A PROOF OF THE EXISTENCE OF ATTRACTORS IN DIMENSION 2

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Abstract. We present a proof of the (folklore) fact that there is a residual subset of $Diff^1(M^2)$ consisting of diffeomorphisms having an hyperbolic attractor. The proof is based on recent results on generic dynamics. We continue by proving some results for generic diffeomorphisms in surfaces with only finitely many sources.

1. Introduction

A main goal in differentiable dynamics is to describe the asymptotic behavior of "most" orbits of "most" diffeomorphisms.

The point of view we adopt is that of generic dynamics, that is, to describe the dynamics of open and dense, or residual (G_δ-dense) subsets of diffeomorphisms and points in the manifold. We work mainly in the C^1 category due to the fact that in higher topologies little is known on how to perturb diffeomorphisms while controlling the dynamical effects of those perturbations.

For a survey on generic dynamics see [BDV] (chapter 10).

A main question in this context would be, is there an attractor for a generic diffeomorphism?, that is, does there exist a residual subset $\mathcal{R} \in \text{Diff}^1(M)$ such that if $f \in \mathcal{R}$ then f has an attractor? (see for example Problem 10.30 in [BDV]).

An attractor is a transitive set Λ such that it admits an open neighborhood U satisfying that $f(\overline{U}) \subset U$ and such that $\Lambda = \bigcap_{n\geq 0} f^n(U)$.

Recently Bonatti, Li and Yang have proved that the question has a false answer in dimensions bigger or equal to 3 ([BLY]). In dimension 2, the result was announced to be true by Araujo ([A]) but the result was never published since there was a gap on its proof. However, with the techniques of [PS] the gap in the proof can be arranged[1] (though this was never written).

We prove here the following theorem which similar to the one by Araujo. The proof we give is quite short but based on the recent developments of generic dynamics (mainly [ABC],[BC], [MP] and [PS]).

Recall that a *sink* is a hyperbolic periodic point whose eigenvalues are all of modulus smaller than one. An *hyperbolic attractor* is an attractor that admits a hyperbolic splitting (we shall define this later).

Theorem 1.1. *There is $\mathcal{R} \subset \text{Diff}^1(M^2)$, a G_δ dense subset of diffeomorphisms in the surface M^2 such that for every $f \in \mathcal{R}$, there is an hyperbolic attractor. Moreover, if f has finitely many sinks, then f is essentially hyperbolic.*

[1] This was communicated by Martin Sambarino.

We say that f is *essentially hyperbolic* if it admits finitely many hyperbolic attractors and such that the union of their basins cover an open and dense subset of M (Araujo proves that the basin of atraction has Lebesgue measure one, his techniques work in this context too). This definition comes from [PT] and is motivated by a new result of [CP] which closes a long standing problem posed by Palis in [PT].

Theorem 1.2. *There is $\mathcal{R} \subset \mathrm{Diff}^1(M^2)$, a G_δ dense subset of diffeomorphisms in the surface M^2 such that for every $f \in \mathcal{R}$ with finitely many sources satisfies that every homoclinic class which is a quasi-attractor (see the definition in the next section) is an hyperbolic attractor.*

In particular we get the following using results in [MP] and [BC]:

Corollary 1.1. *There is $\mathcal{R} \subset \mathrm{Diff}^1(M^2)$, a G_δ dense subset of diffeomorphisms in the surface M^2 such that for every $f \in \mathcal{R}$ with finitely many sources satisfies that generic points converge either to hyperbolic attractors or to aperiodic classes.*

This last Corollary applies for example for the well known Henon attractor, in fact, since hyperbolic attractors which are in a disc which is dissipative are sinks, in the Henon case we get that there are no strange attractors (aperiodic quasi attractors for generic diffeos cannot be attractors).

2. Some preliminaries

We shall present here some recent results we shall use in the proof of Theorem 1.1. To do that, we shall explain more or less the idea of the proof.

