Local and global aspects of almost global stability

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Outline

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Esquema

The context is $\dot{x} = f(x)$, $f(0) = 0$ with $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. The time t of the flow will be denoted by f^t.

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Definition

We say that the origin is almost globally stable (a.g.s.) if and only if $\mathcal{R}^c = \{y \;:\; \lim_{t \to +\infty} f^t(y) \neq 0\}$ *has zero Lebesgue measure.*

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Theorem (Rantzer 2001)

If there exists $\rho \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ non-negative such that $\nabla \cdot (\rho f) > 0$ $m - ae$ and $\frac{\rho f}{|x|}$ is integrable on $B^c(0, \varepsilon)$ for all ε then the origin is a.g.s.

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We call such a ρ a density function.

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μ is a monotone measure if $\mu(B^c(0,\varepsilon)) < +\infty$ and $0 < \mu(Y) < +\infty$ implies

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Theorem

If there exists a monotone measure then the origin is a.g.s.

There are some results relating monotone measures with density functions and showing how they mean the "same". However, to study local properties of dynamical systems near equilibria it is helpfull to obtain differentiability, at least in a neighborhood of the origin.

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We present the following result showing that the existence of a monotone measure implies the existence of a density function (but we do not achieve differentiability without further assumptions that we will show are necessary)

Proposition

If there exists a monotone measure μ then, there exists a density function g in $L^1(m)$ in the sense that there exists a monotone measure ν such that $\nu(E) = \int_E g dm$ and such that

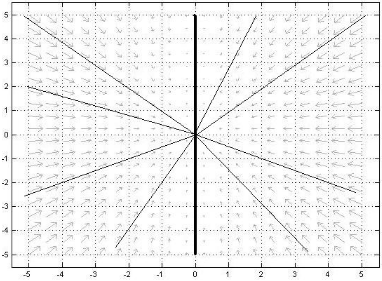
$$\mu = \nu + \lambda \quad , \quad \lambda \perp \mu$$

Esquema

Example

$$\begin{cases} \dot{x_1} &= -x_1^3 \\ \dot{x_2} &= -x_1^2 x_2 \end{cases}$$

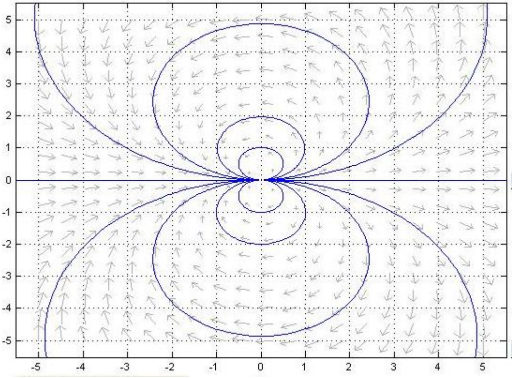
Is a.g.s. but not locally asymptotically stable (l.a.s.)



Example

$$\begin{cases} \dot{x_1} = x_1^2 - x_2^2 \\ \dot{x_2} = 2x_1x_2 \end{cases}$$

Is a.g.s. but does not admit a quadratic density function.



Esquema

Local properties

In [Monzón 2005], some relationships between a.g.s. and l.a.s. were studied for planar systems. We try to extend those results to higher dimensions.

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Proposition:

a) If $\frac{\partial f}{\partial x}(0)$ has at least one eigenvalue λ such that $Re(\lambda) > 0$ then $m(\mathcal{R}) = 0$.

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Proposition:

a) If $\frac{\partial f}{\partial x}(0)$ has at least one eigenvalue λ such that $Re(\lambda) > 0$ then $m(\mathcal{R}) = 0$.

b) If there is one eigenvalue λ with $Re(\lambda) < 0$ and it admits a density function then it is l.a.s.

Example

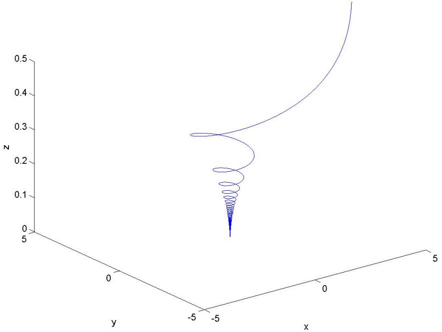
$$\begin{cases} \dot{x_1} &= x_1^2 - x_2^2 \\ \dot{x_2} &= 2x_1x_2 \\ \dot{x_3} &= -x_3 \end{cases}$$

Is a.g.s. but does not admit a density function because it has a negative eigenvalue (-1) and it is not l.a.s.

