PH EXAMPLES IN DIMENSION 3

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ROBUST DYN. PROPERTIES vs. DETECTABLE PROPERTIES. (easy to check conditions)

M = closed 3 manifold.

$f: M \to M$ partially hyperbolic ($TM = E^s \oplus E^c \oplus E^u$ all non trivial)

Why study? Robust transitive, Stably Ergodic.

Examples: → In $\mathbb{T}^3$, matrices with 3 ≠ real eigenvalues.

→ In Nilmanifolds: p.h. automorphisms (skew-products)

These have been classified:

→ BBI: If $\pi_1(M)$ is abelian ⇒ f_* is partially hyperbolic (⇒ no examp in S^3)

Parwani: Extended ↑ to $\pi_1(M)$ with polynomial growth.

(w. A. Hammerlindl) Complete classification modulo what happens in E^c (including integrability & examples of HHU) when $\pi_1(M)$ subexponential.

¿What about $\pi_1(M)$ growing exponentially? (As explained, the "generic" case)

Source of Examples: Anosov flows,

for example: → suspension of linear automorphism of $\mathbb{T}^2$.
→ geodesic flows in negative curvature.

Time one maps are PH.

Problem: Many more examples! Classification is wide open.

Possible dream: Classify "MODULO" the classification of Anosov flows

↳ Pujals, Bonatti-Wilkinson.

What should we prove? → f isotopic to identity & M admits Anosov flow?

→ f has a W^c-foliation & f "fixes" W^c-leafs?

→ Leaf conjugacy? (w. A. Hammerlindl: Yes if $\pi_1(M)$ is solvable)

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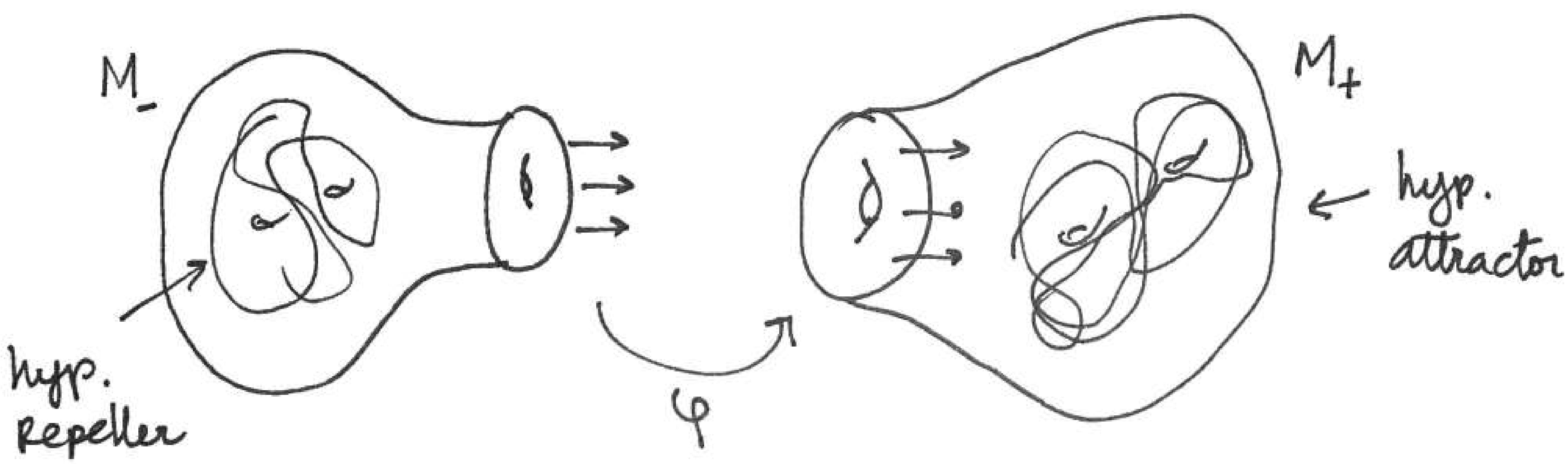
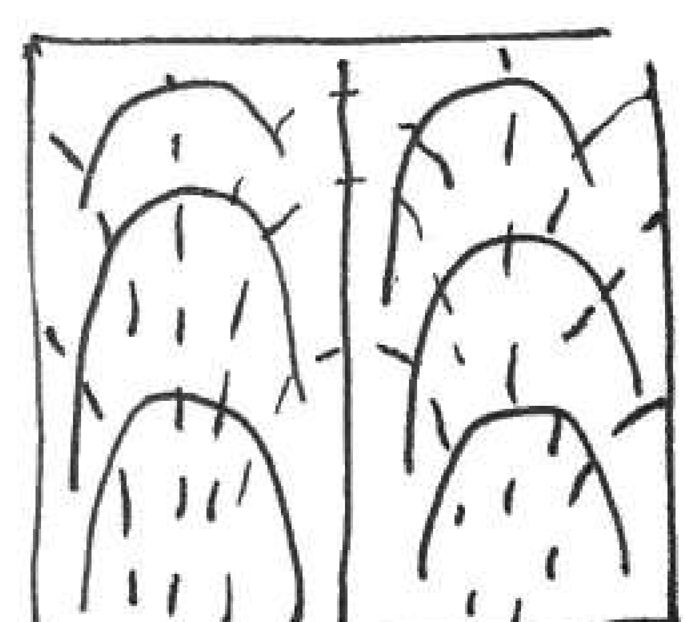
Thm (with C. Bonatti & K. Parwani) $\exists\, f: M \circlearrowleft$ partially hyperbolic, dynamically coherent ($\exists\, \mathcal{W}^c$) difeo with $\pi_1(M)$ having exponential growth s.t. $\forall n > 1$ f^n is not isotopic to id.

