PARTIAL HYPERBOLICITY ON UNIT TANGENT BUNDLES: abundance of mapping classes and lack of integrability of the center bundle.

$f : M^3 \to M^3$ is P.H. if $TM = E^s \oplus E^c \oplus E^u$ continuous Df-invariant (non-trivial) splitting so that $\exists\, \ell > 0$ s.t. $\forall x \in M$ one has:

$$\|Df^\ell/_{E^s(x)}\| \leq \min\{1, \|Df^\ell/_{E^c(x)}\|\} \leq \max\{1, \|Df^\ell/_{E^c(x)}\|\} < \|Df^\ell/_{E^u(x)}\|$$

PUJALS CONJECTURE (c.f. Bonatti-Wilkinson) transitive (dense orbit) P.H. diffeos are (up to finite lift and/or iterate) deformations of:
- algebraic examples
- time one maps of Anosov flows.

To make this more precise, one needs the concept of DYNAMICAL COHERENCE and that of LEAF CONJUGACY (already present in the seminal works of Hirsch-Pugh-Shub and Brin-Pesin).

Def: f is dynamically coherent (DC) if $\exists\, \mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ f-invariant foliations tangent respectively to $E^s \oplus E^c$ and $E^c \oplus E^u$ ($\Rightarrow \exists\, \mathcal{W}^c$ tg to E^c)

Hertz-Hertz-Ures produced an example of (non-transitive) PH diffeo which is not D.C. in $\mathbb{T}^3$ and presented some conjectures extending Pujals'.

Conjecture (HHU) If a PH diffeo in a 3-mfd is not DC $\Rightarrow$ it contains a torus tangent to either $E^s \oplus E^c$ or $E^c \oplus E^u$ (in part, not transitive)

Conjecture 2 (HHU) If f is a PH & DC then, (modulo lift & iterate) it is leaf conjugate to either an algebraic example or time one map of Anosov flow. (won't be defined, kind of 'orbit equivalence')

Thm (j.w. Hammerlindl) If $\pi_1(M)$ is (virtually) solvable $\Rightarrow$ all conjectures are true.

(joint work in progress with T. Barthelme, S. Fenley & S. Frankel is that conjecture 2 also holds for Hyperbolic 3-mfds.)

Recently, some counterexamples started to appear:

→ (j.w. Bonatti & Parwani) ∃ DC & PH diffeos not leaf conjugate to algebraic nor time 1-map of Anosov (Non transitive)

→ (j.w. Bonatti-Gogolev) ∃ transitive (& ergodic) PH diffeos not isotopic to algebraic nor time 1-maps of Anosov.

In this talk I will restrict to a specific (still paradigmatic) familly of 3-mflds: T^1S where S is a higher genus surface.

Goal is to present the following result:

Thm (j.w. Bonatti-Gogolev-Hammerlindl) $\forall \rho \in MCG(S)$ there exits $f: T^1S \to T^1S$ PH diffeo (stably ergodic & robustly transitive) which is isotopic to $P\rho \in MCG(T^1S)$. Moreover, if ρ is a p.A. mapping class $\Rightarrow f$ can be chosen to also be non-D.C.

Notation, $P: MCG(S) \hookrightarrow MCG(T^1S)$ is the (injective) morphism defined by: if $h \in \rho \in MCG(S) \Rightarrow P\rho = \left[\frac{Dh}{\|Dh\|}\right]$ so that $v \in T^1S \mapsto \frac{Dhv}{\|Dhv\|}$

(Contrasts with j.w. with BFF: both conjectures hold for PH ~ id in Seifert mflds)

CONSTRUCTION OF EXAMPLES: If $f, g: M \to M$ are PH diffeos and $h: M \to M$ is any diffeo, we say $f \xrightarrow[h]{} g$ iff:

$$Dh(E^u_f) \pitchfork E^s_g \oplus E^c_g \quad \text{AND} \quad Dh^{-1}(E^s_g) \pitchfork E^c_f \oplus E^u_f .$$

PROPERTIES:
- $f \xrightarrow[h]{} g \iff g^{-1} \xrightarrow[h^{-1}]{} f^{-1}$
- $f \xrightarrow[h]{} g \Rightarrow f \xrightarrow[g^l h f^k]{} g$ for any $l, k \in \mathbb{Z}$.

