Partial hyperbolicity: $\nearrow$ Robust transitivity, $\searrow$ Stable ergodicity (also far from homoclinic tangencies, some homogeneous dynamics...)

More flexible notion than hyperbolicity, good model for: fast-slow systems, IFS, ..

$f: M \to M$ is PH if $\exists TM = E^s \oplus E^c \oplus E^u$ non-trivial, continuous & Df-inv

such that:
- vectors in E^s are contracted by Df.
- vectors in E^u are contracted by Df^{-1}
- vectors in E^c have intermediate behavior

$\forall \mu$ inv. measure, Lyap. exp in E^s are negative, in E^u positive in E^c in between.

When $E^c = \{0\}$ one says f is Anosov

Classification of Anosov diffeos is notorious open problem when $\dim M \geq 4$.

Intermediate behavior seen in Anosov flows $TM = E^s \oplus \mathbb{R}\frac{\partial \varphi_t}{\partial t} \oplus E^u$ $(\dim M \geq 3)$ but despite big progress (c.f. Barbot-Fenley's talks) it is still not completely classified. New examples are still appearing (BBY).

Question: Why attempt to classify PH dynamics? Is it hopeless?

Proposal (Bonatti-Wilkinson/Pujals) Compare PH dynamics to Anosov systems and leave classification of the latter to the experts!

(Rmk: Even if complete classif of Anosov systems is not available, we know a lot about their dynamics, geometry, ...)

The problem has an essentially different nature according to $\pi_1(M)$ small (Polyn growth) or big (exponential). $\boxed{\text{We focus on } \dim M = 3}$

See SURVEY (Hammerlindl-P) for review of case when $\pi_1(M)$ is (virtually) solvable which is completely settled (BBI, BW, HHU, Parwani, HP...)

Recently, Bonatti-Gogolev-Hammerlindl-Parwani-P- constructed new examples of PH dynamics on 3-manifolds with exponential growth of $\pi_1(M)$ but which are not isotopic to identity (remain to be understood...)

I will present j.w. in progress with: T. Barthelme, S. Fenley & S. Frankel

CONTEXT:

- $f : M^3 \to M^3$ P.H. diffeo
- $\pi_1(M)$ exponential (not solvable)
- f homotopic to identity

$\xRightarrow{\text{GOAL}}$ f "looks like" time 1-map of Anosov flow. (mod finite lift & iterate)

Rmk: − If M is hyperbolic, after finite iterate is hom. to identity (Mostow)

Some words on classification (and word foliations in title):
→ dynamical coherence
→ leaf conjugacy.

Thm (w. Barthelme, Fenley, Frankel) Let $f : M \to M$ be a transitive ph diffeo homotopic to identity, then (mod finite lift & iterate) either:

→ f is dynamically coherent & leaf conjugate to time 1 map of Anosov flow

or → f has certain specific features (to be explained).

Without further explaining case 2, it is empty. The following remarks show that it is sufficiently precise to obtain some consequences:

Remark: → If M is Seifert fibered, case 2 cannot happen (and transitivity hyp. can be removed) ⇝ Uses previous work of Hammerlindl-P.-Shannon.

→ If M is hyperbolic and $\exists \gamma$ periodic circle leaf ⇒ case 2 cannot happen. (also one can remove transitivity).

→ Work in progress (with BFF) is to study case 2 in more detail to see if we can push the arguments to hold in more generality.

Disclaimer: We are not sure that case 2 is empty (even in hyperbolic mfds!), but we have a candidate & want to show that every such example looks exactly as this candidate.

FOR THE REST OF THE TALK WE ADD SOME SIMPLIFYING ASSUMPTIONS:

- → f is dynamically coherent
- → W^{cs} & W^{cu} are minimal foliations (always true if f is transitive)

We work in universal cover $\tilde{M} \simeq \mathbb{R}^3$

Choose a lift $\tilde{f}: \tilde{M} \to \tilde{M}$ such that $d(\tilde{f}(x), x) \leq K \quad \forall x \in \tilde{M}$ ($f \sim id$)

Goal: Show that leafs of $\tilde{W}^{cs}$, $\tilde{W}^{cu}$ and their intersections are fixed by $\tilde{f}$ ⇒ $\tilde{f}$ looks like Anosov flow.

