ANOSOV REPRESENTATIONS

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ABSTRACT. This are informal notes on Anosov representations for a minicourse joint with F. Kassel in the conference 'Beyond Uniform Hyperbolicty' (Provo, 2017). They contain more than it will be presented and should (ideally) serve as a map to the references. They have not been revised, and it is likely they contain errors of all types, use with caution. (It is also likely that some important references are missing, please let me know if you find some problems in this regard, or any other.)

1. INTRODUCTION

Let Γ be a subgroup of a (semisimple) Lie group G. We are interested by the study of those subgroups which are discrete in G (i.e. the restriction of the topology to Γ makes it discrete). We will assume throughout that $G \subset \mathrm{GL}(d,\mathbb{R})$ for some $d > 1$ (this is no big restriction) and in several ocasions we will restrict to very specific groups in order to work mainly with linear algebra instead of lie theory. We refer the reader to the excelent notes [Be] (and references therein) for some background on such subgroups.

Some general (and quite imprecise/incomplete) goals can be described as follows:

- Understand the interaction between the geometry of G and that of Γ. This may be particularly rich when Γ is *quasi-isometrically* embedded.
- When $\Gamma = \pi_1(M)$ and M is a (closed) manifold, undertstand to which extent the embedding provides some geometric structure on M.
- *Counting problems*; how many elements of Γ are there in a ball of radius R of G?
- Understand *deformations* of the structure. E.g. how flexible is the embedding? how does the counting vary with the deformation?.
- *Rigidity properties:* A way to see this is the understanding of the Zariski closure of Γ and when it is (much) smaller than G. Describe this closure is a relevant problem and sometimes rigidity is claimed when some weak hypothesis impose some constraints on the Zariski closure.

We are interested in properties which are *stable* under deformations. To study deformations of $\Gamma \subset G$ it is convenient to work with representations: Given Γ an abstract group and G a Lie group, we say that a map $\rho : \Gamma \to G$ is a *representation* if it is a group morphism (i.e. $\rho(\gamma\beta) = \rho(\gamma)\rho(\beta)$ and $\rho(\gamma^{-1}) = (\rho(\gamma))^{-1}$). We say that:

- ρ is *faithful* if it is injective.
- ρ is *discrete* if the image of Γ in G (which is a subgroup) is discrete in G.

We denote $\mathrm{Hom}(\Gamma, G)$ to the set of representations of Γ into G. We give a topology to $\mathrm{Hom}(\Gamma, G)$ stating that a sequence ρ_n converges to ρ if $\rho_n(\gamma) \to \rho(\gamma)$ for every $\gamma \in \Gamma$ (pointwise convergence). We will restrict to the case where Γ is **finitely generated** so that $\mathrm{Hom}(\Gamma, G)$ can be identified with a subset of G^k with the product topology (where $k \geq 1$ is larger than the size of a generator of Γ).

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Remark 1.1. Indeed, when Γ is finitely generated and G is an algebraic group; the set $\mathrm{Hom}(\Gamma, G)$ is an algebraic sub-variety of G. This important fact will be irrelevant in this minicourse.

Of fundamental importance to us will be the existence of representations of a given group Γ which are *robustly* faithful and discrete. Notice that in this case one can consider $G/_{\rho(\Gamma)}$ to be the quotient space that will have the structure of orbifold[1] locally modelled in G.

Remark 1.2. Tipically, one considers the action of Γ into the *symmetric space* of G defined as $G/_K$ where K is a maximal compact subgroup of G. If $\rho(\Gamma)$ is discrete in $G \cong \mathrm{Isom}(G/_K)$ then it is also discrete in $G/_K$.

A well known result, known as Ehresmann-Thurston's principle (see [BeGu]) states that when Γ is cocompact, this is always the case:

Theorem (Ehresmann-Thurston). *If $\rho : \Gamma \to G$ is a faithful and discrete representation so that $G/_{\rho(\Gamma)}$ is compact, then, ρ is robustly faithful and discrete.*

This is a transversality result, and implicitly uses the fact that Γ is finitely generated. Compactness is crucial as it allows to establish this sort of uniform transversality (see the proof in [BeGu] which makes this fact very clear). In fact, Ehresmann-Thurston's theorem is a much more general statement dealing with (G, X)-structures on manifolds and the corresponding statement for cocompact lattices can be shown in many other ways (e.g. in higher rank by appealing to Margulis' rigidity results and in real rank 1 except for $\mathrm{SL}(2,\mathbb{R})$ by appealing to Mostow's rigidity).

We will see later that *Anosov representations*, the object of this notes, are representations satisfying some sort of uniform transversality condition that allows, in a certain way, to prove that they are stably faithful and discrete by a similar argument without using cocompactness (though some form of cocompactness is hidden as shown recently in [DGK]). These representations are interesting, among many other things, because they provide open sets of faithful and discrete representations (which also enjoy other important properties).

Other examples of robustly faithful and discrete representations are given by rigidity results such as Mostow rigidity or Margulis superrigidity, but these are in a sense artificial as the reason for the robustness is tied with the lack of deformations of the group rather than to a way in which the group is embedded in the Lie group. More interesting examples of (non Anosov) representations which are robustly faithful and discrete have recently appeared (see [DGK_2]).

Stil, I find the following open question quite important and a motivation for the study of Anosov representations in the same way as hyperbolic dynamics appears in the study of smooth dynamics:

Question 1. *Is every robustly faithful and discrete representation of a hyperbolic group Γ Anosov? What about a robustly* quasi-isometric *representation of Γ into a higher rank Lie group*[2]*?*

[1] Due to the possible existence of finite order elements in Γ one cannot ensure that the quotient is a manifold, but due to Selberg's theorem [Be, Proposition 2.1] one can take a finite index subgroup Γ' of Γ for which the quotient is a true manifold.

[2] Quasi-isometry will be defined below. Here one needs to add higher rank because as we will see, every quasi-isometric representation into a rank one Lie group is automatically Anosov.

We will see that if a group admits an Anosov representation then it must be hyperbolic. We remark that this question admits a positive answer[3] for $G = \mathrm{PSL}(2,\mathbb{C})$ ([Su]) and into $G = \mathrm{PSL}(2,\mathbb{R})$ ([Go, ABY]). In higher rank, as far as I am aware, the only result in this direction concerns certain connected components of representations of certain Coexeter groups into $\mathrm{PSL}(d,\mathbb{R})$ (see [DGK$_2$, Section 11]).

