Codimension one generic homoclinic classes with interior

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Abstract

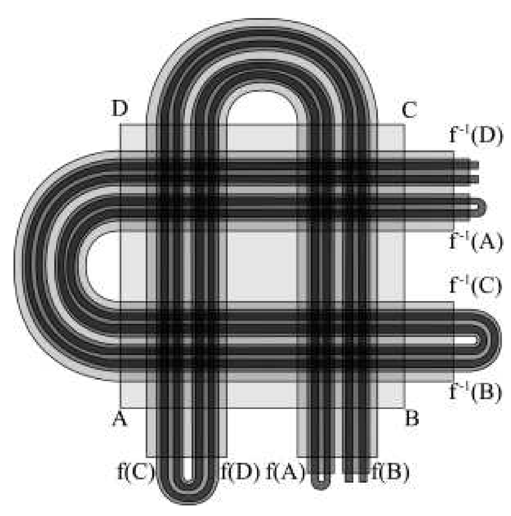
We study generic diffeomorphisms whose nonwandering set has interior. Under some assumptions on the dynamics of the derivative of the diffeomorphisms we prove that it should be transitive. This has some interesting consequences, mainly in low dimensional systems.

1 Introduction

The goal of generic dynamics is to describe the dynamics of a large set of diffeomorphisms. By large set we mean a residual (generic) set of them. For now, the theory is quite far from achieving its goal, but some interesting progress has been made in the C^1 topology where new perturbation results allowed some results to describe some aspects of the dynamics of generic diffeomorphisms. We shall work always in this topology.
The first results gave properties of generic diffeomorphisms which were a priori not related with the dynamics, or its relation was not so direct, but Mañe started a more systematic approach by first studying some consequences of robust dynamics and then giving some explicit results on generic dynamics. A remarkable one is that in surfaces, generic diffeomorphisms may be hyperbolic or else they must exhibit infinitely many distinct limit dynamics. Now, after Pujals and Sambarino's work there is only one remaining question in generic dynamics on surfaces which is if hyperbolic systems are in fact open and dense. However, it was only very recently that some very natural questions were answered. In particular, generically, if a diffeomorphism of a surface has a homoclinic class with interior, then the surface must be the torus and the dynamics conjugated to a linear Anosov diffeomorphism (this was proved in [ABCD] and we also give a different proof in [PS]). So, generic non conservative diffeomorphisms are very different from conservative ones, except in the torus where there are open sets of diffeomorphisms conjugated to conservative ones.
The question we shall be concerned is very specific and attempts to study the same as before, is it possible for a generic non conservative diffeomorphism to have interior in its nonwandering set? The answer to this question should be similar to the one in surfaces, in fact, Abdenur,Bonatti and Diaz ([ABD]) conjecture that a generic diffeomorfism whose nonwandering set has interior should be transitive.

2 Definitions and presentation of results

Let $\mathcal{D}iff^1(M)$ be the set of C^1 diffeomorphisms of M (a closed manifold) with the uniform C^1 topology. With this topology $\mathcal{D}iff^1(M)$ is a Baire space so residual subsets will be dense and intersect each other in a new residual set. We shall say that a diffeomorphism f is *generic* if it belongs to a residual subset $\mathcal{R} \subset \mathcal{D}iff^1(M)$ which we shall omit to mention in general.
A *homoclinic class* of a periodic point p of a generic diffeomorphism f is $H(p,f) = \overline{W^s(p) \cap W^u(p)}$. This is a transitive set that may be very non trivial (see for example the figure below).