Rmk:
- The examples are not transitive, we can do transitive ones (if time allows I will explain). Center leafs are NOT fixed.
- The manifold is DECOMPOSABLE in the sense of Thurston
  (Q: hyperbolic? seifert?)

§ NON TRANSITIVE ANOSOV FLOWS: (FRANKS–WILLIAMS EXAMPLES)



φ glues keeping transversality of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$



Prop: If $\mathcal{W}^{cs} \pitchfork \mathcal{W}^{cu}$ everywhere $\Rightarrow$ the flow is Anosov.

In the diffeo case (c.f. Christian talk) one must show something harder:

every point $R \to A$.

Prop $f: M \circlearrowleft$ C^1-difeo with attractor A repeller R (both P.H.) let E^{cs}_A and E^s_A extensions of $E^s/_A$ and $E^s \oplus E^c/_A$ to $M \setminus R$ and E^{cu}_R and E^u_R extensions of $E^u/_R$ and $E^c \oplus E^u/_R$ to $M \setminus A$. If $E^{cs}_A \pitchfork E^u_R$ and $E^c_A \pitchfork E^{cu}_R \Rightarrow f$ is partially hyperbolic

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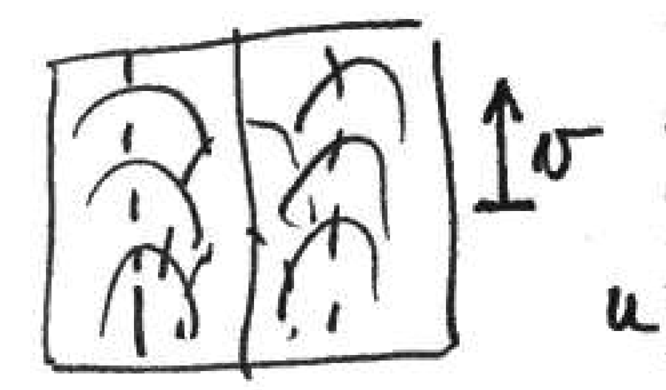
IDEA Let X be FW flow.

Fundamental domain $\mathcal{U}_N = \bigcup_{0 \le t \le N} X_t(T) \overset{\varphi_N}{\simeq} [0,1] \times \mathbb{T}^2$

Given N one considers the diffeo $X_N = F_N$ which has $\mathcal{U}_N$ as fund. domain.