Let me close this introduction by stating what I believe to be the first case I would consider[4]:

Question 2. *Let $A, B \in \mathrm{SL}(3,\mathbb{R})$ be two matrices so that for every pair $(\hat{A}, \hat{B})$ close to (A, B) one has that there exists $c > 0$ such that:*

> *For any $k > 0$ and any W a (reduced) product of k matrices in $\{\hat{A}, \hat{B}, \hat{A}^{-1}, \hat{B}^{-1}\}$ it holds*
> $$\|W\| \geq c^k.$$

Then, is there some $\hat{c} > 0$ so that

> *For any $k > 0$ and any W a (reduced) product of k matrices in $\{A, B, A^{-1}, B^{-1}\}$ it holds*
> $$\frac{\|W\|^2}{\|W^{-1}\|} \geq \hat{c}^k.$$

2. Some preliminaries

2.1. Geometry of finitely generated groups.

When Γ is a finitely generated group and F a finite symmetric generator of Γ we denote, for $\gamma, \beta \in \Gamma$:

$$d_F(\gamma, \beta) := |\beta^{-1}\gamma|_F\ ,$$

where $|\cdot|_F$ denotes the *word length* (i.e. $|\gamma|_F := \min\{k \, : \, \gamma = f_{i_1} \cdots f_{i_k} \text{ where } f_{i_j} \in F\}$). The distance d_F is invariant under left multiplication and it is independent on F up to *quasi-isometry*; i.e. if F' is other finite generating set, then, there exists $C > 1$ such that for every $\gamma, \beta \in \Gamma$:

$$C^{-1} d_{F'}(\gamma, \beta) - C \leq d_F(\gamma, \beta) \leq C d_{F'}(\gamma, \beta) + C.$$

2.2. Geometry of Lie groups.

Semisimple Lie groups can be characterized as the isometry groups of symmetric spaces of non-positive curvature without $\mathbb{R}^k$-factors. If X is such a symmetric space and $G \cong \mathrm{Isom}(X)$ is a semisimple Lie group, it follows that there is an invariant non-degenerate form κ (the *Killing form*) such that if K is a maximal compact subgroup of G then G/K is Riemannian with respect to the induced form of κ in the quotient and is isometric to X. We refer the reader to [BPS, Section 7 and 8] and references therein for a more detailed account in a language not so far from dynamicists. A great general reference is [Eb]; also one can look at the very complete [He].

2.3. Boundaries.

[3] In fact, for $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{R})$ there might be easier ways as the results of [Su] and [ABY] only assume 'robust faithfulness'....

[4] Probably out of ignorance. It makes sense to believe that other groups might have a better way to be attacked too (for example, see [DGK$_2$] where they can even manage the robust faithful and discreteness to imply Anosov). The advantage of $\mathrm{SL}(3,\mathbb{R})$ is that, as we will see, verifies that there is only 'one way' to be Anosov

2.3.1. *Word hyperbolic groups.* A finitely generated group Γ is said to be *word hyperbolic* if its Cayley graph is hyperbolic (in the sense of Gromov) as a metric space with the metric defined by giving unit length to edges of the graph [Gr, CDP, GH]. This metric in the Cayley graph restricts to the word metric in the vertices (i.e. elements of Γ). As being hyperbolic in the sense of Gromov is invariant under quasi-isometries, this does not depend on the choice of the finite (symmetric) generating set (though the graph itself and its topology does depend on this).

Recall that a (geodesic) metric space X is said to be δ-*hyperbolic* if for every triplet $x, y, z \in X$ if one considers $[x, y]$, $[x, z]$ and $[y, z]$ geodesic segments joining such points, then it holds that $[x, y]$ is contained in the δ-neighborhood of $[x, z] \cup [y, z]$. This is sometimes expressed by saying that triangles are δ-*thin*.

Exercise 1. Show that if $d(x, y) > \delta$ then there is a unique geodesic joining x and y.

For a δ-hyperbolic space X it is possible to construct a *boundary* at infinity ∂X which consist of equivalence classes of geodesic rays where the equivalence is given by being at bounded Hausdorff distance apart (this is sometimes called the *visual boundary* and can be defined in more generality, an important aspect of hyperbolic spaces is that many different definitions of boundary coincide).

The working definition of a hyperbolic group in this notes is the one given by the following result of Bowditch [Bow]:

Theorem 2.1 (Bowditch)**.** *Let Γ be a group that acts on a proper and compact (infinite) metric space X. Let $X^{(3)} = \{(x_1, x_2, x_3) \ : \ x_i \neq x_j \ , \ i \neq j\}$ and assume that the diagonal action of Γ is properly discontinuous and cocompact in $X^{(3)}$. Then, Γ is word hyperbolic.*

The converse statement holds if Γ is not elementary (i.e. virtually $\mathbb{Z}$) and the set X for which the statement holds is what it is called the *boundary* at infinity of Γ and denoted by $\partial\Gamma$ as above.

In what follows, the reader can assume that the group is one of the following which already provide a large class of examples:

- free groups in finitely many generators,
- fundamental groups of negatively curved closed manifolds.

For free groups, one can think of the group as lying as a (quasi-isometrically embedded) subgroup of $\mathrm{PSL}(2, \mathbb{R})$ whose boundary $\partial\Gamma$ is a Cantor set and corresponds to the uniformisation of a non-compact surface of finite type and no cusps.

In the case of a negatively curved closed manifold M, one thinks of the boundary of the group as the set of geodesic rays in $\tilde{M}$ through a point x which is identified with a $(\dim(M) - 1)$-dimensional sphere.

We refer the reader to [Led] for an excellent introduction in the setting of fundamental groups of closed manifolds of negative curvature. Classical references to hyperbolic groups are [Gr, GH, CDP, BH].

Remark 2.2. The metric in $\partial\Gamma$ is well defined up to Hölder equivalence, so it makes sense to speak about Hölder maps from and to $\partial\Gamma$ but it does not make sense to speak about more regularity unless some specific choices are made.

Remark 2.3. If Γ is a hyperbolic group, then Γ naturally acts on $\partial\Gamma$. Every non-torsion (i.e. infinte order) element of Γ acts as a north-south dynamics: with two fixed points, one which is a attracts everything except from the other fixed point and one which is an attractor for the inverse. For $\gamma \in \Gamma$, we denote as γ^- and γ^+ to the repeller and attractor respectively.

2.3.2. *Semisimple Lie groups.* The symmetric space of a semisimple Lie group is a non-positively curved space, and, when the real rank of G is larger than 1 there curvature attains 0 at some *flats*. Therefore, there are several possible boundaries one can consider and in general these do not coincide (as in the real rank 1 case), see e.g. [BH, Eb].

The *Furstenberg boundary* is $\mathscr{F} := G/_P$ where P is a minimal parabolic subgroup. A pair of elements of $\mathscr{F}$ in general position correspond to a flat in the symmetric space. Sometimes, one wants boundaries that identify different flats, for this, one considers larger parabolic subgroups (with respect to some roots of the Lie algebra).

To say this in a language familiar to dynamicists (i.e. myself), when $G = \mathrm{SL}(d,\mathbb{R})$ these boundaries correspond to several choices of (not necessarily complete) flags on $\mathbb{R}^d$. Again, we refer the reader to [BPS, Sections 7 and 8] for a (hopefully) gentle introduction to these objects. Also, one can look at [Fil] for other nice introduction making use of not so much Lie theory.

2.4. **Geodesic flow on hyperbolic groups.** Let Γ be a word hyperbolic group. Let $\widetilde{U\Gamma} := \partial^{(2)}\Gamma \times \mathbb{R}$ where $\partial^{(2)}\Gamma = \{(x, y) \in (\partial\Gamma)^2 \ : \ x \neq y\}$.

One can consider a flow $\tilde{\phi}_t : \widetilde{U\Gamma} \to \widetilde{U\Gamma}$ as $\tilde{\phi}_t(x, y, s) = (x, y, s+t)$.

