INTRODUCTION TO PARTIALLY HYPERBOLIC DYNAMICS

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ABSTRACT. These are notes for a minicourse in the School on Dynamical Systems 2015 at ICTP. This version is quite preliminary, incomplete (mainly in the last sections) and probably contains many errors and omissions. We thank the reader who points out mistakes to communicate them to rpotrie@cmat.edu.uy or sylvain.crovisier@math.u-psud.fr.

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July 2015. We thank S. Luzzatto, M. Viana and J.C. Yoccoz for the invitation to give this minicourse. R.P. was partially supported by Grupo CSIC 618.

1. INTRODUCTION

Basic references for dynamics are [KH, R] and for uniform hyperbolicity [Bow, Sh, Y]. For a presentation of the dynamics beyond uniform hyperbolicity and generic behavior, see [BDV, Cr_2]. Historical references for partial hyperbolicity are [BP, HPS]. More recent surveys on partial hyperbolicity are [$BuPSW_1$, HPe, $RHRHU_1$, Pe, Wi_2].

1.1. **Hyperbolic dynamics.**

The dynamics of uniformly hyperbolic systems has been deeply described since the 60's and the richness of its behavior is now well understood. We summarize in this section some of their main properties.

1.1.1. *What is a hyperbolic system?* Let f be a diffeomorphism on a compact manifold M.

Definition 1.1. A *hyperbolic set* for f is a compact f-invariant set K whose tangent bundle admits a splitting into two continuous vector subbundles $T_K M = E^s \oplus E^u$ which satisfy:

- E^s, E^u are invariant: $\forall x \in K$, $D_x f(E^s_x) = E^s_{f(x)}$ and $D_x f(E^u_x) = E^u_{f(x)}$.
- E^s is uniformly contracted and E^u is uniformly expanded: there exist $c > 0$ and $\lambda \in (0,1)$ such that for any $x \in K$, $u \in E^s_x$, and $v \in E^u_x$,

$$\forall n \geq 0,\ \|D_x f^n u\| \leq c\lambda^n \|u\| \text{ and } \|D_x f^n v\| \geq c^{-1}\lambda^{-n}\|v\|.$$

The spaces E^s_x and E^u_x at x are called the *stable and unstable spaces.*

The definition of a hyperbolic diffeomorphism is less universal. In general, one only requires hyperbolicity on a set which satisfies some recurrence. In the late 60's, the attention was focused on the non-wandering set and led Smale to introduce *Axiom A* diffeomorphisms ([Sm]). This class is in general not open and the non-wandering dynamics is not always stable under perturbation (some Ω-explosions may occur).

It seems to us that a (bit stronger) class of systems, involving chain-recurrence, is more natural: once one knows the notion of filtration, this second class is easier to define. It may be equivalently defined as the collection of systems that satisfy the Axiom A and the "no cycle condition". These are these diffeomorphisms that we propose to call *hyperbolic diffeomorphisms.*

1.1.2. *Chain-recurrence and filtrations.* For decomposing the dynamics one introduces the following notion.

Definition 1.2. A *filtration* for f is a finite family of open sets $U_0 = \emptyset \subset U_1 \subset \cdots \subset U_m = M$ such that $f(\overline{U_i}) \subset U_i$ for each i.

The dynamics may then be studied independently in each level of a filtration, that is in restriction to each maximal invariant set in $U_n \setminus U_{n-1}$.

For each *trapping region* U, i.e. any open set which satisfies $f(\overline{U}) \subset U$, the dynamics of points in $U \setminus f(\overline{U})$ is very simple. The collection of points $x \in M$ which do not belong to any such domain is an invariant compact set $\mathcal{R}(f)$, called the *chain-recurrent set.* Each filtration induces a partition of the chain-recurrent set. Considering different filtrations allows to split more. The *chain-recurrence classes* are the maximal invariant compact subsets of $\mathcal{R}(f)$ that can not be decomposed by a filtration.

There is another way to define $\mathcal{R}(f)$ and its decomposition into chain-recurrence classes.

Definition 1.3. Given two points $x, y \in X$ we say that there exists an ε-*pseudo orbit* from x to y if and only if there exists points $z_0 = x, \dots, z_k = y$ such that $k \geq 1$ and

$$d(f(z_i), z_{i+1}) \leq \varepsilon \qquad \text{for any} \quad 0 \leq i \leq k-1.$$

We use the notation $x \dashv y$ to express that for every $\varepsilon > 0$ there exists an ε-pseudo orbit from x to y. We also use $x \vdash\dashv y$ to mean $x \dashv y$ and $y \dashv x$.

Exercise 1. Prove that $x \vdash\dashv y$ for any x, y if and only for any trapping region U, the set $\{x, y\}$ is either included in U or in $M \setminus U$.

Exercise 2. The chain-recurrent set of f coincides with:

$$\mathcal{R}(f) = \{x \in X \ : \ x \vdash\dashv x\}$$

The chain-recurrence classes are the equivalence classes for the relation $x \vdash\dashv y$ on $\mathcal{R}(f)$.

A *chain transitive set* is an f-invariant set K such that $f_{|\mathcal{R}(f)}$ is chain-transitive: for every $x, y \in K$ and for any $\varepsilon > 0$, there exists a ε-pseudo-orbit $z_0 = x, \dots, z_n = y$contained in K such that $n \geq 1$.

1.1.3. *Examples of hyperbolic diffeomorphisms.* One can now define.

Definition 1.4. The diffeomorphism f is *hyperbolic* if there exists a filtration $U_0 \subset \cdots \subset U_m$ for f such that for each $i = 1, \dots, m$, the maximal invariant compact set $\bigcap_{n \in \mathbb{Z}} f^n(U_i \setminus U_{i-1})$ is hyperbolic.

Exercise 3. Equivalently, a diffeomorphism is hyperbolic if each of its chain-recurrence classes is a hyperbolic set.

Example 1: the whole manifold. The whole manifold may be a hyperbolic set, in which case, one say that f is *Anosov*. This is the case for the dynamics induced on $\mathbb{T}^2$ by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. More examples will be discussed in section 3.

Example 2: a finite set. The chain-recurrence set may be reduced to a finite collection of hyperbolic orbits. (Up to a technical assumption), this corresponds to the class of *Morse-Smale* diffeomorphisms.

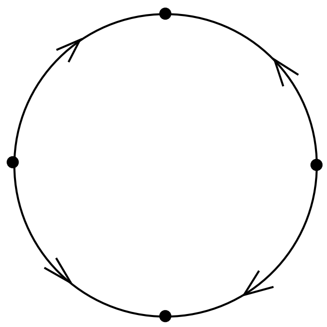

FIGURE 1. Morse-Smale on the circle.

Example 3: a Cantor set. Let R be a rectangle diffeomorphic to $[0,1]^2$, divided in three horizontals subrectangles $R_i = [0,1]\times[(i-1)/3, i/3]$, $i = 1,2,3$. Let f be a diffeomorphism which

– coincides with the map $(x,y)\mapsto(x/3,3y)$ on R_1,
– coincides with the map $(x,y)\mapsto(1-x/3,3-3y)$ on R_3,
– sends R_2 outside R.

The vertical and horizontal directions are preserved, the first is expanded, the second contracted. Hence the maximal invariant set K inside R is a hyperbolic set, homeomorphic to a Cantor set, and called the *horseshoe*.

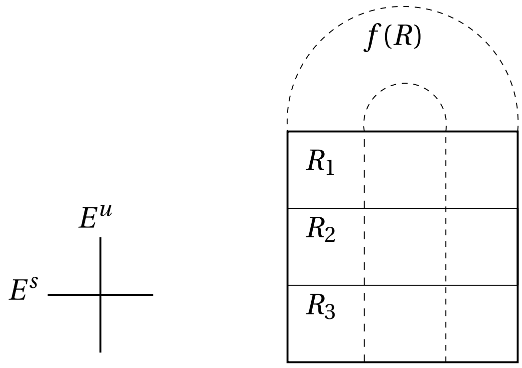

FIGURE 2. The horseshoe.

Example 4: an attractor. We introduce four rectangles R_1, R_2, R_3, R_4, diffeomorphic to $[0,1]^2$, glued along their "vertical" edges. The diffeomorphism f sends the union $\Delta = \bigcup_i R_i$ into its interior as on figure 3 and preserves and contracts the vertical foliation.

