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CLASSIFICATION PROBLEM FOR PH DIFFEOS IN DIM 3

M closed 3 mfd $f: M \to M$ PH diffeo $TM = E^s \oplus E^c \oplus E^u$
all one-dim.

Motivation: → Stable ergodicity
→ Robust transitivity

Examples: → Algebraic: → Automorphisms of $\mathbb{T}^3$ → Anosov
→ Skew-products
→ Automorphisms of nilmanifolds → skew-products
→ $\underbrace{\text{Certain}}_{\text{time 1-maps of}}$ Anosov flows: → suspensions
→ geodesic flow in $K = -1$.
→ Non-Algebraic: Big zoo of Anosov flows, still far from being classified.

So: Why try to classify PH? Hopeless if one does not classify Anosov flows.

Idea (Pujals / Bonatti-Wilkinson) Classify PH systems módulo Anosov systems.

This idea has materialized in plenty of results, conjectures, etc (other than Pujals & BW, we can mention for example HHU, Carrasco, Gogolev, Bonhot, Hammerlindl...)

Two different type of problems appear:

Small manifolds (i.e. "small" fundamental group), here we know how to classify Anosov systems, so there is more hope.

Brin+Burago-Ivanov (also Parwani): If $\pi_1(M)$ is "small" (polynomial growth) ⇒ action in homology is PH (⇒ f homotopic to a model)

Thm (j.w. Hammerlindl) If $\pi_1(M)$ is solvable ⇒ modulo finite lift & iterate f has to be "leaf conjugate" to a model (algebraic).

→ $\exists h: M \to M$ sending $W^c_f \to W^c_{model}$ & conjugates their dynamics.

Big Manifolds: $\pi_1(M)$ is exponential: there is space to grow, so f can be isotopic to identity; however: → f might in principle not be isotopic to id.
→ there are topological obstructions (BBI) but we lack complete understanding (as in Anosov flow case).

§ **NEW EXAMPLES** (joint with: C. Bonatti, A. Gogolev, A. Hammerlindl & K. Parwani)

We have found a new source of examples which have progresively allowed us to provide more & more exotic examples which we still don't fully understand.

Mechanism: $f: M \to M$ is PH with $TM = E^s \oplus E^c \oplus E^u$ and let $h: M \to M$ difeo such that $Dh(E^s) \pitchfork (E^c \oplus E^u)$ and $Dh(E^u) \pitchfork (E^s \oplus E^c)$ then, $\exists n > 0$ such that $F = h \circ f^n$ is PH.

So, if f is the time 1-map of Anosov flow and $h \not\simeq id$ we have constructed examples different from the previously known models.

Proof of Mechanism: Cone field criteria: $\mathcal{E}^u$ narrow cone arround $Dh(E^u)$ and $\mathcal{E}^s$ narrow cone arround E^s. Since $\mathcal{E}^u \pitchfork E^s \oplus E^c$ $\exists n_0 > 0$ such that $Df^{n_0} \mathcal{E}^u$ is very close to E^u $\Rightarrow$ $Dh \circ Df^n(\mathcal{E}^u) \subseteq \mathcal{E}^u$. ∎

Examples 1: Let $\varphi_t : M \to M$ an Anosov flow transverse to an (incompressible) torus $T \subseteq M$. (suspension, but others: Franks-Williams, Bonatti-Langevin, Bonatti-Beguin-Yu)

let $\mathcal{U}_N = \bigcup_{0 \le t \le N} \varphi_t(T)$ and assume $\varphi_t(T) \cap T = \emptyset$ $\forall 0 \le t \le N$

Putting coordinates $\mathcal{U}_N \simeq T \times [0,1]$ via $\varphi_t(x) \mapsto (x, t/N)$ one has that $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ look like $\mathcal{F}^s \times [0,1]$ and $\mathcal{F}^u \times [0,1]$ where $\mathcal{F}^s$ and $\mathcal{F}^u$ are foliations of T.

If N is very large → E^s and E^u in the coordinates $T \times [0,1]$ are close to $T\mathcal{F}^s \times \{t\}$ and $T\mathcal{F}^u \times \{t\}$ everywhere.

this gives hope of constructing h supported in $\mathcal{U}_N$ which is a "Dehn-twist": $h(x,s) = (x+sv, s)$ with $v \in H_1(T,\mathbb{Z})$.