(this works quite nicely in solvmanifold case)

Def: A foliation $\mathcal{F}$ of $\tilde{M}$ is uniform if given $L_1, L_2 \in \mathcal{F}$ there exists $C > 0$ such that $L_1 \subseteq B_C(L_2) = \{y \in \tilde{M} / d(y, L_2) < C\}$ ($\tilde{W}^{cs}$ for geodesic flow / Fenley in hyp. mfds ⇓ $\mathbb{R}$-covered (leaf space is $\mathbb{R}$))

$\uparrow$

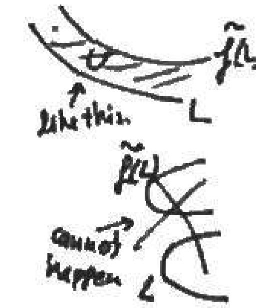
$(d_H(L_1, L_2) < C)$

Proposition There is a dichotomy:

- → Either every leaf of $\tilde{W}^{cs}$ is fixed by $\tilde{f}$ or,
- → $\tilde{W}^{cs}$ is uniform & $\tilde{f}$ acts as a translation on leaf space.

Idea: Take $L \in \tilde{W}^{cs}$ which is not fixed by $\tilde{f}$.

Since $\tilde{f}$ is close to identity & minimal one can see that the region U between L and $\tilde{f}(L)$ is trivially foliated and bounded distance from L.



Take $V = \bigcup_{n \in \mathbb{Z}} \tilde{f}^n(U \cup L)$ and one can show that V is open $\tilde{f}$-invariant, $\tilde{W}^{cs}$ saturated & invariant under deck transf. (proyects to M) ⇒ by minimality one gets $V = \tilde{M}$ and therefore the second possibility of the prop. holds. ☒

Rmk: When W^{cs} not minimal, this is harder and we can prove it only when M is hyperbolic or Seifert, but not yet in the "mixed" case... (non-trivial JSJ dec)

Consequence: $\Rightarrow \tilde{f}$ has no fixed points

$\Rightarrow$ every fixed $\tilde{W}^{cs}$-leaf is a cylinder or a plane (cyclic fund. group)

Pf: 1- If no fixed $\tilde{W}^{cs}$ leafs, trivial. Otherwise  fixed point $\Rightarrow$ all local unstable fixed.

2) Use of Axis of action, the "stable axis" of deck transf coincides with the one of $\tilde{f}$ in the leaf $\rightarrow$ all deck transf commute $\rightarrow$ abelian fund group...

Hölder thm...

($\tilde{f}$ commute with γ_1, γ_2)

The rest of the proof goes as follows:

$\longrightarrow$ One shows that if $\exists \tilde{W}^{cs}$ fixed leaf $\Rightarrow$ the same for $\tilde{W}^{cu}$ (NO MIXED BEHAVIOUR)

$\rightarrow$ If all leafs of $\tilde{W}^{cs}$ and $\tilde{W}^{cu}$ are fixed $\Rightarrow$ so are the connected components of their intersection.

We can now restate our thm in this specific setting:

Thm (BFFP) $f: M \to M$ PH, dyn. coherent (not really necessary, c.f. branch. fol. B.I), homotopic to identity and $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ minimal, then:

- 1) either f is leaf conj to (top) Anosov flow, or
- 2) $\tilde{f}$ is translation on both $\tilde{\mathcal{W}}^{cs}$ and $\tilde{\mathcal{W}}^{cu}$ which are uniform.

Explain: $\rightarrow$ 2) cannot happen if M is Seifert.

$\longrightarrow$ If 2) happens & f has periodic center leaf $\Rightarrow \pi_1(M)$ contains a $\mathbb{Z}^2$.