Example

$$\begin{cases} \dot{x_1} &= x_2 - 2x_1x_3^2 \\ \dot{x_2} &= -x_1 - 2x_2x_3^2 \\ \dot{x_3} &= -x_3^3 \end{cases}$$

Is a.g.s. because it admits $\rho = (x_1^2 + x_2^2 + x_3^2)^{-4}$ as a density function. All the eigenvalues have zero real part but it is not l.a.s. (in the plane $x_3 = 0$, the system is an harmonic oscillator).



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- Then, the set $\mathcal{R}$ must be contained in $\bigcup_{n\geq 0} f^{-n}(W^{cs}_{\varepsilon}(0))$, a countable union of zero measure sets (since there is one eigenvalue with positive real part).

REMARK: Although the center unstable manifold may not be unique, all points converging to the origin must belong to all posible center unstable manifold (Invariant Manifolds, Hirsch-Pugh-Shub) and then that set has zero Lebesgue measure.

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Proposition

If ρ is a density function and $\nabla \cdot f \leq 0$ in a neighborhood of the origin then $V = \rho^{-1}$ is a Lyapunov function.

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Proposition

If ρ is a density function and $\nabla \cdot f \leq 0$ in a neighborhood of the origin then $V = \rho^{-1}$ is a Lyapunov function.

- Together with the fact that the existence of a density function implies that all eigenvalues have non-positive real part (last proposition) we conclude that $\nabla \cdot f < 0$ in a neighborhood of 0 since we have one eigenvalue with negative real part.

Esquema

Necessary conditions for almost global stability

We prove the following theorem generalizing one in [Monzón, 2003].

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Theorem

Given a complete differential equation $\dot{x} = f(x)$ with $f \in C^1$ such that the origin is a almost globally stable and locally asymptotically stable fixed point for the flow f^t, there exists a density ρ differentiable and with continuous derivative up to a set of zero Lebesgue measure. Also, this density can be constructed such that it is zero in the complement of the basin of attraction.

Sketch of the proof:

- As in [Monzón 2003], we conjugate the system with the field $\dot{y} = g(y) = -y$ using Massera's theorem and the fact that the level manifolds of a real valued function are diffeomorphic to a sphere (for small regular values of the function)[1].

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- The conjugation allow us to "send" density and Lyapunov functions from one side to the other.
- The difference with the case of [Monzón 2003] is that the conjugacy is only defined in $\mathcal{R}$ which in our case may not be $\mathbb{R}^n$.

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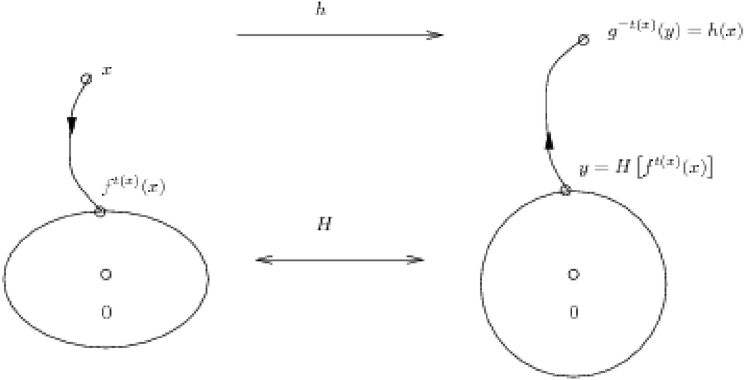

h
x
$g^{-t(x)}(y) = h(x)$
$f^{t(x)}(x)$
$y = H\left[f^{t(x)}(x)\right]$
H
0
0

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- To achieve differentiability, the main idea is to notice that we have a degree of freedom in choosing the density function for the field $y = -y$ (which admits a lot of density function) so we choose a suitable one for our purposes.

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- This implies a lot of work and since to inequalities must be satisfied at the same time it is not posible to ensure de continuity of the derivative of the density function on $\mathcal{R}^c$.

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Conclusions

- We have put together several aspects of almost global stability, monotone measures, density and Lyapunov functions and local and global properties of equilibrium points.

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- We have put together several aspects of almost global stability, monotone measures, density and Lyapunov functions and local and global properties of equilibrium points.
- We have shown how local stability plus almost global stability of the origin can be combined in order to construct a density function.
- We gave a first step into the generalization to higher dimensions of a planar result for a.g.s. and l.a.s., leading to local properties of a.g.s.
- In future works, we will analyze the remaining case of zero divergence, trying to establish conditions for local stability.