A *cocycle* will be a (Hölder) continuous map $c : \Gamma \times \partial^{(2)}\Gamma \to \mathbb{R}$ such that for every $\gamma_1, \gamma_2 \in \Gamma$ and $(x, y) \in \partial^{(2)}\Gamma$:

$$c(\gamma_1\gamma_2, x, y) = c(\gamma_1, \gamma_2 x, \gamma_2 y) + c(\gamma_2, x, y).$$

One says that a cocycle c is *positive* if for every $\gamma \in \Gamma$ one has that $c(\gamma, \gamma^-, \gamma^+) > 0$ (c.f. Remark 2.3).

Given a cocycle c one can define a Γ action on $\widetilde{U\Gamma}$ as $\gamma \cdot (x, y, s) = (\gamma x, \gamma y, s - c(\gamma, x, y))$. This action clearly commutes with the flow $\tilde{\phi}_t$.

Theorem 2.4 (Gromov [Gr, Min])**.** *There exists a positive cocycle $c : \Gamma \times \partial^{(2)}\Gamma \to \mathbb{R}$ such that the action of Γ on $\widetilde{U\Gamma}$ via c is properly discontinuous and cocompact. The quotient flow ϕ_t of $\tilde{\phi}_t$ on the quotient $U\Gamma$ of $\widetilde{U\Gamma}$ is a* topologically Anosov *flow.*

Remark 2.5. The cocycle is not unique; changing the cocycle corresponds to *reparametrising* the flow (which does not affect the topological Anosov nature of it).

Remark 2.6. If $\Gamma = \pi_1(M)$ where M is a closed manifold of negative curvature then one can choose c so that ϕ_t acts on $U\Gamma \cong T^1M$ as the geodesic flow of the metric. In this case, $\widetilde{U\Gamma}$ corresponds to $T^1\tilde{M}$ on which it is well known that geodesics are determined by two different points in the visual boundary of $\tilde{M}$. The cocycle c is the *Busseman* cocycle. See [Led] for an extensive presentation of this in the context of manifolds.

Remark 2.7. For $\Gamma = \pi_1(S)$ where S is a negatively curved surface, then $\widetilde{U\Gamma}$ is a connected component of $(\partial\Gamma)^{(3)}$ in Theorem 2.1 and the diagonal action is a way to choose the cocycle. In higher dimensional negatively curved manifolds M, there is also a relationship as one can think of the space $(\partial\Gamma)^{(3)}$ as a certain frame bundle over M and the diagonal action giving rise to some frame flow but this is more complicated.

2.5. **Dominated splittings and singular values.** Consider $T : X \to X$ a homeomorphism of a compact metric space X and let $A : X \to \mathrm{GL}(d,\mathbb{R})$ a continuous function.

We define the *linear cocycle A* over T to be the pair (T, X) which induces an (invertible) dynamics on $X \times \mathbb{R}^d$ as follows:

$$(T, A) : X \times \mathbb{R}^d \to X \times \mathbb{R}^d \ ; \ (x, v) \mapsto (T(x), A(x)v).$$

We denote $A^{(n)}(x)$ to be the matrix $A(T^{n-1}(x)) \circ \cdots \circ A(x)$ if $n > 0$, the identity if $n = 0$ and the matrix $(A(T^n(x)) \circ \cdots \circ A(T^{-1}(x)))^{-1}$ if $n < 0$.

We remark that similar definitions can be made for flows (see e.g. [BPS, Section 2]). A more complete introduction to dynamics of linear cocycles can be found in [Via]).

We say that (T, A) admits a *dominated splitting of index p* if there exists a pair of maps (a priori not continuous) $E^{cs} : X \to \mathcal{G}_{d-p}(\mathbb{R}^d)$ and $E^{cu} : X \to \mathcal{G}_p(\mathbb{R}^d)$ and $\ell > 0$ such that:

(**equivariance:**) $A(x)E^{cs}(x) = E^{cs}(T(x))$ and $A(x)E^{cu}(x) = E^{cu}(T(x))$,
(**transversality:**) $E^{cs}(x) \oplus E^{cu}(x)$ for every $x \in X$,
(**domination:**) for every unit vectors $v \in E^{cs}(x)$ and $w \in E^{cu}(x)$ one has that:

$$\|A^{(\ell)}(x)v\| \leq \frac{1}{2}\|A^{(\ell)}(x)w\|.$$

Continuity of the bundles follows from these conditions. Other standard characterisations include cone-field criteria (which is important as it shows its robust nature and allows to detect domination with only 'finitely many iterates'), see e.g. [BG] or [BDV, Appendix B].

There is a nice characterisation of dominated splitting in terms of singular values. This was first developed by Yoccoz [Yoc] for 2×2 matrices and then extended by Bochi and Gourmelon in all generality [BG]. This will be a key tool in our proof and it is the only place where 'non-elementary' methods will be used (the proof of Bochi-Gourmelon requires a form of 'Oseledets theorem' for linear cocycles).

We first give some definitions: for $A \in \mathrm{GL}(d,\mathbb{R})$ we define $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_d(A) > 0$ to be the *singular values* of the matrix A (i.e. the square-root of the eigenvalues of the symmetric positive matrix AA^*). It holds that $\sigma_1(A) = \|A\|$ and $\sigma_d(A) = \|A^{-1}\|^{-1}$.

Theorem 2.8 (Bochi-Gourmelon)**.** *A linear cocycle (T, A) admits a dominated splitting of index p if and only if there are constants $C, \lambda > 0$ such that for every $x \in X$ and $n > 0$:*

$$\frac{\sigma_p(A^{(n)}(x))}{\sigma_{p+1}(A^{(n)}(x))} \geq Ce^{\lambda n}.$$

In this setting, one can define the *universal p-dominated* set with constants C, K, λ, more precisely. We say that a sequence $\{A_n\}_{n\in\mathbb{Z}}$ of matrices is a *p-dominated sequence* if there are constants C, K, λ so that it belongs to:

$$\mathcal{D}(C,K,\lambda,p,\mathbb{Z}) := \left\{ \{A_n\}_{n\in\mathbb{Z}} \ : \ \|A_i^{\pm 1}\| \leq K \ ; \ \frac{\sigma_p(A_j \cdots A_i)}{\sigma_{p+1}(A_j \cdots A_i)} \geq Ce^{\lambda(j-i)} \ \forall i < j \right\}.$$

Notice that $\mathcal{D}(C,K,\lambda,p,\mathbb{Z})$ is a compact shift invariant subset of the set of sequences of matrices in $\mathrm{GL}(d,\mathbb{R})^{\mathbb{Z}}$ and therefore we can consider T to be the shift restricted to this subspace and if one considers the cocycle $\{A_n\} \mapsto A_0$ then Theorem 2.8 applies to show that there is a dominated splitting. In particular, by continuity of the bundles and compactness, there is a lower bound on the angle between the 'invariant subspaces'. One can make a further compactness argument to speak about p-dominated sequences which are finite instead of infinite and obtain bounds on the angles of expanded and contracted subspaces (we refer the reader to [BPS, Lemma 2.5] for a precise statement).

Let us remark that in [GGKW, Section 5] the concept of *CLI*-sequences is introduced which is very related to that of p-dominated sequences. Essentially the same results are obtained but by elementary methods (the proof of [BG] uses Oseledets theorem). Other proof of a similar result was also announced in [Mo].