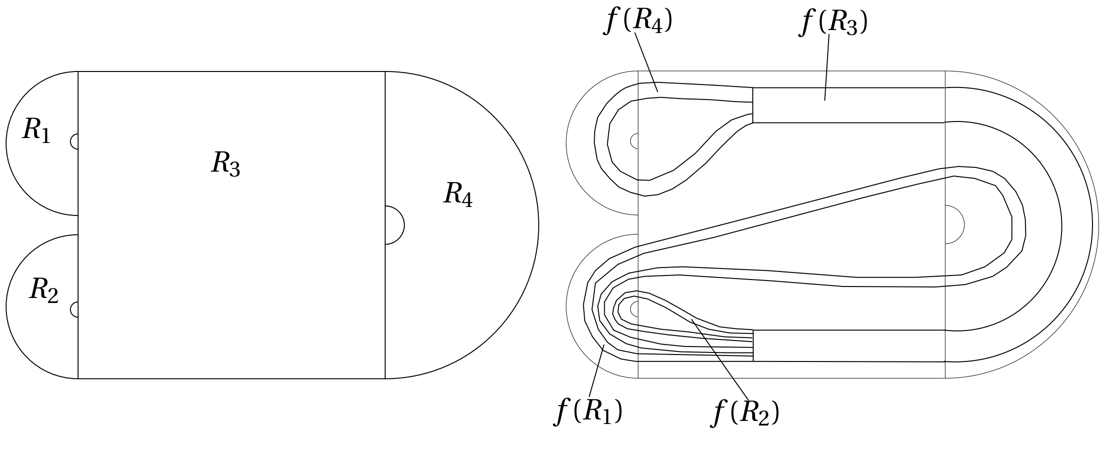

FIGURE 3. Construction of the Plykin attractor.

In this case the hyperbolic set is an attractor which is a union of curves tangent to the unstable direction. A *repeller* is an attractor for f^{-1}.

Exercise 4. The horseshoe is topologically conjugated to the shift on $\{0,1\}^{\mathbb{Z}}$.

1.1.4. *Properties of hyperbolic diffeomorphisms.* Hyperbolicity has many consequences:

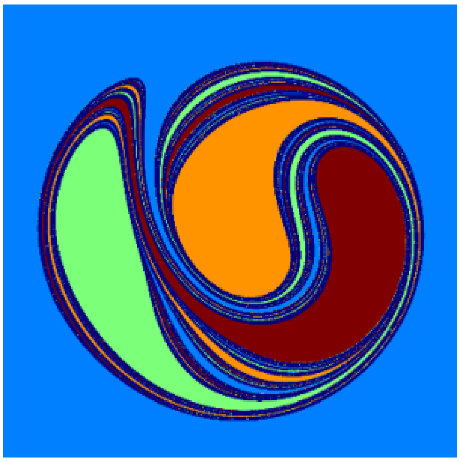

FIGURE 4. The Plykin attractor.

a. Shadowing lemma. *Consider some hyperbolic set K for f. For any $\delta > 0$, there exists $\varepsilon > 0$ such that for any ε-pseudo orbit $(x_i)_{i\in\mathbb{Z}}$ contained in K, there exists a unique point y which satisfies $d(x_i, f^i(y)) < \delta$ for any $i \in \mathbb{Z}$.*

Consider a hyperbolic diffeomorphism f. The shadowing lemma allows to:

- *Code the dynamics* (via Markov partition): the dynamics on $\mathcal{R}(f)$ is Hölder-semi-conjugated to a subshift of finite type.
- *Prove its stability*: for any $\delta > 0$, if g is a diffeomorphism C^1-close to f, the dynamics $(\mathcal{R}(f), f)$ and $(\mathcal{R}(g), g)$ are topologically conjugated.
- *Get a spectral decomposition:* the number of chain-recurrence classes is finite and each of them is transitive, i.e. contains a dense forward orbit.

Exercise 5. For a hyperbolic diffeomorphism, prove the spectral decomposition and that periodic points are dense in $\mathcal{R}(f)$.

b. Stable manifold theorem. *For any diffeomorphism f and any point x in a hyperbolic set, the stable set $W^s(x) := \{y, d(f^n(x), f^n(y)) \xrightarrow[+\infty]{} 0\}$ is an immersed submanifold. (The same holds for the past dynamics and the unstable set $W^u(x) := \{y, d(f^n(x), f^n(y)) \xrightarrow[-\infty]{} 0\}$.)*

This is a key property for studying the geometrical properties of hyperbolic systems. It allows for instance to define the *homoclinic class* of a hyperbolic periodic orbit.

c. Statistical description. *For any hyperbolic C^2-diffeomorphism f, each chain-recurrence class which is an attractor supports a physical ergodic measure μ: the set*

$$B(\mu) := \left\{ y, \frac{1}{n}(\delta_y + \cdots + \delta_{f^{n-1}(y)}) \xrightarrow[+\infty]{} \mu \right\}$$

has positive Lebesgue volume. Moreover, the union of the basins of the physical measures of all the attractors has total volume.

These physical measures satisfy many strong properties: they are Bernoulli, have exponential decay of correlations,... These descriptions apply in particular to Anosov diffeomorphisms which preserve a volume: when the manifold is connected, the volume is the unique physical measure (in particular, it is ergodic , etc).

d. Classification. Hyperbolic dynamics exist on any manifold. (For instance Shub and Smale have shown that any diffeomorphism can be approximated in the C^0-topology by a hyperbolic one whose non-wandering set is totally disconnected, see [Fr$_3$, Appendix B].) However hyperbolicity may be constrained. For instance:

Theorem (Franks-Newhouse [Fr$_1$, Ne])**.** *On a connected manifold, any Anosov diffeomorphism with one-dimensional unstable spaces is conjugated to a linear automorphism of* $\mathbb{T}^d$.

On surface, any hyperbolic chain-recurrence class is described by one of the examples of section 1.1.3. The general classification of Anosov systems or of higher-dimensional hyperbolic chain-recurrence class is still unknown.

1.2. **Partially hyperbolic dynamics.** Partial hyperbolicity is a relaxed form of uniform hyperbolicity which intends to address larger families of dynamics. A main goal of their study consists in understanding how the properties of uniformly hyperbolic systems extends.

Definition 1.5. A *partially hyperbolic set* for f is a compact f-invariant set K whose tangent bundle admits a splitting into three continuous vector subbundles $T_K M = E^s \oplus E^c \oplus E^u$ which satisfy:

- the splitting is dominated,
- E^s is uniformly contracted, E^u is uniformly expanded, one of them is non-trivial.

A splitting $T_K M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is *dominated* if each bundle is invariant and there exist $c > 0$, $\lambda \in (0,1)$ such that for any $x \in K$ and any unit vectors $u \in E_i$, and $v \in E_{i+1}$,

$$\forall n \geq 0,\ \|D_x f^n u\| \leq c\lambda^n \|D_x f^n v\|.$$

In analogy with the hyperbolic case, we will say that a diffeomorphism f is *partially hyperbolic* if it admits a filtration $U_0 \subset U_1 \subset \cdots \subset U_m$ such that the maximal invariant compact set in each set $U_i \setminus U_{i-1}$ is partially hyperbolic. Notice that this allows the presence of sinks and sources, but only finitely many of them.

We now list some of the main problems and/or reasons for studying partially hyperbolic diffeomorphisms.

1.2.1. *Partial hyperbolicity and homoclinic tangencies.* Systems which do not admit any dominated splitting or having an invariant tangent sub bundle which is neither contracted, nor expanded become very delicate to be handled. Typically this occurs for dynamics near systems exhibiting *homoclinic tangencies*, i.e. having a hyperbolic periodic point p such that $W^s(p)$ and $W^u(p)$ have a non-transverse intersection.

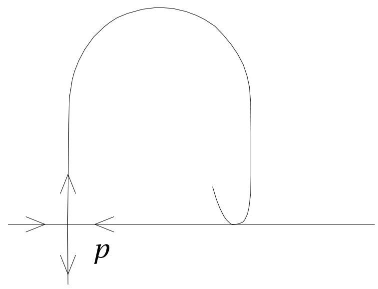

FIGURE 5. An homoclinic tangency.