First of all, we would like to prove the existence of hyperbolic attractors. The strategy will be to concentrate in Lyapunov stable chain recurrence classes. We say that a compact invariant set Λ is *Lyapunov stable* if for every U neighborhood of Λ, there exist $V \subset U$ neighborhood of Λ such that $f^n(V) \subset U$ for every $n \geq 0$. These will be the candidates for attractors.

We must define chain recurrence classes. We note $x \dashv y$ if $\forall \varepsilon > 0$ there is an $\varepsilon-$pseudo orbit $x = x_0, \dots, x_n = y$. We denote as $x \vdash\dashv y$ to the equivalence relation (inside the chain recurrence set, i.e. the points such that $x \dashv x$) given by $x \dashv y$ and $y \dashv x$. The classes of this equivalence relation are called *chain recurrence classes*.

In [MP] it was proved that there exists a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that for every $f \in \mathcal{R}$, there exist a residual subset $R_f \subset M$ such that if $x \in R$ then $\omega(x)$ is a Lyapunov stable set.

This result was improved in [BC], using the connecting lemma for pseudo-orbits that allowed them to prove a conjecture of Hurley. To state that, we shall say that a chain recurrence class Λ is a *quasi-attractor* if there exists a nested sequence of open neighborhoods $\{U_n\}$ such that $\bigcap_n U_n = \Lambda$ and such that $f(\overline{U_n}) \subset U_n$. In [BC] it is proved that a Lyapunov stable chain recurrence class is in fact a quasi-attractor.

Theorem 2.1 ([MP],[BC])**.** *There exist a residual subset $\mathcal{R}_Q \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}_Q$ then there exist a residual subset $R_f \subset M$ such that every point converges to a Lyapunov stable chain recurrence class. Moreover, these chain recurrence classes are quasi-attractors.*

In [BC], also using the connecting lemma for pseudo-orbits they obtain the following very important result. Let us recall that the *homoclinic class* of a hyperbolic periodic point is the closure of the transversal intersections between the stable and unstable manifolds of the point.

Theorem 2.2 ([BC])**.** *There exists a residual subset $\mathcal{R}_C \subset \mathrm{Diff}^1(M)$ such that if a chain recurrence class contains a periodic point p, then it coincides with its homoclinic class. In particular, two homoclinic classes coincide or are disjoint (this consequence is also from* [CMP]*)*

We shall study Lyapunov stable chain recurrence classes and we want to prove that far away from sinks they are homoclinic classes and in fact, hyperbolic attractors. To do that, we shall study the invariant measures supported in the quasi-attractor.

To obtain the hyperbolic splitting we shall use the notion of dominated splitting. We say that a compact invariant set Λ admits a *dominated splitting* if there is a decomposition $T_\Lambda M = E \oplus F$ on $Df-$invariant non-trivial subspaces such that for every $v \in E$ and $w \in F$ we have that there exists $n > 0$ such that

$$\frac{\|Df^n v\|}{\|v\|} \leq \frac{1}{2}\frac{\|Df^n w\|}{\|w\|}$$

We say that Λ is *hyperbolic* if it admits a dominated splitting and there exists $n_0 > 0$ such that for every $v \in E$ and $w \in F$ we have that $\|Df^{n_0}v\| < \frac{1}{2}\|v\|$ and $\|Df^{-n_0}w\| < \frac{1}{2}\|w\|$. A *hyperbolic attractor* is an attractor which is hyperbolic (notice that a sink is an hyperbolic attractor).

We have the following Theorem of [ABCD] which deduces a generic property from the results of [PS] and shows that to find hyperbolicity it suffices to obtain a dominated splitting

Theorem 2.3 ([PS], [ABCD] Theorem 2)**.** *There exist a residual subset $\mathcal{R}_H \subset \mathrm{Diff}^1(M^2)$ where M^2 is a surface such that if $f \in \mathcal{R}_H$ and Λ is a chain recurrence class admitting a dominated splitting, then, Λ is hyperbolic.*

To obtain the dominated splitting in the class we shall use the following results, the first one is new, but the others are quite classical

Theorem 2.4 ([ABC] and [C], Proposition 1.4)**.** *Let μ be an ergodic hyperbolic measure (that is, all the Lyapunov exponents different from zero) such that the Oseledet's splitting admits a dominated splitting. Then, the support of μ is contained in a homoclinic class.*

The following proposition follows easily from arguments from [BDP]. We shall give a proof for completeness. For a point $x \in M$ we denote its orbit by $\mathcal{O}_x$ and for a periodic point p we denote its period by $\pi(p)$.