3. ANOSOV REPRESENTATIONS

3.1. **Classical definition.** Anosov representations were introduced in [Lab] for fundamental groups of closed manifolds with negative curvature. This was generalised by [GW] to work in any hyperbolic groups. Today, there exist several equivalent definitions, some of which do not require (a priori) that the group is hyperbolic (see for instance: [BCLS, GGKW, KLP, BPS, DGK, DGK$_2$]).

Let Γ be a word hyperbolic group with boundary $\partial\Gamma$.

Let $\rho : \Gamma \to G$ be a representation. We start by assuming that $G = \mathrm{GL}(d,\mathbb{R})$ (or $\mathrm{SL}(d,\mathbb{R})$ to be able to give a definition independent of Lie theory. We denote as $\mathscr{G}_p(\mathbb{R}^d)$ to the Grassmanian of p-planes in $\mathbb{R}^d$ endowed with its natural topology.

The representation is said to be *p-Anosov* if the following conditions are verified:

- There exist continuous maps $\xi_p : \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ and $\xi_{d-p} : \partial\Gamma \to \mathscr{G}_{d-p}(\mathbb{R}^d)$ which are:
  - *equivariant*, i.e. $\rho(\gamma)\xi_p(x) = \xi_p(\gamma x)$ and $\rho(\gamma)\xi_{d-p}(x) = \xi_{d-p}(\gamma x)$ for every $\gamma \in \Gamma$ and $x \in \partial\Gamma$,
  - *transverse*, i.e. if $x \neq y \in \partial\Gamma$ then one has that $\xi_p(x) \oplus \xi_{d-p}(y) = \mathbb{R}^d$.
- Moreover, the maps verify a uniform contraction-expansion property that we will explain below.

Let us remark that it is shown in [GW] that the contraction-expansion property is immediate if the image of ρ is *irreducible* (i.e. there is no proper subspace of $\mathbb{R}^d$ invariant under all $\rho(\gamma)$ with $\gamma \in \Gamma$). Also, in [GGKW] it is shown that one can express the contraction and expansion of the bundles in terms of the expansion and contraction of $\rho(\gamma)$ along $\xi_p(\gamma^+)$ and $\xi_{d-p}(\gamma^-)$ where γ^+ and $\gamma^- \in \partial\Gamma$ are the attractor and repeller of γ in $\partial\Gamma$.

The expansion and contraction property is expressed in terms of *linear cocycles* over the geodesic flow of Γ. Define $\tilde{V}$ to be the trivial vector bundle over $\widetilde{U\Gamma}$ with fiber $\mathbb{R}^d$, i.e. $\tilde{V} = \widetilde{U\Gamma} \times \mathbb{R}^d$. Consider the linear flow over the geodesic flow in $\tilde{V}$ defined by $\tilde{\psi}_t(x,v) = (\tilde{\phi}_t, v)$. Given a representation ρ of Γ one can define the bundle V_ρ over $U\Gamma$ by making the quotient of $\tilde{V}$ by the action of Γ given by:

$$\gamma((x,y,s),v) \mapsto (\gamma(x,y,s),\rho(\gamma)v).$$

(The action in the first coordinate $(x,y,s) \in \partial^{(2)}\Gamma \times \mathbb{R} \cong \widetilde{U\Gamma}$ is one given by Theorem 2.4.)

When ρ admits equivariant boundary maps ξ_p, ξ_{d-p} as above, these give rise to invariant bundles $\tilde{E}(x,y,s) = \xi_{d-p}(y)$ and $\tilde{F}(x,y,s) = \xi_p(x)$. It follows that these bundles project to bundles E, F which are invariant by the flow ψ_t due to the equivariance. We endow V_ρ with a riemannian metric (varying continuously on the base point).

We will say that they verify a contraction/expansion property whenever there are constants $C, \lambda > 0$ such that for every $z \in U\Gamma$, $v \in E(z)$ and $w \in F(z)$ unit vectors and $t > 0$ one has:

$$\frac{\|\psi_t v\|}{\|\psi_t w\|} \leq C e^{-\lambda t}.$$

In the language of linear cocycles, one can say equivalently that the bundle F dominates the bundle E. Indeed, it is shown in [BPS, Proposition 4.6] that if the flow ϕ_t admits a dominated splitting of index p, then when lifted to $\tilde{V}$ the bundles only depend on one of the points at infinity and therefore give rise to equivariant maps (and in particular, the representation must be p-Anosov).

Exercise 2. Show that if $p \leq d/2$ then $\xi_p(x) \subset \xi_{d-p}(x)$ for all $x \in \partial\Gamma$.

When the group G is not $\mathrm{GL}(d,\mathbb{R})$ but is semisimple, then one can define the notion of P-Anosov where P is a parabolic subgroup of G. For this, one needs the parabolic subgroup to be invariant under the Cartan involution and one defines a map $\xi : \partial\Gamma \to G/_P$. (In the case of $G = \mathrm{SL}(d,\mathbb{R})$ for each p there is such a parabolic subgroup and the quotient $G/_P$ is exactly the set of pairs of p-planes and $d-p$-planes in general position. See [BPS, Section 7] for a 'translation' from one setting to the other.)

We also remark that if one has a representation to $\mathrm{GL}(p,\mathbb{R})$ then one can use exterior powers to define a representation into another $\mathrm{GL}(k,\mathbb{R})$ (in fact, k is the dimension of $\bigwedge^p \mathbb{R}^d$) so that the representation becomes 1-Anosov (sometimes also called *proyectively Anosov*). Using some representation theory, it is shown in [GW, Section 4] that this procedure is completely general and every P-Anosov representation into a semisimple Lie group P can be 'transformed' into a 1-Anosov representation into $\mathrm{SL}(d,\mathbb{R})$. See [BPS, Section 8] for more information on these equivalences.

3.2. **Some equivalences.** The above definition is very useful, and enjoys several very nice properties explored in the papers mentioned above. One clear drawback of the definition is that finding equivariant maps is not so trivial in general. As in the theory of dominated splittings, it is however natural to expect that if some 'cone contraction' condition is verified, then the maps will exist (this allows, e.g. to detect domination by just knowing finitely many images of ρ).....

One of the main purposes of this notes is to give an indication of the proof of the following result from [KLP] which was obtained by other means[5] in [BPS]. See also [GGKW] for similar results. This allow to give characterisation of Anosov representations without need to find the equivariant maps and without knowledge a priori on the geometry of the group Γ. We will work with $G = \mathrm{GL}(d,\mathbb{R})$ but of course all extends to semisimple Lie groups using the above comments.

Given a group Γ with a finite generating set F we say that a representation $\rho : \Gamma \to \mathrm{GL}(d,\mathbb{R})$ is *p-dominated* if there exists $C, \lambda > 0$ such that for every $\gamma \in \Gamma$ one has that:

$$\frac{\sigma_p(\rho(\gamma))}{\sigma_{p+1}(\rho(\gamma))} > Ce^{\lambda|\gamma|_F}. \tag{3.2.1}$$

We remark that other related notions of domination appear in the literature (see e.g. [DT, GKW] or [GGKW, Section 7]) and can be related to this one too.