A compact invariant set Λ is said to admit a *dominated splitting* if $T_\Lambda M = E_1 \oplus \ldots \oplus E_k$ where the E_i are continuous and $Df-$invariant subbundles satisfying the following domination condition:

$$\|Df|_{E_i(x)}\| \|Df^{-1}|_{E_j(f(x))}\| < \lambda < 1 \qquad \forall x \in \Lambda \quad i < j$$

The dominated splitting is said to be of *codimension one* if one of the extremal subbundles is one dimensional. A subbundle E is uniformly contracting (expanding) if there exists $n > 0$ ($n < 0$) such that for every $x \in M$ one has $\|Df^n|_{E(x)}\| < 1/2$. A compact invariant set is *hyperbolic* (*partially hyperbolic*) if it admits a dominated splitting $T_\Lambda M = E^s \oplus E^u$ ($T_\Lambda M = E^s \oplus E^c \oplus E^u$) where E^s is uniformly contracting and E^u is uniformly expanding. If a set is partially hyperbolic and both E^s and E^u are non trivial one says that the set is *strongly partially hyperbolic*.

2.1 The Conjecture

In their paper [ABD], Abdenur, Bonatti and Diaz make the following conjecture:

CONJECTURE (Abdenur, Bonatti & Diaz 2004) Any generic homoclinic class with nonempty interior is the whole manifold.

2.2 Our Theorem

In [PS] we prove the following result in an attempt to attack the conjecture stated above under some conditions on the dominated splitting the class admits (a result of Bonatti, Diaz and Pujals asserts that such an homoclinic class must admit some dominated splitting).

THEOREM: Any generic homoclinic class with nonempty interior admiting a codimension one dominated splitting satisfies that the one dimensional subbundle is hyperbolic.

Using a result of [ABD] which asserts that generically, a strongly partially hyperbolic homoclinic class with interior is the whole manifold we get the following corollary which in particular recovers a result previously obtained in [ABCD] (the conjecture in dimension 2).

COROLLARY: If a generic homoclinic class with nonempty interior admits a dominated splitting whose extremal bundles are both one dimensional then the conjecture holds. In particular, it holds in dimension 2.