Proposition 2.1. *Let H be homoclinic class of a periodic point with $|det(Df_p^{\pi(p)})| < 1$, then there is a dense subset of periodic points in H having the same property.*

Proof. Let U be an open set in H. There is a periodic point $q \in U$ homoclinically related to p. Consider $x \in W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$ and $y \in W^s(\mathcal{O}_q) \cap W^u(\mathcal{O}_p)$. The set $\mathcal{O}_p \cup \mathcal{O}_q \cup \mathcal{O}_x \cup \mathcal{O}_y$ is a hyperbolic set. So, using the shadowing lemma we can obtain a periodic point $r \in U$, homoclinically related to p such that its orbit spends most of the time near $\mathcal{O}_p$. Thus, it will satisfy that $|det(Df_r^{\pi(r)})| < 1$.

□

We shall also use Franks' lemma and Mañe's ergodic closing lemma (see [F] and [M]). The first one asserts that one can change by a small C^1-perturbation the derivative of a periodic point, and the latter implies that given an ergodic measure for f, there exists diffeomorphisms $g_n \to f$ and periodic orbits $\mathcal{O}_n$ such that $\mathcal{O}_n$ converge in the Hausdorff topology to the support of μ, in particular, if $\int log|det(Df)|d\mu = a$ then for these periodic orbits the logarithm of the determinant will be arbitrarily close to a.

3. Proof of the Theorem 1.1

Let $\mathcal{K}$ be the set of all compact subsets of M with the Hausdorff topology.

Let $\Gamma : \text{Diff}^1(M^2) \to \mathcal{K}$ be the map such that $\Gamma(f) = S(f)$ where $S(f)$ is the clousure of the set of sinks of f.

Since Γ is lower semicontinuous, there exists a residual subset $\mathcal{R}_0$ of $\text{Diff}^1(M)$ such that for every $f \in \mathcal{R}_0$, f is a continuity point of Γ. This implies that we can write $\mathcal{R}_0 = \mathcal{F} \cup \mathcal{I}$ open sets in $\mathcal{R}_0$ such that for every $f \in \mathcal{F}$ the number of sinks is locally constant and finite (that is, there is a neighborhood $\mathcal{U}$ of f in $\text{Diff}^1(M)$ such that for every $g \in \mathcal{U}$ the number of sinks is the same and they vary continuously), and such that for every $f \in \mathcal{I}$ there are infinitely many sinks.

To prove the Theorem it is enough to work inside $\tilde{\mathcal{F}}$ (an open set in $\text{Diff}^1(M)$ such that $\mathcal{F} = \tilde{\mathcal{F}} \cap \mathcal{R}_0$) since the Theorem is satisfied in $\mathcal{I}$.

Let $\mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_Q \cap \mathcal{R}_C \cap \mathcal{R}_H$. Let $f \in \tilde{\mathcal{F}} \cap \mathcal{R}$. We must show that f is essentially hyperbolic.

We shall prove that every Lyapunov stable chain recurrence class is an hyperbolic attractor.

Since $f \in \mathcal{R}_Q$ we have that if Λ is a non trivial (not a sink) Lyapunov stable chain recurrence class then Λ is a quasi attractor, thus, it admits open neighborhoods U_n such that $\Lambda = \bigcap_{n\geq 0} U_n$, $\overline{U_{n+1}} \subset U_n$ and such that $f(\overline{U_n}) \subset U_n$.