So we can now state the announced result:

Theorem 3.1. *If* Γ *admits a* p*-dominated representation into* $\mathrm{GL}(d,\mathbb{R})$ *then* Γ *is word hyperbolic and* ρ *is* p*-Anosov.*

We will use sometimes the following fact: the (left invariant) Riemannian metric on the symmetric space $\mathrm{PSL}(d,\mathbb{R})/_{\mathrm{PSO}(d,\mathbb{R})}$ of $\mathrm{PSL}(d,\mathbb{R})$ is given by $d(\mathrm{id}, g) = \sqrt{\sum_{i=1}^d (\log(\sigma_i(g)))^2}$. This implies that

$$d(g,h) \sim \log \|g^{-1}h\|.$$

(Notice that if $k_1, k_2 \in \mathrm{PSO}(d,\mathbb{R})$ then $d(gk_1, hk_1) = d(g,h)$.)

In $\mathrm{GL}(d,\mathbb{R})$ one needs to normalize through the determinant to get that

[5] In particular, by using linear cocycles and an interpretation of domination due to [Yoc, BG] explained in section 2.5.

$$d(\mathrm{id},g) = \sqrt{(\log(\hat{\sigma}_1(g)))^2 + \ldots + (\log(\hat{\sigma}_d(g))^2) + (\log(\det(g)))^2}$$

where $\hat{\sigma}_i(g) = \sigma_i(g)(\det(g))^{-1}$, as $\det(g) = \sigma_1(g)\cdots\sigma_d(g)$ may be different from one as is the case in $\mathrm{PSL}(d,\mathbb{R})$. Similar formulas exist for general Lie groups (see e.g. [BPS, Section 8]). As we've mentioned, one can always reduce to consider $\mathrm{PSL}(d,\mathbb{R})$. In the case of $\mathrm{GL}(d,\mathbb{R})$ this consists on dividing by the determinant which does not affect the definitions (see [BPS, Section 3]).

Once one shows that if a group Γ admits a p-dominated representation then Γ is word hyperbolic ([BPS, Section 3]), then the equivalence between definitions is not hard but still requires some amount of work (see [BPS, Section 4] or [GGKW, Section 5]).

4. Properties

We prove some direct properties which follow from the definition of p-dominated or p-Anosov representations.

4.1. **Quasi-isometry.** First we show the following direct property:

Proposition 4.1. *If $\rho : \Gamma \to \mathrm{GL}(d,\mathbb{R})$ is p-dominated, then, it is quasi-isometric.*

Proof. We assume the image of ρ is in $\mathrm{PSL}(d,\mathbb{R})$ for simplicity. Notice that in this case we have that $\sigma_1(\rho(\gamma))\cdots\sigma_d(\rho(\gamma)) = 1$ for all $\gamma \in \Gamma$. Recall that $d(g,h) \sim \log\|g^{-1}h\| = \log\sigma_1(g^{-1}h)$.

Choose $\gamma,\beta \in \Gamma$. We want to estimate $d(\rho(\gamma),\rho(\beta)) \sim \log\|\rho(\gamma^{-1})\rho(\beta)\|$ in terms of $d(\gamma,\beta) = |\gamma^{-1}\beta|$. So, it is enough to estimate $d(\mathrm{id},\rho(\gamma))$ in terms of $|\gamma|$. Notice that if F is a symmetric generating set and $K > 0$ is the maximum norm of $\|\rho(f)\|$ with $f \in F$ then it follows immediately that:

$$d(\mathrm{id},\rho(\gamma)) \sim \log\|\rho(\gamma)\| \leq |\gamma|\log K.$$

To estimate from below, choose $\gamma \in \Gamma$ and fix the constants $c,\lambda > 0$ from p-domination. Let $q_\gamma \in \{1,\ldots,d-1\}$ be the largest q for which $\sigma_q(\rho(\gamma)) \geq 1$

The fact that ρ is p-dominated implies that: $\sigma_p(\rho(\gamma)) > c\lambda^{|\gamma|}\sigma_{p+1}(\rho(\gamma))$

- if $q_\gamma \geq p+1$ it follows that $\sigma_{p+1}(\gamma)) \geq 1$ and therefore $\sigma_1(\rho(\gamma)) \geq \sigma_p(\rho(\gamma)) \geq ce^{\lambda|\gamma|}$ showing that $\log\|\rho(\gamma)\| \geq \lambda|\gamma| + \log c$.
- the same holds if $q_\gamma \leq p-1$ by using the fact that $\sigma_{d-i}(\rho(\gamma)^{-1}) = \sigma_i(\rho(\gamma))$ and that $|\gamma| = |\gamma^{-1}|$.
- if $q_\gamma = p$ we get (using either γ or γ^{-1} that $\log\|\rho(\gamma)\| \geq \frac{\lambda}{2}|\gamma| + \log c$.

This completes the proof. □

4.2. **Robustness.** To prove robustness of p-dominated representations, we shall use Theorem 3.1 (which does not need robustness in the proof) by assuming at some point that Γ is hyperbolic.

Let Γ be a finitely generated group with symmetric generator F and $\rho : \Gamma \to \mathrm{GL}(d,\mathbb{R})$ a p-dominated representation. Let:

$$\mathscr{G}_F := \{\{f_i\}_{i\in\mathbb{Z}} \subset F^{\mathbb{Z}} \ : \ |f_i \cdots f_{i+j}| = j+1 \text{ for } i \in \mathbb{Z},\ j \in \mathbb{Z}_{>0}\}.$$

The set $\mathscr{G}_F$ parametrizes all infinite geodesics passing through id in the group Γ: if $\{\gamma_n\}_{n\in\mathbb{Z}}$ is such a geodesic (with $\gamma_0 = \mathrm{id}$), then the sequence $\{g_n\}_{n\in\mathbb{Z}} \in \mathscr{G}_F$ with $g_n =$

$\gamma_n^{-1}\gamma_{n+1}$. Indeed, since $d(\gamma_n,\gamma_{n+1}) = 1$ it follows that $\gamma_n^{-1}\gamma_{n+1} \in F$ and one has that $g_n \cdots g_{n+m} = \gamma_n^{-1}\gamma_{n+m+1} = d(\gamma_n,\gamma_{n+m+1})$ therefore $|g_n \cdots g_{n+m}| = m+1$. The fact we force $\gamma_0 = \mathrm{id}$ is to have a bijective correspondence.

Denote $\sigma : F^{\mathbb{Z}} \to F^{\mathbb{Z}}$ the shift map $\{x_n\}_n \mapsto \{x_{n+1}\}_n$. Notice also that:

Claim 4.2. *The set $\mathcal{G}_F$ is closed and shift invariant in $F^{\mathbb{Z}}$.*

Proof. The fact that it is shift invariant is immediate as the definition does not involve the position of the elements (just its relative position). To show it is closed, choose a sequence $\{f_n\}_{n\in\mathbb{Z}}$ not in $\mathcal{G}_F$ and numbers i, j so that $|f_i \cdots f_{i+j}| < j+1$. There is a neighborhood of the sequence which coincides with $\{f_n\}_n$ for $n \in \{i,\dots,i+j\}$ therefore, the complement of $\mathcal{G}_F$ is open. This completes the proof. □

Define the following linear cocycle over σ. $A : F^{\mathbb{Z}} \to \mathrm{GL}(d,\mathbb{R})$ is such that $\{f_n\}_n \mapsto \rho(f_0)^{-1}$. It follows that $A^{(k)}(\{f_n\}_n) = \rho(f_{k-1})^{-1}\cdots\rho(f_0)^{-1} = \rho(f_0\cdots f_{k-1})^{-1}$ (if $k > 0$, for negative k it is similar).