Consider a Dehn-twist $G: [0,1] \times \mathbb{T}^2 \circlearrowleft \quad (t, x) \mapsto (t, x + \rho(t) v)$

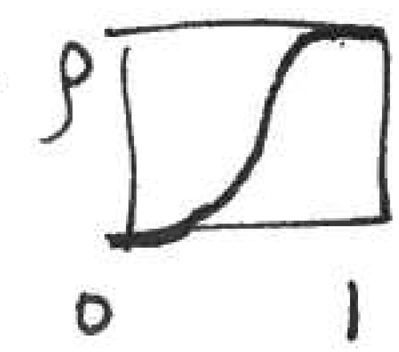
where



So, for large N we will consider

$$\mathcal{G} = \varphi_N^{-1} \circ G \circ \varphi_N \quad \text{and} \quad f = G \circ F_N.$$

and ρ

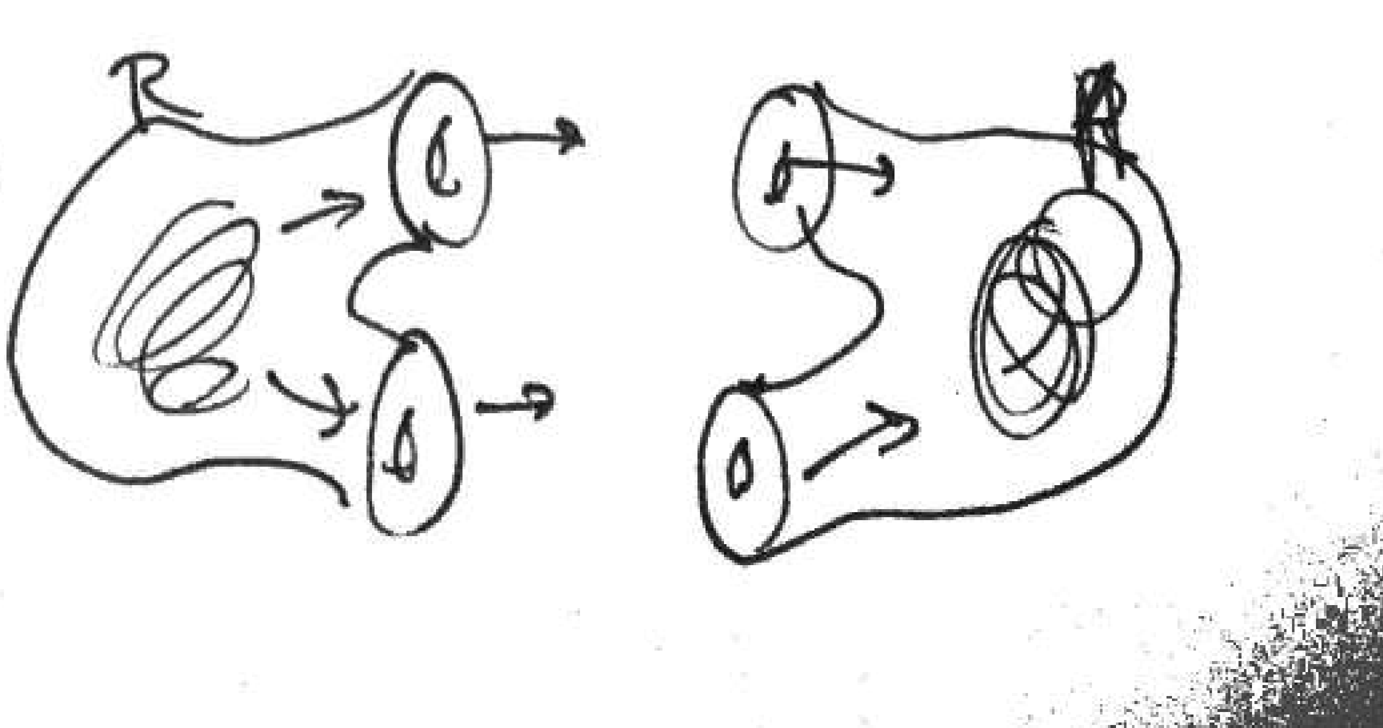


Since φ_N "pushes" E^s_A and E^u_R to the $\{t_0\} \times \mathbb{T}^2$ directions, for large enough N f will satisfy the criteria and therefore it will be partially hyperbolic. $\boxed{\Xi}$

§ INTEGRABILITY : Idea : $f = X_N$ out of $\mathcal{U}_N \Rightarrow$ it is almost isometry in $E^c_f \Rightarrow$ curves cannot "reach" the attractor in finite length.