IF TIME ALLOWS: $\longrightarrow$ Some comments on mixed behaviour (next page)

$\longrightarrow$ Brief ideas on fixed centers (more delicate proof, but easier to believe.)

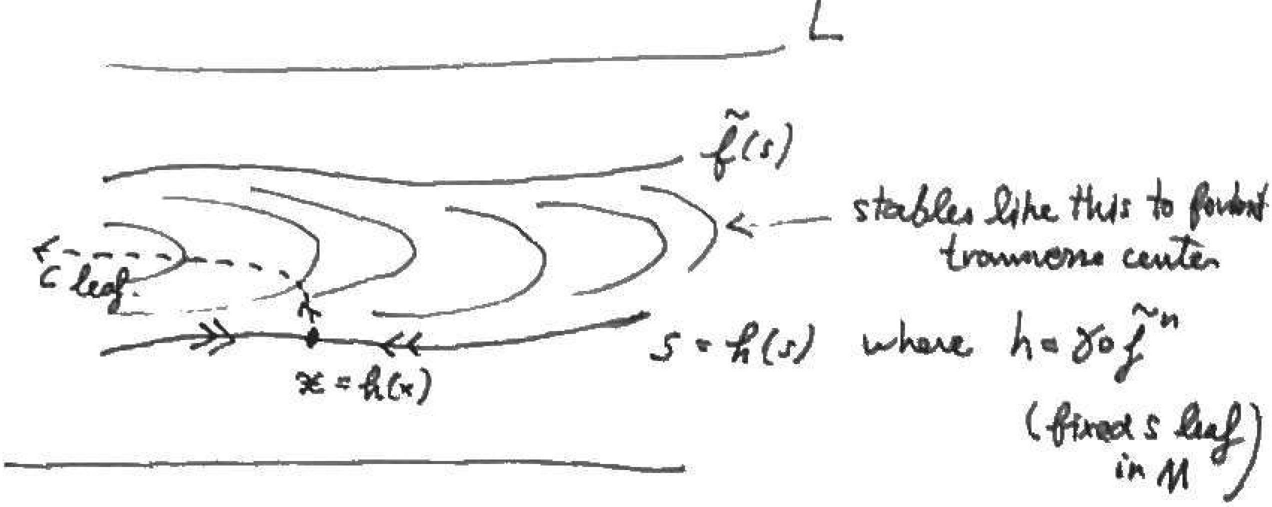
MIXED BEHAVIOUR

By contradiction, assume $\widetilde{W}^{cs}$ fixed & $\widetilde{W}^{cu}$ translation.
We work on some $L \in \widetilde{W}^{cs}$ which projects to cylinder ($\exists \gamma \in \pi_1(M)$ $\gamma \cdot L = L$)

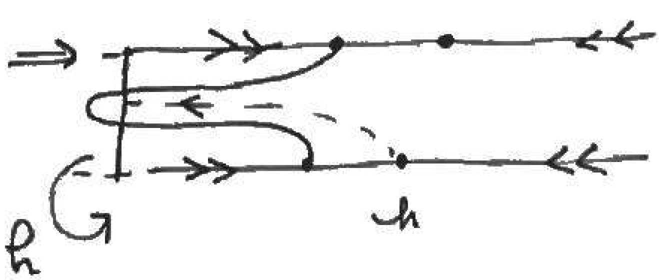
As $\tilde{f}$ translates $\widetilde{W}^{cu}$, there are no fixed center leafs.
As $\tilde{f}$ has no fixed points, there are no fixed stable leafs.

Using a graph transform argument one can show $\nexists$ γ curve transverse to W^s in a cylinder leaf.

One obtains a picture like the following:



Dynamics by h on the c leaf goes opposite to the stable



This contradicts that
h is isometry composed with something close to identity.....

FIXED CENTERS

Key steps: $\longrightarrow$ fixed centers is open (easy)
$\longrightarrow$ fixed centers is empty of $\tilde{M}$ (hardest part)
$\longrightarrow$ there is at least one fixed center (quite involved too..)