The domination condition translated to this setting[6] is exactly the condition of [BG] and therefore we obtain:

Theorem 4.3 (Bochi-Gourmelon, see section 2.5)**.** *The cocycle A over $\sigma|_{\mathcal{G}_F}$ admits a dominated splitting of index p.*

Dominated splittings are robust, therefore, we know that for $\rho' \sim \rho$ the induced cocycle will also admits a dominated splitting of index p. It is in this moment that we need to assume that Γ is hyperbolic to deduce from this that ρ' is p-dominated. For this we will use the following fact (which is a very weak version of the automatic property of hyperbolic groups, see [CDP] or [BPS, Section 5]):

Fact 4.4. *There exists $k > 0$ so that for every $\gamma \in \Gamma$ there is an infinite geodesic through* id *that lies at distance $\leq k$ of γ.*

With this fact in mind, it is easy to extend domination along elements in $\mathcal{G}_F$ (i.e. that belong to infinite geodesics through the origint) to any element in Γ just by loosing some uniform constants. Notice that there are many groups beyond hyperbolic groups that enjoy this property (e.g. $\mathbb{Z}^d$).

In [BPS, Section 5] stronger properties of the subset $\mathcal{G}_F$ are discussed (having to do with Cannon's cone types and the automatic structure of hyperbolic groups, see e.g. [CDP]). In particular, the restriction $\sigma|_{\mathcal{G}_F}$ is a *sofic shift*.

5. EXAMPLES

The following list is not exhaustive and is meant to provide references for examples rather than developing them. It turns out that each of the examples can be viewed as a topic of research by itself and one of the beautiful features of Anosov representations is that it unifies several aspects of these rather different settings.

5.1. **Schottky subgroups.** This is kind of the easiest family of examples. Schottky subgroups consist on representations of the free group into G. As before, we restrict to $\mathrm{SL}(d,\mathbb{R})$ for simplicity.

Whenever there are two matrices $A, B \in \mathrm{SL}(d,\mathbb{R})$ which verify that A, B, A^{-1} and B^{-1} are *proximal* in the sense that they have a largest eigenvalue with multiplicity one and

[6]Notice that since we took inverses, the p-domination becomes $d-p$-domination. But it is clear that if a representation is p-dominated then it is also $d-p$-dominated.

whose modulus is larger than the second eigenvalue, it follows from a simple application of cone-fields that if the eigenspaces of A and B are in general position, then there is $n > 0$ so that the group generated by A^n and B^n is 1-Anosov. This is easy to show using cone-fields (and a kind of ping-pong argument).

See [Be, Be_2] as well as [KLP, CLS] for more examples and proofs.

5.2. **Fuchsian representations of surface groups.** Let Σ be a higher genus closed orientable surface and $\pi_1(\Sigma)$ be its fundamental group. Recall that a Fuchsian representation of $\pi_1(\Sigma)$ is a faithful and discrete representation which then induce a hyperbolic metric on Σ. It has been shown by Goldman that (given an orientation on Σ) these representions are connected. Moreover, they constitute a whole connected component of $\text{Hom}(\pi_1(\Sigma), \text{PSL}(2,\mathbb{R}))$. In this way, it is natural to identify the Teichmuller space of Σ with this component of $\text{Hom}(\pi_1(\Sigma), \text{PSL}(2,\mathbb{R}))/_{\text{PSL}(2,\mathbb{R})}$.

The group $\text{PSL}(2,\mathbb{R})$ acts naturally on the circle $\mathbb{P}(\mathbb{R}^2)$ by the linear action on subspaces. The stabiliser of a line is exactly the (unique strict) parabolic subgroup of $\text{PSL}(2,\mathbb{R})$ up to conjugacy .

Given a point $x \in \partial\pi_1(\Sigma)$ one can choose a geodesic ray $\{\gamma_n\}_{n\geq 0}$ in $\pi_1(\Sigma)$ converging to x and associate $\xi(x)$ in $\mathbb{P}(\mathbb{R}^2)$ via the limit of $U(\rho(\gamma_n))$ the most expanded direction of the matrix $\rho(\gamma_n)$. It is an exercise (using e.g. the Morse Lemma in $\mathbb{H}^2$) to show that this direction is well defined, continuous and independent of the chosen geodesic. In particular, this gives a continuous mapping from $\partial\pi_1(\Sigma)$ to $\mathbb{P}(\mathbb{R}^2)$ which for periodic elements (i.e. geodesics of the form $\gamma_n = \gamma^n$) corresponds to the expanding eigendirection of γ.

Notice that if ρ a Fuchsian representation then $\rho(\gamma)$ is diagonalizable for all $\gamma \in \partial\pi_1(\Sigma)$, therefore, this gives that the map ξ verifies that $\xi(x) \oplus \xi(y) = \mathbb{R}^2$ at least when x and y correspond to the attractor and repeller of elements γ. One can push this to show that it is indeed an Anosov representation (and it is a quite nice way to see that being faithful and discrete is an open property[7]!)

5.3. **Quasi-Fuchsian representations of surface groups.** When $\rho : \pi_1(\Sigma) \to \text{PSL}(2,\mathbb{R})$ is faithful and discrete, it follows that the action is *cocompact*[8] and therefore *quasi-isometric* via the Svarc-Milnor Lemma. This allows to show that being faithful and discrete is an open propery (as it implies that the representation is Anosov!). Closedness of faithful and discrete representations is typically simpler by Margulis Lemma (or in the case of $\text{PSL}(2,\mathbb{R})$ or $\text{PSL}(2,\mathbb{C})$ by Jorgensen's inequality).

However, when a representation is discrete and faithful into a rank one Lie group but not quasi-isometric, there is no reason for the discretness and faithfullness to resist perturbations. This is key in the proof of geometrisation of 3-manifolds that fiber over the circle on which the following notion appears:

A representation $\rho : \pi_1(\Sigma) \to \text{PSL}(2,\mathbb{C})$ is *quasi-Fuchsian* if it is quasi-isometric. Being quasi-isometric, it is easy to construct a continuous limit map from $\partial\pi_1(\Sigma)$ into the boundary at infinity of the symmetric space $\mathbb{H}^3$ of $\text{PSL}(2,\mathbb{C}) \cong \text{Isom}_+(\mathbb{H}^3)$ (notice that in rank one, the symmetric space is hyperbolic and therefore 'all' notions of boundary coincide). This gives the Anosov property and opennes of such representations follow.

It is relevant to remark though, that Anosov representations from $\pi_1(\Sigma)$ to $\text{PSL}(2,\mathbb{C})$ do not represent the whole connected component, indeed, one can go to the 'limit' to

[7]To show closedness, one uses Jorgensen's inequality

[8]One way to see this is that the symmetric space is homeomorphic to $\mathbb{R}^2$ and then, by a cohomological dimension argument it follows that the quotient must be compact.

find representations for which the 'limit curve' is a Peano curve (also called Cannon-Thurston's curve) and corresponds to the inclusion of the fundamental group of a fiber in the case of a hyperbolic 3-manifold which fibers over the circle.