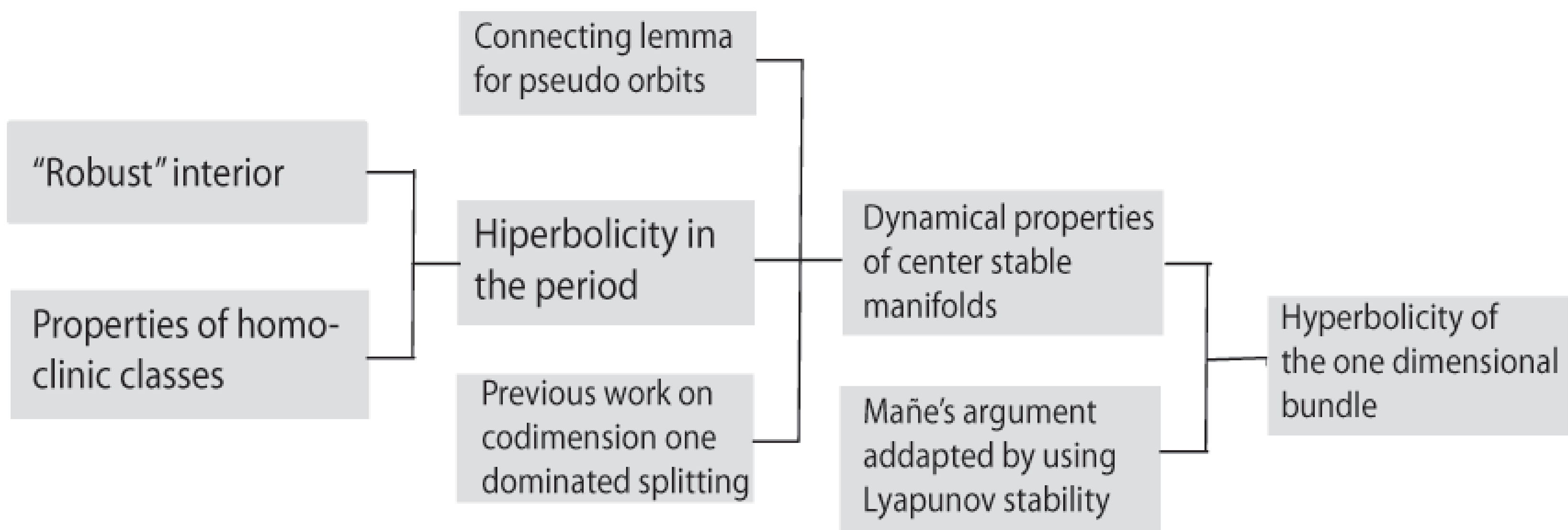
3 Idea of the proof

3.1 Previous Related Results

In [ABD] the conjecture is proved in some situations for example, they prove it to be true if the class is strongly partially hyperbolic .
Also they prove that if there is such a class and the diffeomorphism is not transitive, then the class cannot be isolated (a set Λ is isolated if and only if there exists U neighborhood of Λ such that $\Lambda = \bigcap_{n\in Z} f^n(U)$) .
Finally, they prove that the class must be Lyapunov stable for f and f^{-1} (this means, in the generic context that the class is saturated by stable and unstable sets), and that the interior of the class is in some sense robust (given an open set U whose closure is contained in the interior of the class, there exist a neighborhood of f, such that any generic diffeomorphism in that neighborhood has a homoclinic class which contains U) .
A result of Bonatti, Diaz and Pujals asserts that if a generic homoclinic class does not admit any non trivial dominated splitting, then the class should be contained in the closure of the union of the set of sinks and sources of the diffeomorphism. Since sinks and sources cannot intersect the homoclinic class (the homoclinic class of a sink or a source is always trivial) one gets that the class must admit a non trivial dominated splitting. The fact that sinks and sources cannot appear in the interior of the homoclinic class is one of the main tools in the proof of our theorem.

3.2 Some techniques used in the proof

We use some classic techniques in generic dynamics, in particular the connecting lemma for pseudo-orbits of Bonatti and Crovisier. Also, we use some results of Pujals and Sambarino on codimension one dominated splittings and a new generic result of Lessa and Sambarino asserting that the curves that integrate one dimensional extremal bundles are in some sense stable (i.e. points in the center stable manifold remain near for future iterates). We also make use of Pliss and Liaos lemmas which allow us to control the expansion and contraction of periodic points in the bundles of the dominated splitting.



3.3 Rough idea of the proof

Let us assume that the class admits a dominated splitting of the form $T_H M = E \oplus F$ where $dim(E) = 1$.
First we prove that if the homoclinc class has interior, the periodic points in it (which are saddles) have eigenvalues (in the E direction) exponentially far from 1 in the period. Otherwise we manage to obtain a sink or a source in the interior of the class and thus contradicting the fact that the interior of the homoclinic class for generic diffeomorphisms is, roughly speaking, robust.
Using the previous fact and some results of Lessa, Pujals and Sambarino we manage to prove that the manifolds tangent to E have dynamical properties. To do this, we first prove the assertion for the periodic points (using that they have good eigenvalues and the result of Lessa and Sambarino) and then we extend it to the rest of the points using the ideas of Pujals and Sambarino and the connecting lemma for pseudo orbits.
Now, if the one dimensional bundle is not hyperbolic, the other one should have some hyperbolicity from domination. This, together with the dynamical properties allow us to make an Anosov Closing Lemma type argument to find periodic points near the class with bad hyperbolicity in the period. The difficult part is to ensure that the periodic point we construct is in fact in the class (reaching a contradiction and proving the theorem). To do this we use Lyapunov stability of the class and the Lemma of Liao which gives uniform size on the stable and unstable sets of the periodic points we find and thus we prove the theorem.

4 References

[ABD] F.Abdenur, C.Bonatti and L.Diaz, Non wandering sets with non empty interior. *Nonlinearity* (2004)
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