Exercise 6. Consider a diffeomorphism and chain-recurrence class C which contains a hyperbolic periodic point exhibiting a homoclinic tangency. Prove that C is not hyperbolic and that $T_C M$ has no dominated splitting $E \oplus F$ such that $\dim(E_p) = \dim(E^s_p)$.

For instance Newhouse has shown that near any surface diffeomorphism exhibiting an homoclinic tangency, there exists an open set $\mathscr{U}$ of non-hyperbolic C^2-diffeomorphisms and the systems in a dense subset of $\mathscr{U}$ have infinitely many sinks: it shows that homoclinic tangencies generates dynamics with infinitely many chain-hyperbolic classes. From the works initiated by Newhouse, Benedicks-Carleson,... one knows that the dynamics which occurs close to systems exhibiting some types of homoclinic tangencies become much harder to describe.

Homoclinic tangencies do not appear near a system whose chain-recurrence classes are partially hyperbolic and whose center bundle has a dominated splitting into one-dimensional sub bundles. One may expect that these partially hyperbolic diffeomorphisms represent the natural world of systems that could be described by generalizing technics developed for hyperbolic systems. For instance, as a major problem, one wonders if this partial hyperbolicity ensures (for typical systems) the finiteness of the chain-recurrence classes (see [Bon, Conjecture 11] and Section 6 below).

The following result from [CSY, CPuS] shows that these two kinds of dynamics cover the whole systems and settles partial hyperbolicity as a natural boundary for the space of systems with low complexity.

Theorem 1.6 (Crovisier-Pujals-Sambarino-D. Yang)**.** *Any diffeomorphism may be approximated in* $\mathrm{Diff}^1(M)(M)$ *by one which:*

- *either is partially hyperbolic;*
  moreover there are at most finitely many sinks and sources and any other chain-recurrence class has a partially hyperbolic dominated splitting $T_K M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u$, *where* E^s, E^u *are non-trivial and the bundles* E^c_i *are one-dimensional;*
- *or exhibits a homoclinic tangency.*

In [Cr_1] a technique to study the dynamics of partially hyperbolic sets with center bundles of dimension one was introduced. The idea is to study the dynamics along the center-direction as a skew product over the initial dynamics and profit from the one-dimensionality to control the possible behavior; the technique is called *centrer models*.

In some cases, the study of centrer models allows to mimic the ideas in the proof of the spectral decomposition in hyperbolic dynamics. However, there are some cases where even with this understanding of the dynamics along the center is not enough to decide if chain transitive sets which accumulate must be in the same chain-recurrence class and more detailed study is needed. Unfortunately, we have not presented in this notes the study of centrer models (although we hope to do it some day), but we can refer the reader to [Cr_2, Chapitre 9] for a presentation of this technique.

In section 6 we give a glimpse on some new techniques which allow in some cases to obtain results in the direction of a spectral decomposition for partially hyperbolic quasi-attractors.

1.2.2. *Robust transitivity.* It has been known since the work of Franks and Mañe ([Fr_2, Ma_2]) that absence of a dominated splitting allows one to make a C^1-small perturbation in order to change the index of the periodic points. An important application of this fact describes C^1-robustly transitive diffeomorphism, i.e. diffeomorphisms such that any system C^1-close has a dense orbit.

Mañe ([Ma_1, Ma_2]) first devised that a C^1-robustly transitive diffeomorphism of a closed surface has to be Anosov. Generalizations of this result to higher dimensions were then found ([DPU, BDP, We, ABC]). Let us state a general version of this line of research.

Theorem 1.7 ([ABC])**.** *There exists a G_δ-dense subset $\mathcal{G}_{VH}$ of* $\text{Diff}^1(M)$ *such that if $f \in \mathcal{G}_{VH}$ and C is a chain-recurrence class which is not a periodic sink, then the following dichotomy holds:*

- *either C is accumulated by infinitely many periodic sinks, or,*
- *the tangent space $T_C M$ admits a dominated splitting of the form $T_C M = E \oplus F$ where the bundle F is volume-expanded.*

Exercise 7. Show that if f is C^1-robustly transitive then it cannot be perturbed in order to have periodic sinks or sources. Deduce that if f belongs to some dense G_δ set of diffeomorphisms and is C^1-robustly transitive diffeomorphism of M then f is *volume partially hyperbolic* (i.e. it admits a dominated splitting $TM = E^{vs} \oplus E^c \oplus E^{vu}$ such that $Df|_{E^{vs}}$ contracts volume uniformly and $Df|_{E^{vu}}$ expands volume uniformly). Show that in dimensión 3 this implies that f is *partially hyperbolic.*

In the conservative setting, an analogous statement has been proved in [BFP]: if f belongs to some dense G_δ subset of the space of C^1-diffeomorphisms and if the volume is C^1-stably ergodic, then f is *non-uniformly Anosov,* i.e. there exists a dominated splitting $TM = E \oplus F$ such that for Lebesgue-almost every point x, any vector in E is exponentially contracted in the future and any vector in F is exponentially contracted in the past. Precise definitions on this type of domination will appear in section 2. In any case, Theorem 1.7 highlights the importance of understanding partial hyperbolicity in order to understand robust dynamical behavior.

1.2.3. *Stable ergodicity.* A major source of questions and problems in partially hyperbolic dynamics comes from the study of stable ergodicity. The problem goes back to the seminal works of Brin,Pesin, Pugh and Shub (see [BP, PSh_2] and references therein). This has been enhanced by the conjectures of Pugh and Shub suggesting that stable ergodicity is open and dense among partially hyperbolic diffeomorphisms (see [PSh_2]). This conjecture is divided in two important subconjectures:

- *accessibility* is dense among partially hyperbolic diffeomorphisms (this will be partially discussed in section 5),
- accessible partially hyperbolic diffeomorphisms are ergodic (the state of the art on this problem is [BuW_2]).

We remark here that recently, the full conjecture was established in the C^1-topology [ACW]. We refer the reader to [BuPSW_1, PSh_2, RHRHU_1, Wi_2] for more detailed account on the problem of stable ergodicity.

1.2.4. *Natural examples.* An important reason for studying partially hyperbolic diffeomorphisms is that they are quite ubiquitous and related to several different fields inside and outside math. An incomplete, yet rather comprehensive presentation of examples can be found in section 3 of this notes.

Let us nonetheless give a few natural examples in this introduction without entering into details:

- Algebraic examples appear naturally when studying geometry, number theory, etc. If one studies a structure preserving diffeomorphism of a homogeneous space, then one expects that the derivative at each point behaves essentially the same.

Therefore, whenever there is one eigenvalues of modulus different from one, a partially hyperbolic example will arise.

- The study of the geometry of negatively or non-positively curved spaces is quite an active topic of research. Understanding the properties of the geodesic flow or the frame flow might be important to understand the underlying geometry. Just to cite a (quite surprising) example, recently the exponential mixing of the frame flow on hyperbolic 3-manifolds was used to prove a long standing conjecture regarding the existence of surface groups as subgroups of the fundamental group of a hyperbolic 3-manifold ([KM]).
- Skew-products appear "everywhere". It is quite common to encounter situations where the dynamics which models a given system is coupled and the "base dynamics" is hyperbolic. Partially hyperbolic systems sometimes serve as good models for *fast-slow* systems ([dSL])...

1.3. **Objectives of the notes.** Many works have been devoted to partially hyperbolic systems and we can only address few aspects of this subject. These notes mainly discuss results towards a generalization of the spectral decomposition, i.e. the finiteness and the robustness of the decomposition.

Complementary presentations may be found in [BDV, BuPSW_1, CRHRHU, Cr_3, HPe, HPS, Pe, PSh_2, RHRHU_3, Wi_2] were classifications, SRB/Gibbs states and stable ergodicity are discussed.

2. DEFINITIONS AND FIRST PROPERTIES

We consider M a closed connected d-dimensional Riemannian manifold and let TM its tangent bundle. We also consider f in the space $\text{Diff}^r(M)$ of C^r-diffeomorphisms endowed with the C^r-topology, $r \geq 1$.

Notations. We denote by $Df : TM \to TM$ the derivative of f and by $D_x f : T_x M \to T_{f(x)} M$ the derivative at a point x.