2 problems: → h might be isotopic to identity (i.e. suspension of Anosov in $\mathbb{T}^2$)
→ in general one needs N to be huge so, might be difficult
→ h must satisfy the transversalities ↳ so that one gets transversality.

In the non-transitive case, the second problem does not exist, the first one is rather easy to solve for specific examples (where $v \in H_1(T,\mathbb{Z})$ is $\neq 0$ in $H_1^*(M,\mathbb{R})$) and the same for the third one.

Also, in this case one can show by hand that the example is dynamically coherent (i.e. $E^s \oplus E^c$ and $E^c \oplus E^u$ integrate to invariant foliations).

Thm (w. Bonatti-Parwani): $\exists f: M \to M$ dynamically coherent not isotopic to identity such that M admits an Anosov flow and f is not leaf conjugate to the flow.

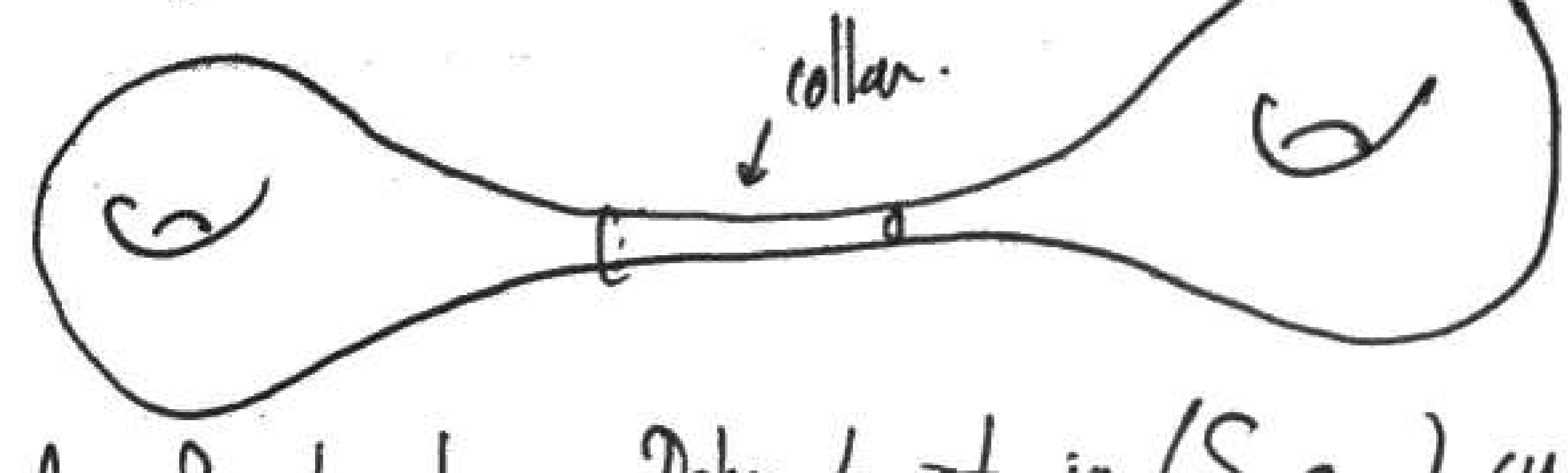
With more attention, starting with examples introduced by Bonatti-Langevin we can prove:

Thm (w. Bonatti & Gogolev) $\exists f: M \to M$ p.h. stably ergodic and robustly transitive not isotopic to identity where M is such that it admits an Anosov flow.

One may ask if it is possible to construct examples where action in homology is large (Dehn-twists have eigenvalues $= 1$). For this, I will try to explain a different construction:

Let S be a surface of genus ≥ 2. Fix γ a simple closed curve in S

One can choose hyperbolic metrics g_n in S (the geodesic flows will be uniformly Anosov) where $\ell(\gamma, g_n) \to 0$.