Not all leafs fixed: Center leafs in $\mathcal{U}_N$ are $W^{cs}_X \cap G(W^{cu}_X)$

§ ISOTOPY CLASS Do with two tori:



Curve is homologically non trivial

$\Rightarrow$ its image is not homologous.

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Transitive examples: (w. C. Bonatti & A. Gogolev)

Dificulties: → One has to adapt criterium: Cone fields

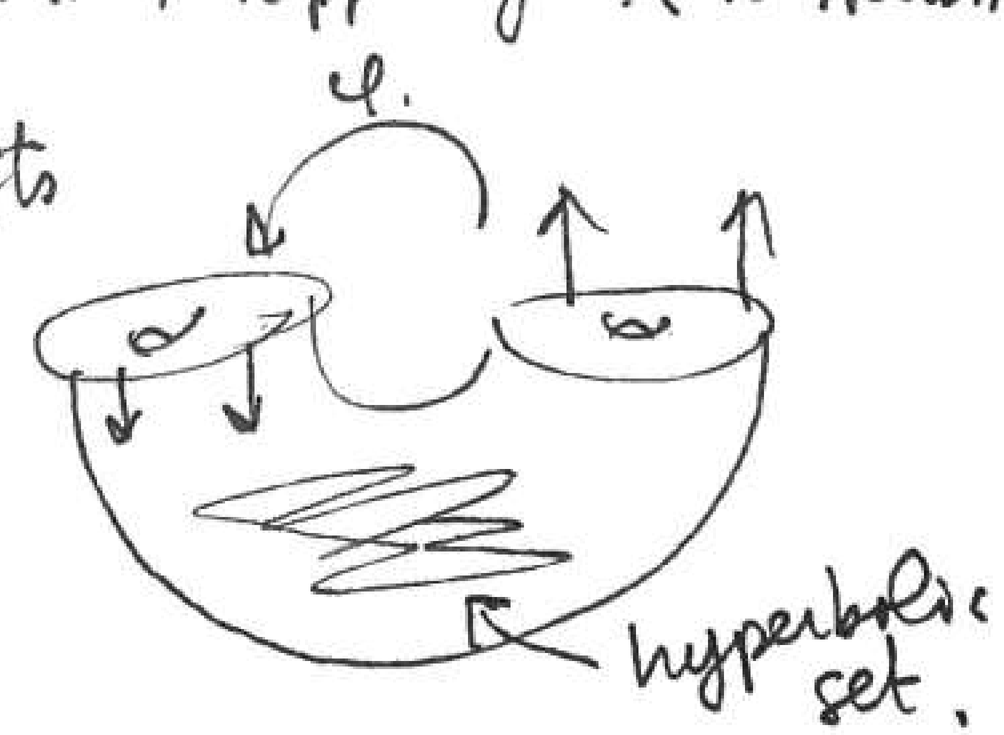
Prop Suppose $\exists$ cone fields $\mathcal{C}^{ss}$, $\mathcal{C}^{uu}$ such that $Df^{N}\overline{\mathcal{C}^{uu}_x} \subseteq \mathcal{C}^{uu}_{f^N x}$ and $Df^{-N}\overline{\mathcal{C}^{ss}_x} \subseteq \mathcal{C}^{ss}_{f^{-N}(x)}$ and vectors in $\mathcal{C}^{ss}$ are contracted in N-iterates and in $\mathcal{C}^{uu}$ expanded $\Rightarrow f$ is PH.

Then: Let $f: M \to M$ PH & conservative which preserves cone-fields $\mathcal{C}^{ss}$ and $\mathcal{C}^{uu}$ and let $g: M \to M$ such that $Dg\, Df\, \overline{\mathcal{C}^{uu}} \subseteq \mathcal{C}^{uu}$ and $Dg^{-1} Dg^{-1}\overline{\mathcal{C}^{ss}} \subseteq \mathcal{C}^{ss} \Rightarrow g \circ f$ is PH.

This is essentially what we do in the previous example.

→ Problem: We need $X_t(T) \cap T = \emptyset$ for $t \in [0,N]$ and N arbitrarily large. This cannot happen if X is transitive.

Solution: Consider finite lifts



Volume preserving ⇒ Stably ergodic / robustly transitive

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