For $K \subset M$ one denotes $T_K M = \bigcup_{x \in K} T_x M \subset TM$ with the topology induced by the inclusion. If $E_x \subset T_x M$ is a subbundle of $T_x M$ one denotes $Df|_{E_x}$ to the linear map $Df|_{E_x} : E_x \to D_x f(E_x)$ which is the restriction of $D_x f$ to E_x.

For a linear map $L : V \to W$ between normed vector spaces V, W, one defines $\|L\| = \max\{\frac{\|Lv\|}{\|v\|} : v \neq 0\}$ and $m(L) = \min\{\frac{\|Lv\|}{\|v\|} : v \neq 0\}$.

2.1. Dominated splittings.

For $K \subset M$, a *splitting* of $T_K M = E_1 \oplus \ldots \oplus E_\ell$ is a linear decomposition $T_x M = E_1(x) \oplus \ldots \oplus E_\ell(x)$ at each $x \in K$ such that $\dim E_i(x)$ does not depend on x for any $1 \leq i \leq \ell$.

If $K \subset M$ is an f-invariant set (i.e. $f(K) = K$) then we say that a splitting $T_K M = E_1 \oplus \ldots \oplus E_\ell$ is a *dominated splitting* iff:

- **Invariance:** The bundles are Df-invariant. This means that for every $x \in K$ and $1 \leq i \leq \ell$ one has $D_x f(E_i(x)) = E_i(f(x))$.
- **Domination:** There exists constants $c > 0$ and $\lambda \in (0,1)$ such that, for every $1 \leq i \leq \ell - 1$, for every $x \in K$ and vectors $u \in E_i(x) \setminus \{0\}$ and $v \in E_{i+1}(x) \setminus \{0\}$ one has:

$$\frac{\|D_x f^n u\|}{\|u\|} \leq c\lambda^n \frac{\|D_x f^n v\|}{\|v\|} \quad , \quad \forall n \geq 0 \tag{2.1.1}$$

Domination can be also expressed by saying that for any $x \in K$ and $1 \leq i \leq \ell - 1$ one has that $\|Df^n|_{E_i(x)}\| \leq c\lambda^n m(Df^n|_{E_{i+1}(x)})$.

Remark 2.1.
(a) If $\ell = 1$ we say that the splitting is *trivial.* Sometimes, when one says that an f-invariant subset admits a dominated splitting one implicitly assumes that it is not trivial.
(b) One can replace condition (2.1.1) by asking for the existence of $N > 0$ such that for any $x \in K$ and vectors $u \in E_i(x) \setminus \{0\}$ and $v \in E_{i+1}(x) \setminus \{0\}$ one has:

$$\frac{\|D_x f^N u\|}{\|u\|} \leq \frac{1}{2} \frac{\|D_x f^N v\|}{\|v\|}. \tag{2.1.2}$$

In any case, in such a situation one says that E_{i+1} *dominates* E_i and one someone denotes this as $E_1 \oplus_< \ldots \oplus_< E_\ell$ to emphasize the order of the domination.
(c) If one replaces the bundles E_i, E_{i+1} by their direct sum $E_i \oplus E_{i+1}$ the splitting remains dominated.

Exercise 8.
- Show that the definition of dominated splitting is independent of the Riemannian metric in M (the constants may change with the change of the metric).
- Show properties (b) and (c) of Remark 2.1.

If one fixes the dimensions of the bundles, then a dominated splitting must be unique.

Proposition 2.2 (Uniqueness)**.** *For $f \in \text{Diff}^r(M)$, if $K \subset M$ is f-invariant and admits two dominated splittings $T_K M = E \oplus F$ and $T_K M = G \oplus H$ and* $\dim(E(x)) \leq \dim(G(x))$*, then $E(x) \subset G(x)$ for every $x \in K$. In particular, the splitting $T_K M = E \oplus (F \cap G) \oplus H$ is dominated.*

PROOF. For a vector $u \in E(x)$ one decomposes $u = u_G + u_H$ in a unique way where $u_G \in G(x)$ and $u_H \in H(x)$. Similarly, one can decompose $u_H = u'_E + u'_F$ with $u'_E \in E(x)$ and $u'_F \in F(x)$. Notice that both u'_F must be zero since by domination one would get that $\|D_x f^n u\|$ would grow at the same speed as $\|D_x f^n u'_F\|$ which is impossible since $u \in E(x)$. Therefore, one deduces that $u_H \in E(x) \cap H(x)$ and then $u_G \in E(x) \cap G(x)$. Symmetrically one deduces that if $v \in G(x)$ is decomposed as $v = v_E + v_F$ with $v_E \in E(x)$ and $v_F \in F(x)$ one has $v_E \in E(x) \cap G(x)$ and $v_F \in F(x) \cap G(x)$.

Assume by contradiction that $E(x)$ is not contained in $G(x)$. One can choose $u \in E(x)$ such that $u_H \neq 0$ is contained in $(E(x) \cap H(x)) \setminus \{0\}$. Since $\dim E(x) \leq \dim G(x)$ and $E(x), G(x)$ do not coincide, one gets a non zero vector $v \in F(x) \cap G(x)$ by the same argument. Using the fact that H dominates G one deduces that $\|D_x f^n u\|$ grows faster than $\|D_x f^n v\|$ contradicting the fact that F dominates E.

□

Corollary 2.3. *For any $n \neq 0$, a splitting $T_K M = E_1 \oplus \ldots \oplus E_\ell$ is dominated for f if and only if it is dominated for f^n (with the order of the bundles reversed if $n < 0$).*

PROOF. The only thing one must show is that if the splitting is dominated for f^n $(n > 1)$ then it is invariant under Df. To see this, notice that the image of the splitting by Df is still dominated for f^n and therefore using uniqueness one gets the result.

□

Corollary 2.4 (Finest Dominated Splitting). *For $f \in \text{Diff}^r(M)$ and $K \subset M$ an f-invariant subset, there exists a finest dominated splitting $T_K M = E_1 \oplus \ldots \oplus E_\ell$ (with possibly $\ell = 1$ when the splitting is trivial) such that every dominated splitting on K is obtained by considering the sum of some consecutive subbundles of this finest dominated splitting.*

Exercise 9. Prove Corollary 2.4.

The following property is very useful since it allows one to check the domination property considering a dense subset of the desired space. Indeed, dominated splitting allows sometimes to extend the information one has on hyperbolic periodic points to its closure.

Proposition 2.5 (Continuity and extension to the closure). *Let $f \in \text{Diff}^r(M)$ and $K \subset M$ an f-invariant set with a dominated splitting $T_K M = E_1 \oplus \ldots \oplus E_\ell$. Then, the bundles E_i vary continuously with the point $x \in K$ and the closure $\overline{K}$ of K admits a dominated splitting which coincides with $E_1 \oplus \ldots \oplus E_\ell$ in K.*

Continuity in this context means that when considering local coordinates so that the tangent bundle becomes trivial, the bundles depend on the point continuously as subspaces of $\mathbb{R}^d$. Equivalently, the continuity can also be expressed by seeing the bundles as sections of some Grassmanian bundle.

PROOF. It is enough to prove it for dominated splitting into 2 sub-bundles since any other dominated splitting is obtained from this case by intersection.

Let $T_K M = E \oplus F$ be a dominated splitting. Let us consider a sequence (x_n) in K converging to $x \in \overline{K}$ and any limits E'_x, F'_x of the spaces $E(x_n), F(x_n)$.

By continuity of Df, for any $u \in E'_x \setminus \{0\}$ and $v \in F'_x \setminus \{0\}$, the property (2.1.2) still holds, proving that $T_x M$ still decomposes as $E'_x \oplus F'_x$.

One may build an invariant splitting by replacing (or defining if $x \notin K$) each $E(f^n(x))$ by $D_x f^n(E'_x)$ and the limit argument proves that it is still dominated. The uniqueness given by Proposition 2.2 implies that $E'_x = E(x)$. Hence E is continuous. Similarly, F is continuous.

□

2.2. **Cone-criterion.** We adopt the convention that if V is a vector space, a *cone* C in V is a subset such that there exists non-degenerate quadratic form Q_C such that

$$C = \{v \in V \,:\, Q_C(v) \geq 0\}$$

The interior of a cone is $\mathrm{int} C = \{v \in V \,:\, Q_C(v) > 0\} \cup \{0\}$.

