & 0 \end{pmatrix}$$

and Z_t as the family of cubic threefolds with equation

$$Z_t: \quad x_1^2 + x_1 x_2^2 + x_2 x_3^2 - \tfrac{t}{M} x_3 x_4^2 + x_4^2 + x_1 x_2 x_3 = 0, \qquad t \in \mathbb{C}^\times, \quad M := \frac{11^{11} 2^2}{3^3 4^4 5^5}.$$

Taking the Zariski closure $\overline{Z}_t \subseteq \mathbb{P}^4$ we obtain a family of projective cubic threefolds, smooth for $t \neq 1$. It is a classical fact that the Hodge numbers of a smooth projective cubic threefold for its middle cohomology are

$$(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (0, 5, 5, 0).$$

This matches the calculation of $\delta^\#(\Delta, T)$ using (4) and the values below.

N	m_-	m_+	$\delta_N^\#$
1	2	4	1
2	1	1	T^2
3	0	1	$T^3 + T^2$
4	0	1	$T^3 + T^2$
5	0	1	$2T^3 + 2T^2$
11	1	0	$5T^3 + 5T^2$

.

We also obtain $\delta(\Delta, T) = 5T^3 + 6T^2 + T + 1$ and

$$E(\Delta; a, b) = \begin{matrix} 0 & 5 & 0 \\ 0 & 1 & 5 \\ 1 & 0 & 0 \end{matrix}$$

Here the hypergeometric motive $\mathcal{H}(\gamma|t)$ is a Tate twist of H^1 of the intermediate Jacobian of $\overline{Z}_t$ (see [12] and [7]) and

$$\#\overline{Z}_t(\mathbb{F}_q) = q^3 + q^2 + q + 1 - q\mathcal{H}(t), \qquad q = p^k, \quad p > 11.$$

We can independently check that consistent with Theorem 5.1 the following factorial ratios

$$c_n := \frac{(11n)!(2n)!}{n!(3n)!(4n)!(5n)!},$$

are integral for $n = 0, 1, \dots$

As a final example consider

$$\gamma := (-63, -8, -2, 1, 4, 16, 21, 31).$$

Now we have

$$q_\infty = \Phi_{63}\Phi_3, \qquad q_0 = \Phi_1^2\Phi_4\Phi_{16}\Phi_{31}.$$

We may take the projective closure of Z_t in $\mathbb{P}^5$

$$\overline{Z}_t: \quad x_1^2x_3 + x_0x_1x_2 + x_1x_2^2 - \tfrac{t}{M}x_2x_6^2 + x_3x_5 + x_4^2x_6 + x_4x_5^2 + x_0^3, \qquad t \in \mathbb{C}^\times,$$

a smooth family of cubic fivefolds (for $t \neq 1$). The only non-zero Hodge numbers of the middle cohomology of $\overline{Z}_t$ are $h^{4,1} = h^{1,4} = 21$ and $\mathcal{H}(\gamma|t) = H^5(\overline{Z}_t)$ is a pure motive of rank 42 and weight 5.

The data on the polytope Δ is

N	m_-	m_+	$\delta^\#$
1	3	5	1
2	2	2	T^2
3	1	1	$T^4 + T^2$
4	1	2	$T^3 + T^2$
7	1	1	$6T^3$
8	1	1	$4T^3$
9	1	0	$3T^4 + 3T^3$
16	0	1	$4T^4 + 4T^3$
21	1	1	$12T^3$
31	0	1	$15T^4 + 15T^3$
63	1	0	$18T^4 + 18T^3$

giving $\delta^\#(\Delta, T) = 21T^4 + 21T^3$, $\delta(\Delta, T) = 22T^4 + 45T^3 + 4T^2 + T + 1$ and

$$E(\Delta; a, b) = \begin{matrix} 0 & 1 & 21 & 0 \\ 0 & 1 & 23 & 21 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}$$

In this case,

$$\#\overline{Z}_t(\mathbb{F}_q) = q^5 + q^4 + q^3 + q^2 + q + 1 - q^2\mathcal{H}(t), \qquad q = p^k, \quad p > 31.$$

Again we find that

$$c_n := \frac{(63n)!(8n)!(2n)!}{n!(4n)!(16n)!(21n)!(31n)!},$$

is integral for $n = 0, 1, \dots$

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