Lemma 3.1. *Let Λ be a chain recurrent quasi-attractor. Then, there exist an ergodic measure μ supported in Λ such that $\int log(|det(Df)|)d\mu \leq 0$.*

Proof. Let m_n be the normalized Lebesgue measure in U_n. Consider μ_n a limit point in the weak-$*$ topology of the sequence of measures given by $\nu_k = \frac{1}{k}\sum_{i=1}^{k} f_*^i(m_n)$ which is an invariant measure supported in $f(\overline{U_n})$. Recall that $f_*(\nu)(A) = \nu(f^{-1}(A))$.

Since $f(\overline{U_n}) \subset U_n$ we have that $\int \log(|\det Df|)dm_n < 0$, and since this can be iterated, we have that $f^k(\overline{U_n}) \subset f^{k-1}(U_n)$ thus $\int \log(|\det Df|)df_*^k(m_n) < 0$. This implies that $\int \log(|\det Df|)d\nu_k \leq 0$. So, we get that $\int \log(|\det Df|)d\mu_n \leq 0$.

Now, consider a measure μ which is a limit point in the weak-$*$ topology of the measures μ_n. This must be an invariant measure, supported on Λ satisfying that $\int \log(|\det Df|)d\mu \leq 0$. Using the ergodic decompostion theorem (see [M2]) one can assume that μ is ergodic.

□

Since the set of sinks varies continuously with f and there are finitely many of them, we can choose n such that there are no sinks in U_n. Using the ergodic closing lemma [M] we get that the support of the measure must admit a dominated splitting. Otherwise we get periodic points converging to the support of the measure and with $\log(|\det Df^{\pi(p)}|)$ close to zero. If they don't admit a dominated splitting, using a classical argument (see Lemma 7.7 of [BDV]) one can convert them into sinks by applying Franks' lemma, [F], a contradiction.

So, the support of the measure μ admits a dominated splitting. Also, the measure must be hyperbolic since if no positive exponents exist one can create a sink with the ergodic closing lemma and since $\int \log(|\det Df|)d\mu \leq 0$ then one negative exponent must also exist. Using Theorem 2.4, we deduce that the support of μ is contained in a homoclinic class, in particular, Λ is a homoclinic class. Also we get periodic points inside the class such that $\log(|\det Df^{\pi(p)}|) < \varepsilon$ for any $\varepsilon > 0$.

Using Proposition 2.1 we get that periodic points with this property are dense in the homoclinic class and so we get a dominated splitting $T_\Lambda M = E \oplus F$ in the whole class. In fact, since we are far from sinks, we get that F must be uniformly expanding.

Since we are in $\mathcal{R}_H$ we get that Λ is hyperbolic, and thus, using classical arguments from hyperbolic theory, we get that Λ is a hyperbolic attractor.

This proves the first assertion of the Theorem.

Now, suppose there are infinitely many hyperbolic attractors. Assume $\Lambda_n \to K$ in the Hausdorff topology, clearly $K \cap S(f) = \emptyset$.

Notice that there are measures μ_n supported in Λ_n such that $\int \log(|det(Df)|)d\mu_n \leq 0$. Consider a weak-$*$ limit μ of these measures, so we have that μ is supported in K and verifies that $\int \log(|det(Df)|)d\mu_n \leq 0$. So using the same argument as before we deduce that K is contained in a hyperbolic homoclinic class, and thus isolated, a contradiction.

Since $f \in \mathcal{R}_C$, generic points in the manifold converge to Lyapunov stable chain recurrence classes and we get that there is an open and dense subset of M in the basin of hyperbolic attractors. This finishes the proof of the Theorem.

□

4. Proof of the Theorem 1.2

We shall use the following Theorem from [Pot]:

Theorem 4.1. *Let Λ be a Lyapunov stable homoclinic class of a generic diffeomorphism f. So, if Λ doesn't admit any dominated splitting then every periodic point p in Λ satisfies that $det(D_p f^{\pi(p)}) > 1$.*

If Λ doesn't admit any dominated splitting, then the previous theorem toghether with Frank's Lemma allow us to create new sources. The genericity hypothesis implies that Λ must admit dominated splitting.

Theorem 2.3 concludes.

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