A *cone-field* $\mathscr{C}$ on $K \subset M$ is then a choice of a cone $\mathscr{C}(x) \subset T_xM$ for each point in M such that in local charts the quadratic forms can be chosen in a continuous way and have the same signature (d_+, d_-).

Equivalently, a cone-field $\mathscr{C}$ in K is given by:

- a (not necessarily invariant) splitting $T_KM = \hat{E} \oplus \hat{F}$ into continuous subbundles wose fibers have dimension d_- and d_+ respectively,
- a continuous family of Riemannian norms $\|.\|$ defined on T_KM (not necessarily the ones given by the underlying Riemannian metric).

In this setting, for $x \in K$, one associates

- the cone $\mathscr{C}_x = \{v = v_{\hat{E}} + v_{\hat{F}} \in T_xM,\ \|v_{\hat{F}}\| \geq \|v_{\hat{E}}\|\}$,
- the dual cone $\mathscr{C}^*_x = \{v = v_{\hat{E}} + v_{\hat{F}} \in T_xM,\ \|v_{\hat{E}}\| \geq \|v_{\hat{F}}\|\}$,

The *dimension* $\dim \mathscr{C}$ of the cone-field $\mathscr{C}$ is the dimension d_+ of the bundle $\hat{F}$.

We say that a cone-field $\mathscr{C}$ defined in K is Df*-contracted* if there exists $N > 0$ such that for every $x \in K \cap \cdots \cap f^{-N}(K)$ one has that

$$D_x f^N(\mathscr{C}_x) \subset \mathrm{int}\mathscr{C}_{f^N(x)}$$

(Equivalently, the dual cone field $\mathscr{C}^*$ is Df^{-1}-contracted.)

Recall that the Perron-Frobenius Theorem[1] in linear algebra states that if a linear map $A : V \to V$ sends a cone C in its interior, then there is an invariant splitting $V = E \oplus F$ with $\dim(F) = \dim(C)$ and the eigenvalues of A along F are larger than those along E. The following result can be thought of as a fibered version of Perron-Frobenius Theorem.

Theorem 2.6 (Cone-field criterion)**.** *Let $f \in \mathrm{Diff}^r(M)$, let K be an invariant compact set and fix $d_+ \geq 1$. Then K is endowed with a Df-contracted cone-field $\mathscr{C}$ with dimension d_+ if and only if there exists a dominated splitting $T_KM = E \oplus_< F$ with $d_+ = \dim(F)$.*

PROOF. One direction is easy: let us assume that $T_K = E \oplus F$ is dominated. By Proposition 2.5, the bundles E, F are continuous. Equation 2.1.2 implies that the cone-field defined by

$$\mathscr{C}_x := \{v = v_E + v_F;\ \|v_F\| \geq \|v_E\|\}$$

is Df-contracted. Obviously it has the same dimension as the fibers of F.

Let us now assume that $\mathscr{C}$ is a Df-contracted cone field on K. There exists $N > 0$ such that $D_x f^N(\mathscr{C}_x) \subset \mathrm{int}\mathscr{C}_{f^N(x)}$ for any $x \in K$. Thanks to Corollary 2.3, we know that finding a

[1]Usually, Perron-Frobenius Theorem is stated for one-dimensional cones. However, by considering exterior powers one can always reduce the general case to the one-dimensional one.

dominated splitting for f^N gives the same for f. In particular, we can assume that $N = 1$ without loss of generality.

Now, we consider the following subset of T_xM for any given $x \in K$.

$$V_x^+ = \bigcap_{n\geq 1} D_{f^{-n}(x)} f^n(\mathscr{C}_{f^{-n}(x)}) \subset \mathscr{C}_x$$

which defines a non-trivial subset of T_xM because it is a decreasing intersection of non-trivial compact subsets (in the projective space). This set is *cone-like* in the sense that if $v \in V_x^+$ then $\lambda v \in V_x^+$ for every $\lambda \in \mathbb{R}$. If follows from their definition that the subsets V_x^+ are invariant in the sense that:

$$D_x f(V_x^+) = V_{f(x)}^+.$$

One defines symmetrically the subset V_x^- using the dual cone-field and Df^{-1}. One has, for every $x \in K$ that $V_x^+ \cap V_x^- = \{0\}$ and $T_xM = V_x^- + V_x^+$.

Claim. *The sets V_x^+ and V_x^- are transverse subspaces with dimensions d_+ and d_-.*

Proof. We shall make use of the well known cross ratio (sometimes called the Hilbert metric) and follow the proof given in [BG]. Given a two-dimensional vector space V with a basis $\{e_1, e_2\}$ there is a map $\Gamma_{e_1,e_2} : \mathbb{P}(V) \to \mathbb{R} \cup \{\infty\}$ given by

$$\Gamma_{e_1,e_2}(\mathbb{P}(ae_1 + be_2)) = \frac{b}{a}$$

For $a, b, c, d \in \mathbb{R} \cup \{\infty\}$ one defines the cross ratio as:

$$[a, b, c, d] = \frac{c-a}{b-a}\frac{d-b}{d-c}$$

Let A be a linear transformation $A : V \to W$ where V and W are vector spaces with bases $\{b_1, b_2\}$ and $\{b_1', b_2'\}$ respectively. We still denote by A the induced action on the projective spaces($A : \mathbb{P}(V) \to \mathbb{P}(W)$). Therefore one gets a map $\hat{A} : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ defined by

$$\hat{A}(x) = \Gamma_{b_1',b_2'} A \Gamma^{-1}_{b_1,b_2}(x) = \frac{\alpha_{22}x + \alpha_{21}}{\alpha_{12}x + \alpha_{11}}$$

Where α_{ij} are the coefficients of the matrix A in the aforementioned basis. A simple computation (see Exercise 10) shows that $\hat{A}$ preserves the cross ratio: if $a, b, c, d \in \mathbb{R} \cup \{\infty\}$ then $[a, b, c, d] = [\hat{A}(a), \hat{A}(b), \hat{A}(c), \hat{A}(d)]$. Consequently the cross ratio is defined on lines of V independently from the choice of a basis. Below, we will sometimes denote $[a, b, c, d]$ and $[A(a), A(b), A(c), A(d)]$ when $a, b, c, d \in V \setminus \{0\}$ instead of $\mathbb{R} \cup \{\infty\}$ by abuse of notation.

Exercise 10. Show that linear maps preserve the cross ratio. (Hint: Decompose the transformation as one does with Möbius transformations and see that each part of the decomposition preserves the cross ratio.)

Let us suppose then that V_x^+ is not a subspace. This means that there exists a plane $V \in T_xM$ whose intersection with $\partial\mathscr{C}_x$ is not a linear space: in the projective space, it is a closed interval bounded by two lines generated by some vectors $v, w \in V \setminus \{0\}$.

By definition of V_x^+ this implies that there are vectors $v_n \to v$ and $w_n \to w$ in V such that $v_n \in Df^n(\partial\mathscr{C}_{f^{-n}(x)})$ and $w_n \in Df^n(\partial\mathscr{C}_{f^{-n}(x)})$.

One deduces that $[v_n, v_{n+1}, w_n, w_{n+1}] \to \infty$ and therefore

$$[Df^{-n}v_n, Df^{-n}v_{n+1}, Df^{-n}w_n, Df^{-n}w_{n+1}] \to \infty$$

Notice that vectors $Df^{-n}v_n$ and $Df^{-n}w_n$ belong to $\partial\mathscr{C}_{f^{-n}(x)}$ while the vectors $Df^{-n}v_{n+1}$ and $Df^{-n}w_{n+1}$ belong to $Df(\mathscr{C}_{f^{-n-1}(x)})$. Since K is compact and $Df(\mathscr{C}) \subset \mathrm{int}\mathscr{C}$, we know that the angle between $\partial\mathscr{C}_{f^{-n}(x)}$ and $Df(\mathscr{C}_{f^{-n-1}(x)})$ is uniformly bounded independently of n. This implies that the cross-ratio cannot diverge and gives a contradiction. Hence V_x^+ (and similarly V_x^-) is a subspace.