5.4. **Convex cocompact subgroups in rank 1.** Rank one lie groups correspond to the isometry groups of symmetric spaces of negative curvature. These include $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$, $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$, $\mathrm{SO}(1,n) \cong \mathrm{Isom}^+(\mathbb{H}^n)$, $\mathrm{PU}(1,n) \cong \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{C}})$, etc.... (see [He, Qui]).

A finitely generated subgroup Γ of a rank one Lie group G is said to be *convex-cocompact* if it is quasi-isometrically embedded. This implies directly that it is Anosov with respect to the unique possible parabolic. The name has to do with another equivalent characterisation: there is a convex set in the symmetric space $X = G/K$ of G which is preserved by Γ with compact quotient (for this characterisation it is not needed to assume a priori that Γ is finitely generated as it is a consequence of the definition). See [GW, Bou] for a proof of these equivalences. These generalise some Schottky subgroups, Fuchsian and quasi-Fuchsian representations of surface groups, etc.

A particularly important class of examples is given by (finitely generated) Fuchsian and Kleinian groups without parabolic elements. (See [Su].)

5.5. **Hitchin representations.** Hitchin components ([Hi]) of representations of surface groups into $\mathrm{PSL}(d,\mathbb{R})$ generalize to higher rank the well known *Fuchsian* representations (see [Lab]).

5.6. **Benoist representations.** We refer the reader to [Be_4] for a survey. We will just indicate one specific example which is in some sense, the analogue of quasi-fuchsian representations from fundamental groups of surfaces in $\mathrm{PSL}(2,\mathbb{C})$ but instead of growing the dimension and keeping the rank, we increase the rank (this example is also an example of a Hitchin representation, and also the easiest one).

In fact, one can see $\mathrm{PSL}(2,\mathbb{R})$ as $\mathrm{SO}(1,2)$. If one has a fuchsian representation $\rho : \pi_1(\Sigma) \to \mathrm{SO}(1,2)$ then one can think of quasi-fuchsian representations as deformations of $\hat{\rho} : \pi_1(\Sigma) \to \mathrm{SO}(1,3) \cong \mathrm{PSL}(2,\mathbb{C})$ via the natural inclusion of $\mathrm{SO}(1,2)$ into $\mathrm{SO}(1,3)$. This increases the dimension while keeping the rank equal to one.

Another way to extend this is to use the embedding $\mathrm{SO}(1,2) \hookrightarrow \mathrm{SL}(3,\mathbb{R})$. This way, deformations of a *fuchsian* representation give rise to *convex projective structures* on Σ. Notice that $\mathrm{SL}(3,\mathbb{R})$ acts on the projective space $\mathbb{P}(\mathbb{R}^3)$ which is a surface.

It is not hard to check that the composition of a fuchsian representation with the natural embedding $\mathrm{SO}(1,2) \hookrightarrow \mathrm{SL}(3,\mathbb{R})$ is 1-Anosov and the boundary map defines an injective circle in $\mathbb{P}(\mathbb{R}^3)$. It is not hard to show that the curve after perturbation will still bound a strictly convex set included in an affine chart on which the surface group will act properly discontinously and therefore induce a 'convex projective structure' on the surface. It can be shown that the whole connected component of such representations represent convex projective structures but this is more involved (see [Be_4, Lab] and references[9] therein).

5.7. **Maximal representations.** See [BIW, BILW]. Also we recommend the survey [BIW].

[9] In fact, opennes of such representations follows for a much more general theorem of [Ko] which provides opennes of convex projective structures (notice that the fact that the structure remains convex after perturbation does not follow from Ehresmann-Thurston's principle, in the case of 'strictly convex' structures such as it is the case here, there is a simpler argument for opennes due to Benoist [**?**]). For closedness of convex projective structures on surfaces, see [**?**]. This follows in more generality too, see [**?**].

5.8. **Quasi-fuchsian representations in higher rank.** See e.g. [Ba_2, BaM]

5.9. **Other examples.** Recently, several new examples have appeared, notably those in [DGK, DGK_2]. This will maybe the subject of one of Fanny's talks.

Let us also point out other examples [Ba] where there is a very nice construction of a geometric structure in terms of the representation.

6. GEOMETRIC MEANING

Anosov representations are now known to have induce some proper actions in certain homogeneous spaces giving rise to some geometric structures on the groups. See [GW, GGKW]. Fanny will discuss some of this in one of her lectures.

7. WORD HYPERBOLICITY OF GROUPS ADMITTING ANOSOV REPRESENTATIONS

In this section we indicate the main steps of the proof of Theorem 3.1 and refer the reader to [BPS, Section 3] for a complete (elementary) proof. It is worth indicating that some ideas here also appear in [GGKW, Section 5].

7.1. **Strategy of the proof.** We will use Theorem 2.1 as working definition of (non-elementary) word hyperbolic group. We will assume for simplicity (and without loss of generality) that $p = 1$.

We recall that an action of Γ in a topological space X is:

- *properly discontinuous* if given any compact subset $K \subset X$ there exists n such that if $|\gamma| > n$ then $\gamma K \cap K = \emptyset$;
- *cocompact* if there exists a compact subset $K \subset X$ such that $\Gamma x \cap K \neq \emptyset$ for every $x \in X$.

The strategy is then to use condition (3.2.1) in order to construct a set X in $\mathbb{P}(\mathbb{R}^d)$ containing all the 'expanding' directions of large elements in Γ and show that the action of Γ in this set satisfies the desired properties. Using some linear algebra, one can reduce the proof to the following:

- Show that the set X is perfect, and for this it will be enough to show that $X^{(3)}$ is non-empty and use correctly condition (3.2.1).
- Show that the action on $X^{(3)}$ is properly discontinuous, this is probably the easiest part: we will just use that if one chooses a very large element satisfying (3.2.1) and considers 3 sufficiently separated points in X then at least two of them will be mapped very close-by.
- Show that the action is cocompact, this is a bit subtler, but the idea is simple, if one finds an element whose 'repeller' is close to some very small angle between a triplet of points in X then this element will make all angles between elements sufficiently large, obtaining a compact fundamental domain for the action.

In what follows we give some more details on these steps and prove the key technical statement that uses domination (and the main result of [BG]) crucially.

7.2. **Some preliminaries in linear algebra and the key lemma.** For a matrix A we will say that it has a *gap* if $\sigma_1(A)/\sigma_2(A) > 1$ and a *large gap* if $\sigma_1(A)/\sigma_2(A)$ is much larger than 1.

When a matrix A has a gap, one can define $U(A) \in \mathbb{P}(\mathbb{R}^d)$ to be $A(\mathbb{R}v)$ for the unique direction v such that $\frac{\|Av\|}{\|v\|} = \|A\| = \sigma_1(A)$. Similarly, we define $S(A)$ to be the $d-1$-dimensional subspace orthogonal to v.