These are trivial, but the most interesting property is a 'kind' of transitivity of the relation' that allows to 'compose'.

Prop: $f_i \xrightarrow[h_i]{} f_{i+1}$ with $i = 1, \dots, n-1 \Rightarrow \exists \hat{h} \,/\, f_1 \xrightarrow[\hat{h}]{} f_n$

and $\hat{h}$ can be chosen $\hat{h} = h_{n-1} \circ f_{n-1}^{k_{n-1}} \circ h_{n-2} \circ \dots \circ f_2^{k_2} \circ h_1$ for suff. large k_i.

Proof: enough to do $f_1 \xrightarrow[h_1]{} f_2 \xrightarrow[h_2]{} f_3$

As $Dh_1(E^u_{f_1}) \pitchfork E^s_{f_2} \oplus E^c_{f_2}$, for large k_2 one has that:

$$Df_2^{k_2}(Dh_1(E^u_{f_1})) \text{ very close to } E^u_{f_2}$$

$\Rightarrow D(h_2 \circ f_2^{k_2} \circ h_1)(E^u_{f_1}) \pitchfork E^s_{f_3} \oplus E^c_{f_3}$. The rest is similar. $\blacksquare$

Corollary: If $\{f_t\}_{t \in [0,1]}$ is a path of PH diffeos $\Rightarrow f_0 \xrightarrow[h]{} f_1$ for some h which is isotopic to f_0^N for some large N (and if $\{f_t\}$ are volume preserving, h can be chosen to send the volume preserved by f_0 to the one of f_1)
(standard: 'Moser's trick')

proof: Notice that bundles vary continuously, so, one takes $f_i = f_{i/n}$ for large n and $h_i = id$ in the above proposition. $\blacksquare$

We obtain the following criteria:

Proposition: Let $f: M \to M$ be a PH diffeo and $h: M \to M$ so that $f \xrightarrow[h]{} f \Rightarrow$ for large enough $n, m > 0$ one has that $F_{n,m} = f^n \circ h \circ f^m$ is PH.
Moreover, the bundles of $F_{n,m}$ converge to those of f as $n, m \to \infty$.

Pf: Cone fields again. $\blacksquare$

Now, by Dehn-Lickorish, every $\rho \in MCG(S)$ can be written as

$$\rho = \tau_{\gamma_1}^{\varepsilon_1} \circ \dots \circ \tau_{\gamma_m}^{\varepsilon_m}$$ where γ_i are (not nec distinct nor disjoint) simple closed curves in S.

$$\varepsilon_i = \pm 1 .$$

So, we want $h \sim P\rho$ (volume preserving) for $f = \phi_1$ ← time-1-map of geodesic flow in T^1S.

Thm (j.w. Bonatti-Gogolev) Given γ simple closed curve in S, there exists a metric μ of curvature -1 in S so that if ϕ_1^{μ} is the time one map of its geodesic flow $\rightsquigarrow \exists\, h$ isotopic to $P\tau_\gamma^{\pm 1}$ (and vol. preserving) so that $\phi_1^{\mu} \xrightarrow{h} \phi_1^{\mu}$.

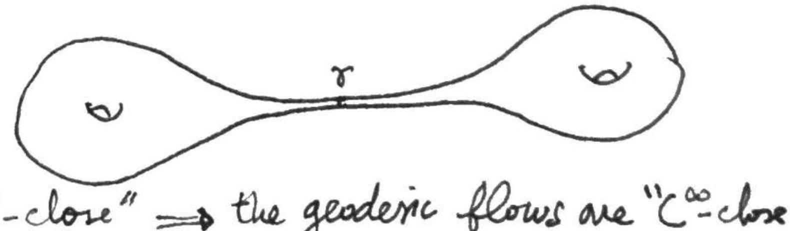
Using this and connectedness of Teichmuller space one can show:

if $\rho = \tau_{\gamma_1}^{\varepsilon_1} \circ \dots \circ \tau_{\gamma_m}^{\varepsilon_m}$ and $\mu_1, \dots, \mu_m$ are the metrics of Thm for $\gamma_1, \dots, \gamma_m$

$$\Rightarrow \phi_1^{\mu_1} \xrightarrow[h_1]{} \phi_1^{\mu_1} \xrightarrow[g_1]{} \phi_1^{\mu_2} \xrightarrow[h_2]{} \phi_1^{\mu_2} \rightarrow \dots \xrightarrow[g_{m-1}]{} \phi_1^{\mu_m} \xrightarrow[h_m]{} \phi_1^{\mu_m} \xrightarrow[g_m]{} \phi_1^{\mu_1}$$

where $\phi_1^{\mu_i} \sim id$ $g_i \sim id$ and $h_i \sim P\tau_{\gamma_i}^{\varepsilon_i} \Rightarrow \phi_1^{\mu_1} \xrightarrow[\hat{h}]{} \phi_1^{\mu_1}$ with $\hat{h} \sim P\rho$.

SOME WORDS ON PROOF OF THM ABOVE



If $\ell_\mu(\gamma)$ is very short $\Rightarrow$

the metric μ and $(\tau_\gamma)_*\mu$ are "C^∞-close" $\Rightarrow$ the geodesic flows are "C^∞-close" ($\Rightarrow$ bundles are close) and as $\frac{D\tau_\gamma}{\|D\tau_\gamma\|}$ conjugates both geodesic flows, it sends "bundles to bundles" $\Rightarrow$ preserves transversalities.

Rmk: To make this work we actually used that the metrics are of curvature -1 to make all computations in $\mathbb{H}^2$ and "essentially" as $\ell_{\mu_m}(\gamma) \to 0$ $\tilde{\tau}_\gamma \to id$ in C^∞ top. But this should work in more generality ....

Rmk: This shows why changing metrics at each step is relevant (c.f. collar lemma)

SOME WORDS ON NON-DYNAMICAL COHERENCE

We assume now that $f : T^1S \to T^1S$ is PH with bundles $E^s_f \oplus E^c_f \oplus E^u_f$ very close to those of the geodesic flow ϕ^μ_t for μ hyp metric on S.

Assume also that $f \sim P\varphi$ with φ p.A. mapping class

Thm f is NOT D.C.

Idea: By contradiction assume $\exists\, \mathcal{W}^c$ invariant center foliation. (we assume everything is orientable)

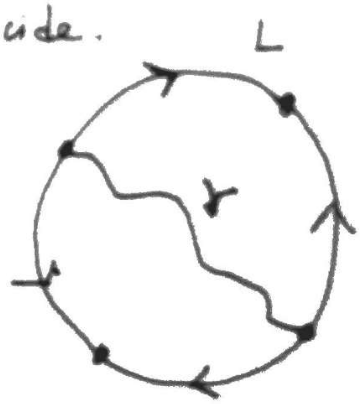
As bundles are close, if one takes X^c_f vector field so that $\mathbb{R}X^c_f = E^c_f$

$\rightsquigarrow$ one can apply shadowing lemma.

Claim: the flow of X^c_f is orbit equivalent to ϕ^μ_t.

idea: semiconjugacy is shadowing. Uniqueness follows from P.A. ∎

(after iterate)
Choose $\tilde{f} : T^1\mathbb{D} \to T^1\mathbb{D}$ a lift associated to a regular periodic orbit of φ (the p.A. map) $\rightsquigarrow$ actions 'at infinity' coincide.



Choose γ a center curve joining repelling points and L its center unstable leaf (it projects to $\mathbb{D}^2$ as covering)

Using dynamics of $\varphi \rightsquigarrow \gamma$ is 'coarsely contracting'

So, the fixed points in γ contribute positive index in L.

However, the global index in L is -1, but this is impossible since all other possible fixed points are in center curves that must be 'coarsely saddle-node' (therefore 0-index). A contradiction.