Taking limits of subspaces contained in $Df^n(\mathscr{C}_{f^{-n}(x)})$ one know that $\dim(V_x^+) \geq \dim\mathscr{C} = d_+$ and similarly $\dim(V_x^-) \geq \dim\mathscr{C}^* = d_-$. Since $V_x^+ \cap V_x^- = \{0\}$, one deduces $T_xM = V_x^- \oplus V_x^+$. □

Now, to show that domination is verified in this invariant splitting, it is enough to check condition (2.1.2). By compactness, for a vector outside a small cone field around V_x^-, the number of iterates needed to belong to $\mathscr{C}$ is uniformly bounded (see exercise 11). Let us consider two unit vectors $u \in V_x^-$, and $v \in V_x^+$. The sum $u+v$ does not belong to a small cone around V_x^-. Hence, there exists a uniform $m \geq 1$ such that $Df^m(x).(u+v)$ belongs to a small cone around $V^+_{f^m(x)}$: This implies that $\|Df^m(x).v\|$ is larger than $\frac{1}{2}\|Df^m(x).u\|$ as required.

□

For proving a consequence of the cone-file criterion, we shall need the following result from differential geometry.

Proposition 2.7. *Any continuous linear bundle $E \subset T_KM$ over a compact set $K \subset M$ admits a continuous extension to a neighborhood of K.*

PROOF. Let us denote $d = \dim(M)$ and k the dimension of the spaces $E(x)$. For each point $x \in K$, one can choose a chart $\psi\colon U \to \mathbb{R}^d$ defined on a neighborhood of x such that $\psi_*(E)$ is transverse to $\{0\} \times \mathbb{R}^{d-k}$.

One can thus find a continuous family of linear maps $L_z\colon \mathbb{R}^k \to \mathbb{R}^{d-k}$ for $z \in K \cap U$ such that $\psi_*(E_z)$ is the graph of L_z for each z. One can extend continuously L_z to U (up to reduce U if necessary) by extending each of its coordinate.

Let us choose a finite number of charts $\psi_i\colon U_i \to \mathbb{R}^d$ and of compact sets $K_i \subset U_i \cap K$ whose union cover K and of continuous maps $L_i\colon U_i \to L(\mathbb{R}^k, \mathbb{R}^{d-k})$ whose graphs extend the bundle E on $K \cap U_i$. We will prove inductively that one can extend E on a neighborhood of $K_1 \cup \cdots \cup K_n$. On U_1, neighborhood of K_1, we already have extended E. Let us assume that E has been extended on a neighborhood V of $K_1 \cup \cdots \cup K_n$ as a bundle E'. Reducing V if necessary, the space $\psi_{n+1\,*}E'_x$ for $x \in V \cap U_{n+1}$ is the graph of a linear map $L'_x \in L(\mathbb{R}^k, \mathbb{R}^{d-k})$. Let us consider a continuous function $\varphi_V\colon [0,1]$ which equals 1 on a neighborhood of $K_1 \cup \cdots \cup K_n$ and 0 near the boundary of V, and another continuous function $\varphi_{n+1}\colon [0,1]$ which equals 1 on K_{n+1} and 0 near the boundary of U_{n+1}, We extend E on U_{n+1} as E'' such that E''_x for $x \in U_{n+1}$ is the graph of

$$(\varphi_{n+1}(x) + \varphi_V(x))^{-1} \times (\varphi_{n+1}(x)L_x + \varphi_V(x)L'_x).$$

It is continuous, coincides with E at points of $K \cap V_{n+1}$, We define $E'' = E'$ on a neighborhood of $K_1 \cup \cdots \cup K_n \setminus U_{n+1}$. By construction E'' is continuous. We have thus extended continuously E on a neighborhood of $K_1 \cup \cdots \cup K_{n+1}$ as required.

□

Now we prove the robustness of the domination.

Corollary 2.8. *Let K be a compact set invariant by a diffeomorphism f with a dominated splitting $T_K M = E_1 \oplus \cdots \oplus E_\ell$. Then, there exists a neighborhood U of K and a neighborhood $\mathscr{U}$ of f in* $\mathrm{Diff}^1(M)$ *such that any invariant compact set $K' \subset U$ for any diffeomorphism $g \in \mathscr{U}$ has also a dominated splitting $T_K M = E'_1 \oplus \cdots \oplus E'_\ell$, with* $\dim(E'_i) = \dim(E_i)$.

Moreover the splitting at a point x for a diffeomorphism g depends continuously on (x,g).

PROOF. It is enough to prove it for two bundles: $T_K M = E \oplus F$. By Proposition 2.5, these bundles are continuous. Let us consider a (non-invariant) continuous extension $\hat{E} \oplus \hat{F}$ of the splitting $E \oplus F$ to a neighborhood $\hat{U}$ of K, as given by Proposition 2.7. Considers $N > 0$ as in equation 2.1.2: by continuity, this condition still holds for the extended bundles. This implies that the cone-field defined by

$$\mathscr{C}_x := \{v = v_{\hat{E}} + v_{\hat{F}};\ \|v_{\hat{F}}\| \geq \|v_{\hat{E}}\|\}$$

is Dg-contracted in a possibly smaller neighborhood U of K for any diffeomorphism g that is C^1-close to f.

The cone field criterion is thus satisfied by g and g-invariant compact sets contained in U has a dominated splitting with the same dimensions as $E \oplus F$.

To obtain continuity, notice that given $\varepsilon > 0$ there exists $N > 0$ such that for every point $x \in \bigcap_{i=-N}^{N} f^i(U)$ one can consider the cone-field $Df^N(\mathscr{C})$ which is as narrow as one desires and contains F. Choosing g sufficiently C^1-close to f the same property holds, the cone-field is still Df-contracted and therefore the bundle F for g must also be contained in $Df^N(\mathscr{C})$.

□

Exercise 11. Let us assume that $\lambda := \max_{x \in K} \left\{ \|Df|_{E(x)}\| \,/\, m(Df|_{F(x)}) \right\}$ is smaller than one. Prove that spaces in $T_x M$ close to $F(x)$ get closer by iterations.

If F_1, F_2 are close to F_x, they are graphs of linear maps $L_1, L_2 \colon F_x \to E_x$ and their distance may be defined as $d(F_1, F_2) := \|L_2 - L_2\|$. Prove that $d(Df(F_1), Df(F_2)) < \lambda d(F_1, F_2)$.

2.3. **Uniform bundles.**

A Df-invariant sub-bundle $E \subset T_K M$ on K is *uniformly contracted* by Df if and only if there exist $c > 0$ and $\lambda \in (0,1)$ such that for any $x \in K$ and $u \in E(x)$,

$$\forall n \geq 0,\ \|D_x f^n u\| \leq c\lambda^n \|u\| \tag{2.3.1}$$

Similarly, the bundle is *uniformly expanded* if there exist $c > 0$ and $\lambda \in (0,1)$ such that for any $x \in K$ and $u \in E(x)$,

$$\forall n < 0,\ \|D_x f^n u\| \leq c\lambda^n \|u\| \tag{2.3.2}$$

As in the dominated case, one has equivalent definitions.

Exercise 12. A bundle E is uniformly contracted (resp. uniformly expanded) if there exists $N \geq 1$ (resp. $N \leq -1$) such that for $x \in K$, $u \in E(x)$,

$$\|D_x f^n u\| \leq \frac{1}{2}\|u\|$$

Exercise 13. Show that the definition of uniformly contracted bundle is independent of the Riemannian metric. Show that there always exists an *adapted norm*, i.e. such that one has condition (2.3.1) with $c = 1$.

2.4. **Adapted metrics.** In some situations it is comfortable to work with metrics on which one can see the domination in only one iterate. In some occasions, it is enough to work with an iterate, but sometimes it is better if one can do this for the diffeomorphism itself.

Definition 2.9. A Riemannian metric is *adapted* to a dominated splitting $T_K M = E_1 \oplus \cdots \oplus E_\ell$ if one can choose $c = 1$ in equation (2.1.1).

A Riemannian metric is *adapted* to a Df-uniformly contracted sub-bundle $E \subset T_K M$ if if one can choose $c = 1$ in equation (2.3.1).