Therefore, if f_n denotes a Dehn-twist in (S, g_n) supported in the collar, one can see that

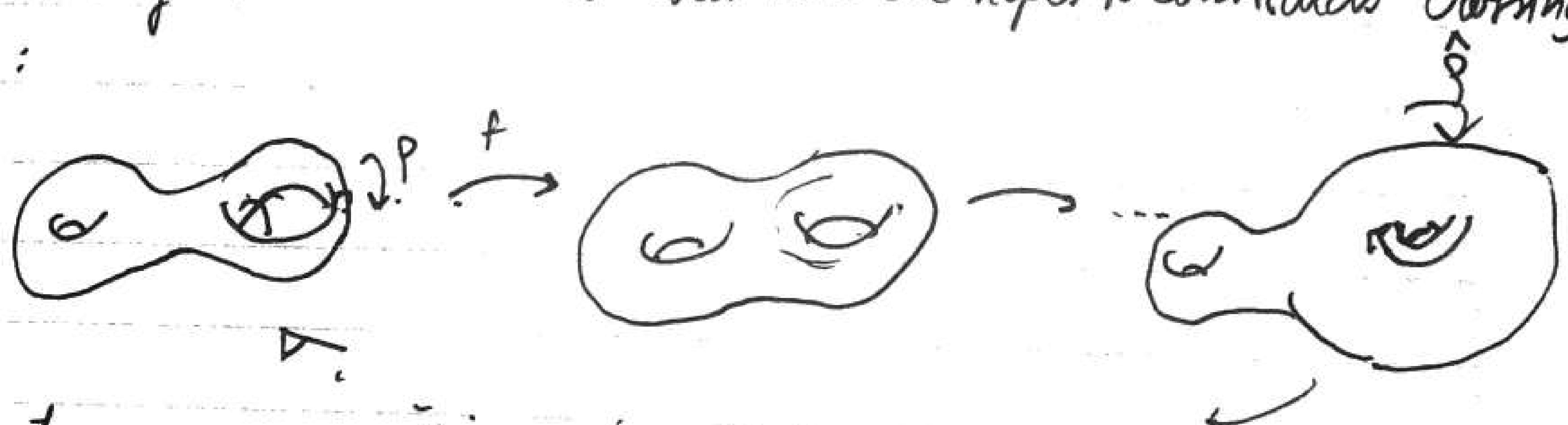
$$(f_n)_* g_n \sim g_n$$

Notice that $Df_n : T^1S \circlearrowleft$ conjugates the geodesic flows $\Rightarrow$ preserves transversalities

$\Rightarrow$ for large n, if φ_t^n is the geodesic flow of g_n one has that

$F = Df_n \circ \varphi_T^n$ with large T is PH (one can do volume preserving, etc)

The isotopy class of F is still "small" but now one hopes to constructs "crossing" Dehn twists:



One can create any mapping class group of the base.

Thm (w. Bonatti, Gogolev & Hammerlindl) Given $h \in MCG(S)$ there exists $f : T^1S \circlearrowleft$ PH, stably ergodic, robustly transitive such that $f \sim Dh$.

Question: $\rightarrow$ If M admits a PH diffeo, does it admit an Anosov flow?

$\rightarrow$ If $f : M \circlearrowleft$ PH isotopic to id $\Rightarrow$ leaf conjugate to Anosov (expansive) flow?

Some progress:

[[**Thm** (w. A. Hammerlindl & M. Shannon) Let M be a circle bundle over a surface S $\Rightarrow$ M admits a PH diffeo (with orientable bundles) if and only if M admits an Anosov flow.]]

Corollary: $S \times S^1$ does not admit PH diffeos.

Let me explain the proof of this consequence of Ghys work with our proof:

Prop: $S \times S^1$ does not admit transitive Anosov flows.

Proof: A result of Thurston & Roussarie (extended by Britenham & for C^0-foliations) implies that (after changing coordinates) every leaf of $\mathcal{F}^{cs}$ is either horizontal or vertical (consists of entire fibers).

Existence of vertical leafs contradicts transitivity.

If every center & stable leaf is horizontal $\rightarrow$ the vector field X generating the Anosov flow is horizontal (if $\pi : S \times S^1 \to S$ $\pi(x,t) = x$ $\Rightarrow$ $D\pi X \neq 0$ $\forall (p,t) \in S \times S^1$)

$\Rightarrow$ if $\sigma : S \to S \times S^1$ is a section, one would get a non vanishing vector field in S, a contradiction.

$\boxtimes$