The following observations will be crucial (for precise computations and complete proofs see e.g. [BPS, Appendix A]):

Lemma 7.1. *Let B be a given matrix, and A a matrix with a large gap (with respect to B) then:*

(1) $U(AB) \sim U(A)$,
(2) $U(BA) \sim BU(A)$,
(3) *if P makes a good angle with $S(A)$ then $A(P)$ is very close to $U(A)$ and the expansion along P is comparable to $\|A\|$.*

Proof. For the first two items, notice that it is trivial if B is an isometry. The general case looks exactly the same as the norm of B and B^{-1} is chosen much smaller than the gap of A.

The last item is kind of the definition of $U(A)$ and $S(A)$, one can compute exactly the expansion along P with respect to the angle that P makes with $S(A)$ and the gap of A. □

Using this estimate and the main result of [BG] we can prove the following key (technical) statement:

Lemma 7.2. *Let $\rho : \Gamma \to \mathrm{SL}(d,\mathbb{R})$ be a 1-dominated representation and $\gamma, \beta \in \Gamma$ sufficiently large elements, then, if the angle between $U(\rho(\gamma))$ and $U(\rho(\beta^{-1}))$ is big, then so is the angle between $U(\rho(\gamma))$ and $S(\rho(\beta))$.*

Notice that this is non-trivial as even if $U(\rho(\beta^{-1})) \subset S(\rho(\beta))$ the subspace $S(\rho(\beta))$ has dimension $d-1$ instead of 1. The lemma tells us that if the attractor of an element and the attractor of the inverse of another element are far away, then so is the repeller of the element. This knowledge is crucial for the proof of hyperbolicity of the group and its proof needs to use domination.

Sketch of the proof. Consider γ and η^{-1} and write $\gamma = g_1 \dots g_n$ where g_i are elements of the fixed symmetric generator F and $\beta^{-1} = h_1 \dots h_m$ with $h_i \in F$. Notice that we can choose them in order to have $|g_i \dots g_{i+j}| = j$ and $|h_i \dots h_{i+j}| = j$ for every i, j which make sense. Therefore, the sequences $\{\gamma_i = g_1 \dots g_i\}_{i=0}^n$ and $\{\hat\beta_i = h_1 \dots h_i\}_{i=0}^m$ are geodesics (WARNING: $\hat\beta_i^{-1}$ and γ_i^{-1} have no reason to be geodesics).

The first important remark is that if $U(\rho(\gamma))$ and $U(\rho(\beta^{-1}))$ form a good angle, then one can kind of 'concatenate' the geodesics into (uniform) quasi-geodesics. More precisely (still not so precise), we can show that [10]:

$$d(\gamma_i, \hat\beta_j) = |h_j^{-1} h_{j-1}^{-1} \dots h_1^{-1} g_1 \dots g_i| \sim i + j.$$

This is implied essentially by the domination property and Lemma 7.1 item (1) applied (if say $i > j$) to $A = \rho(\gamma_i)$ and $B = \rho(\gamma_i^{-1}\hat\beta_j)$: Indeed, if the quantities were not *quasi*-the same, then the matrix B would be 'small' compared to A and therefore the angle between $U(A) = U(\rho(\gamma_i))$ and $U(AB) = U(\rho(\hat\beta_j))$ would be small contradicting our hypothesis (notice that for i, j large enough one has that $U(\rho(\gamma_i)) \sim U(\rho(\gamma))$ and $U(\rho(\hat\beta_j)) \sim U(\rho(\beta^{-1})$.

[10] Here we will use $\sim$ to denote that the quantities are *quasi* the same in the sense of quasi-geodesics or quasi-isometries.

Using this fact, the Lemma follows by noticing that the sequence

$$\{\rho(g_n),\ldots,\rho(g_1),\rho(h_1^{-1})\ldots,\rho(h_m^{-1})\}$$

is a *dominated sequence* (in the sense of subsection 2.5), this guarantees the good angle between $S(\rho(\beta))$ and $U(\rho(\gamma))$.

□

7.3. **Limit set.** We define the following set:

$$X=\bigcap_{n>0}\overline{\{U(\rho(\gamma))\,:\,|\gamma|>n\}}\subset\mathbb{P}(\mathbb{R}^d).$$

This set has been previously considered in [Be$_2$] for the Zariski dense case and in [GGKW] in the general case.

It is easy from Lemma 7.1 item (2) to show that it is equivariant and it is clearly compact and non-empty.

7.4. **The limit set is perfect.** The main point here is to show that the limit set has at least 3 points. Once one has this, then applying Lemma 7.2 and Lemma 7.1 item (3) one can easily show that every point in the limit set is an accumulation point (see [BPS, Lemma 3.12]).

Let us sketch the main ingredients to show that there are more than 2 points in X:

- Assuming there are finitely many points $X=\{P_1,\ldots,P_k\}$ one can (up to passing to a finite index subgroup which does not affect domination) define a morphism $\varphi:\Gamma\to\mathbb{R}^k$ defined as $\varphi(\gamma)=(\|\rho(\gamma)v_1\|,\ldots,\|\rho(\gamma)v_k\|)$ where v_i are unit vectors in P_i.
- The image of this morphism is abelian and it is possible to show using Lemma 7.2 and Lemma 7.1 item (3) that its kernel is finite.
- This will force the image to be $\mathbb{Z}$ as a simple argument shows that $\mathbb{Z}^\ell$ cannot admit dominated representations (very briefly: one can connect γ and γ^{-1} through paths which are away from identity but whose length is comparable to $|\gamma|$, this allows to exchange the contracting and repelling direction 'continuously' showing that they must coincide and this contradicts Lemma 7.2 for well chosen elements).

See [BPS, Section 3.4] for more details.

7.5. **Proper discontinuity.** Given a triple $T=(x_1,x_2,x_3)\in X^{(3)}$ we denote

$$|T|=\min_{i\neq j}d(x_i,x_j)>0.$$

Notice that for any $\delta>0$ the set of triples so that $|T|\geq\delta$ is a compact subset of $X^{(3)}$ and any compact set is contained in a set of this form.

To show proper discontinuity it is enough to show that:

Proposition 7.3. *For every $\delta>0$ there exists $\ell>0$ such that if $|\gamma|>\ell$ one has that for every T with $|T|\geq\delta$ one has that $|\rho(\gamma)T|<\delta$.*

The proof follows again by combining Lemma 7.2 and Lemma 7.1 item (3). Indeed, for a large enough element γ using Lemma 7.2 one has that $S(\rho(\gamma))$ can be close to at most one of the elements of T. Therefore, by Lemma 7.1 item (3) the image of $\rho(\gamma)$ of the other two will be very close. See [BPS, Section 3.5] for more details.

7.6. **Cocompactness.** We want to show that there exists $\delta > 0$ so that for every triple $T = (x_1, x_2, x_3)$ there exists $\gamma \in \Gamma$ such that $|\gamma T| > \delta$. This combines similar ideas as before, one assumes that some points, say x_1, x_2 in T are closer than δ and chooses elements γ_n so that $U(\gamma_n) \sim x_1, x_2$. One expects that iterating by $\rho(\gamma)^{-1}$ these points will separate but one needs to be careful not to choose a very large element which could make other points to get closer... We refer the reader to [BPS, Section 3.6] for the details.

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