Theorem 2.10 (Gourmelon [Gou$_1$])**.** *For any f-invariant compact set endowed with a dominated splitting $T_K M = E_1 \oplus \cdots \oplus E_\ell$, there exists a smooth Riemannian metric $\|.\|$ which is adapted to the dominated splitting and to any Df-uniformly contracted sub bundle and to any Df^{-1}-uniformly contracted sub bundle.*

PROOF. Let us consider a dominated splitting $T_K M = E \oplus F$. These bundles can be extended continuously to a small neighborhood U of K (but they are not invariant, see Proposition 2.7). We choose $N \geq 1$ large enough (and U small enough) so that

$$\|Df^N|_{E(x)}\| < \frac{1}{2} m(Df^N|_{F(x)})$$

This allows us to construct a function $r : U \to \mathbb{R}$ such that for every $x \in U$:

$$\|Df^N|_{E(x)}\|^{1/N} < r(x) < m(Df^N|_{F(x)})^{1/N}$$

Denote $R_n(x) = \prod_{i=0}^{n-1} r(f^i(x))$. Note that $\frac{\|Df^n|_{E(x)}\|}{R_n(x)}$ as well as $\frac{R_n(x)}{m(Df^n|_{F(x)})}$ converge exponentially to 0 as $n \to +\infty$ for every $x \in K$ (see Lemma 2.12 below).

Therefore, for each $u \in E(x) \setminus \{0\}$ one can set

$$\|u\|_E^2 = \sum_{n=0}^{+\infty} \frac{\|D_x f^n u\|^2}{R_n(x)^2}$$

which is well defined. One can then compute:

$$\|D_x f u\|_E^2 = r(x)^2 \left(\|u\|_E^2 - \|u\|^2\right) < r(x)^2 \|u\|_E^2$$

A symmetric argument provides a metric $\|\cdot\|_F$ in $F(x)$ such that for $v \in F(x) \setminus \{0\}$:

$$\|D_x f v\|_F > r(x)\|v\|_F.$$

Let $\|.\|'$ be the (continuous) metric which coincides with $\|.\|_E$ on E and with $\|.\|_F$ on F and which makes the bundles E and F orthogonal. This gives

$$\frac{\|D_x f v\|'}{\|v\|'} > r(x) > \frac{\|D_x f u\|'}{\|u\|'}$$

and by compactness, there exists $\lambda \in (0,1)$ such that the desired condition holds:

$$\frac{\|D_x f u\|'}{\|u\|'} \leq \lambda \frac{\|D_x f v\|'}{\|v\|'}.$$

The metrics $\|.\|$ and $\|.\|'$ can be glued together outside a neighborhood of K (as a barycenter). Any smooth metric close to the obtained metric is adapted to the splitting.

Finally, let us notice that if, for instance, the bundle E admits a finer splitting $E = E_1 \oplus E_2$ and if the initial metric $\|.\|$ was adapted to this splitting, then the new metric is still adapted to this splitting (since it is obtained by averaging the iterates of the initial metric).

Similarly if E_1 is uniformly contracted and $\|.\|$ is adapted to this bundle, then the new metric is also adapted to the contraction of E_1.

Let us now consider a dominated splitting $T_K M = E \oplus F$ such that E is uniformly contracted (the case F is uniformly expanded is similar).

If the bundle E is contracted, one chooses N large enough such that $\|Df^N|_{E(x)}\| < \frac{1}{2}$.

One deduces that $\|D_x f u\|' \leq 2^{-1/N}\|u\|'$, hence the new metric is adapted to the contracted bundle E. Similarly it is adapted to F if Df^{-1} contracts F.

One can thus obtain a metric adapted to the finest dominated splitting by applying inductively the process above to the different decompositions into two bundles.

□

Remark 2.11. Instead of constructing a unique function r one could have constructed functions r_1, r_2 such that $r_1 < \frac{1}{2} r_2$:

$$\|Df^N|_{E(x)}\|^{1/N} < r_1(x) < r_2(x) < m(Df^N|_{F(x)})^{1/N}.$$

This allows one to estimate the value of λ in about $2^{-1/N}$.

We end this section by proving the following lemma which we used in the proof of the previous theorem.

Lemma 2.12. *With the notations of the proof of Theorem 2.10 one has that* $\frac{\|Df^n|_{E(x)}\|}{R_n(x)}$ *as well as* $\frac{R_n(x)}{m(Df^n|_{F(x)})}$ *converge exponentially to* 0 *as* $n \to +\infty$ *for every* $x \in K$.

PROOF. We prove $\frac{\|Df^n|_{E(x)}\|}{R_n(x)} \to 0$ exponentially as the other result is analogous. One chooses $q \geq 0$ such that for any $0 \leq i \leq N-1$ one can write $n = i + qN + \ell$ with $0 \leq \ell \leq 2N-1$. Hence:

$$\|Df^n|_{E(x)}\| \leq \|Df^\ell|_{E(f^{qN+i}(x))}\| \left(\prod_{j=0}^{q-1} \|Df^N|_{E(f^{jN+i}(x))}\| \right) \|Df^i|_{E(x)}\|.$$

By compactness one knows that there is $c > 0$ such that $\|Df^j|_{E(y)}\| \leq c$ and $r(y)c > 1$ for any $0 \leq j \leq N-1$ and $y \in K$. Therefore, for every $0 \leq i \leq N-1$ one gets

$$\|Df^n|_{E(x)}\| \leq c^2 \left(\prod_{j=0}^{q-1} \|Df^N|_{E(f^{jN+i}(x))}\| \right). \tag{2.4.1}$$

Notice that by the definition of r and compactness of K there exists $\lambda \in (0,1)$ such that $r(y)^N \geq \lambda \|Df^N|_{E(y)}\|$ so:

$$\prod_{i=0}^{N-1} \left(\prod_{j=0}^{q-1} \|Df^N|_{E(f^{jN+i}(x))}\| \right) \leq \lambda^{qN} R_{qN}(x)^N \leq (c/\lambda)^{2N} \lambda^n R_n(x)^N. \tag{2.4.2}$$

Putting equations (2.4.1) and (2.4.2) together one obtains:

$$\frac{\|Df^n|_{E(x)}\|}{R_n(x)} \leq \frac{c^4}{\lambda^2} \lambda^{n/N}$$

as desired.

□

2.5. **Partial hyperbolicity.** We shall define the notion of partial hyperbolicity which is the center of these notes.

Definition 2.13. A splitting $T_K M = E_1 \oplus \cdots \oplus E_\ell$ is *partially hyperbolic* if it is dominated and either E_1 is uniformly contracted by Df or E_ℓ is uniformly contracted by Df^{-1}. An invariant set K is partially hyperbolic if it admits a partially hyperbolic splitting.

In a similar way as for dominated splittings, the following properties are verified:

Exercise 14. Show that:

(a) The definition is independent of the metric.
(b) A set K is partially hyperbolic for f if and only if it is partially hyperbolic for f^n, $n \neq 0$.
(c) The partial hyperbolicity is *robust*: it is satisfied for the invariant sets in a neighborhood of $\overline{K}$ for the diffeomorphisms C^1-close to f.

Notation and terminology. To emphasize the uniform contraction we will denote the dominated splitting as

$$TM = E^s \oplus E_1^c \oplus \cdots \oplus E_\ell^c \oplus E^u.$$

The bundles E^s and E^u are the *stable* and *unstable bundles*. The bundles E_i^c are the *center bundles*. Since the dominated splitting is not unique in general, there may exist different partially hyperbolic splittings. The bundles E^s and E^u are also sometimes called *strong stable* and *strong unstable* bundles and denoted E^{ss} and E^{uu} (or E^{su}) to distinguish them from stable/unstable bundles with larger dimensions.

Remark 2.14. a) When the center bundles are degenerate the set K is *hyperbolic*.

b) *Variants:* Sometimes one may ask for both extremal bundles to be uniform. We shall call that notion *strong partial hyperbolicity*.

Sometimes an even stronger form of partial hyperbolicity is considered (called *absolute partial hyperbolicity*). It requires the existence of constants $\lambda_1 < \mu_1 < \cdots < \lambda_\ell < \mu_\ell$ such that for each $x \in K$ and each vectors $u_i \in E_{i,x}$, one has $\lambda_i \|u_i\| < \|D_x f u_i\| \leq \mu_i \|u_i\|$.

c) *Volume partial hyperbolicity:* Under some robust dynamical conditions, the following variant appears naturally as the first obstruction to the coexistence of infinitely many sinks or sources. We say that a set K is *volume partially hyperbolic* if its finest dominated splitting $T_K M = E_1 \oplus \ldots \oplus E_\ell$ verifies that E_1 is *volume contracting* and E_ℓ is *volume expanding*. Volume contraction (resp. expansion) means that the jacobian $\mathrm{Jac}(D_x f^n|_{E_1})$ of $D_x f^n$ along E_1 is uniformly contracted (resp. expanded): there exist $c > 0$ and $\lambda \in (0,1)$ such that $|\mathrm{Jac}(D_x f^n|_{E_1})| < c\lambda^n$ (resp. $> c\lambda^{-n}$).

One then defines:

Definition 2.15. A diffeomorphism f is *globally partially hyperbolic* if the whole manifold M admits a partially hyperbolic splitting for f. Similarly, f is *globally strongly partially hyperbolic* if the whole manifold is a strongly partially hyperbolic set.

Remark 2.16. As in the introduction, any diffeomorphism admitting a filtration $U_0 \subset \cdots \subset U_m$ such that for each $i = 1, \ldots, m$, the maximal invariant compact set $\bigcap_{n\in\mathbb{Z}} f^n(U_i \setminus U_{i-1})$ is partially hyperbolic will be called *partially hyperbolic*.

Exercise 15. (1) Show that if a partially hyperbolic set K has a splitting $T_K M = E^s \oplus E^c \oplus E^u$ but E^c and E^u are $\{0\}$ then K is a finite union of periodic sinks.

(2) Show that if f is a partially hyperbolic diffeomorphism (with the definition given in Remark 2.16) then it has finitely many sinks and sources.

2.6. **Appendix: Dominated splitting and non-uniform hyperbolicity.** Another way to relax the uniform hyperbolicity is to introduce a measurable notion (called *non-uniform hyperbolicity*). At the end of this section we describe some interactions between dominated splitting and non-uniform hyperbolicity.

Invariant ergodic measures can be thought of as a generalization of periodic orbits.

Theorem 2.17 (Oseledets)**.** *Let $f : M \to M$ be a C^1-diffeomorphism and μ an ergodic measure. Then, there exists $k \in \mathbb{Z}^+$, real numbers $\chi_1 < \chi_2 < \ldots < \chi_k$ and for x in a f-invariant full measure set $R^\mu(f)$ a splitting $T_xM = E_1(x) \oplus \ldots \oplus E_k(x)$ with the following properties:*

- *(***Measurability***) The functions $x \mapsto E_i(x)$ are measurable.*
- *(***Invariance***) $D_xf(E_i(x)) = E_i(f(x))$ for every $x \in R^\mu(f)$.*
- *(***Lyapunov exponents***) For every $x \in R^\mu(f)$ and $v \in E_i(x) \setminus \{0\}$ one has*
$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_xf^n v\| = \chi_i$$
- *(***Subexponential angles***) For every $x \in R^\mu(f)$ and vectors $v_i \in E_i(x)$ and $v_j \in E_j(x)$ one has that:*
$$\lim_{n\to\pm\infty} \frac{1}{n} \log \sin \measuredangle\left(\frac{D_xf^n v_i}{\|D_xf^n v_i\|}, \frac{D_xf^n v_j}{\|D_xf^n v_j\|}\right) = 0$$

Some explainations are in order:

Lyapunov exponents. The numbers χ_i appearing in the statement of Theorem 2.17 are usually called *Lyapunov exponents* of μ.

In general, for any diffeomorphism f a point $x \in M$ is called *regular* (or *Lyapunov regular*) if there exists a splitting $T_xM = E_1(x) \oplus \ldots \oplus E_{k(x)}(x)$ and numbers $\chi_1(x) < \chi_2(x) < \ldots < \chi_{k(x)}(x)$ such that for any vector $v \in E_i \setminus \{0\}$ one has

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_xf^n v\| = \chi_i(x).$$

Exercise 16. Show that if $x \in M$ is a regular point and $v \in \bigoplus_{j=1}^{i} E_j(x) \setminus \bigoplus_{j=1}^{i-1} E_j(x)$ then

$$\lim_{n\to+\infty} \frac{1}{n} \log \|D_xf^n v\| = \chi_i(x).$$

In particular, every regular point verifies that every vector has a well defined Lyapunov exponent for the future (and the past). The bundles E_i are the sets on which both coincide.

The set of regular points $R(f)$ is f-invariant and Oseledets theorem implies that it has measure 1 for every f-invariant probability measure. It also holds that all the involved functions are measurable on $R(f)$ which is a measurable set.

Notice that every periodic point has positive measure for an invariant measure (namely the one that gives equal weight to each point in the orbit) and therefore must be regular. Of course, one does not need Oseledets theorem to prove this. Notice that if $f^n(p) = p$, then the Lyapunov exponents of p are the logarithms of the modulus of the eigenvalues of D_pf^n divided by n.

The *non-uniformly hyperbolic set* of f is the set of regular points for which all Lyapunov exponents are different from 0, that is, the set of points $x \in R(f)$ such that $\chi_i(x) \neq 0$ for all $1 \leq i \leq k(x)$. A measure μ is called *(non-uniformly) hyperbolic* if all its Lyapunov exponents are non-zero. One should be careful with this name, the *non* applies to the uniformity and not to the hyperbolicity and it should be understood as "*not necessarily uniformly hyperbolic but still with a non-uniform form of hyperbolicity*".

Consider an ergodic (non-uniformly) hyperbolic measure μ for which one has Lyapunov exponents $\chi_1 < \ldots < \chi_i < 0 < \chi_{i+1} < \ldots < \chi_k$ one can group the bundles depending on the sign of the Lyapunov exponent. In this case, we denote $E^s(x) = E_1(x) \oplus \ldots \oplus E_i(x)$ and $E^u(x) = E_{i+1}(x) \oplus \ldots \oplus E_k(x)$. Note that if $v^s \in E^s(x) \setminus \{0\}$ and $v^u \in E^u(x)$ then:

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v^s\| < 0 < \lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n v^u\|$$

So that vectors in $E^s(x)$ are the ones which are exponentially contracted in the future by Df and vectors in E^u are exponentially contracted in the past by Df.

Angles and measurability. We remark that, differently from the case of periodic orbits, the concept of norm and angle are essential in this setting as they provide a way to compare vectors which do not belong to the same vector space. However:

Exercise 17. The values of the Lyapunov exponents are independent of the choice of the Riemannian metric in TM.

The Riemannian metric also provides a way to compute angles between vectors and this is the sense one has to give to the last part of the statement of Theorem 2.17. It is possible to show that this last part is a consequence of the rest, but it is so important that it merits to appear explicitly in the statement.

Another relevant comment is about the notion of measurability of the functions $x \mapsto E_i(x)$. This should be understood in the following way: the arrow defines a function from M to the space of subspaces of TM. This can be thought of as a fiber bundle over M in the following way, for a given $j \leq d = \dim M$ one considers $G_j(M)$ to be the fiber bundle over M such that the fiber in each point is the Grasmannian space of T_xM of subspaces of dimension j. This is well known to have a manifold structure and provide a fiber bundle structure over M([2]). This gives a sense to measurable maps from M to some of these Grasmannian bundles, and since one does not a priori require that the bundles have constant dimension one can think of the function E_i to be a function from M to the union of all these bundles and then the measurability of the function makes sense as both the domain and the target of the function are topological spaces.

Non-ergodic measures. There is a statement for non-ergodic measures which is very much like the one we stated but for which the constants k and χ_i become functions of the points and some other parts become more tedious. Look [KH, Supplement] or [Ma$_4$, Chapter IV.10] for more information and proofs of this result.

Relationship with dominations. We propose the following:

Exercise 18. Let $f : M \to M$ be a C^1-diffeomorphism, $K \subset M$ a compact invariant set and $T_KM = E \oplus F$ a continuous Df-invariant splitting.

(a) Show that if μ is an ergodic measure which has simple spectrum (i.e. all Lyapunov exponents have multiplicity one), then the Oseledet's splitting respects $E \oplus F$ (i.e. every bundle is contained either on E or F).

[2]For example, if $j = 1$ this is the projective bundle over M.

(b) Show that E is uniformly contracted if and only if for every ergodic measure μ supported on K, the largest Lyapunov exponent of μ along E is negative.
(c) Show that $E \oplus F$ is dominated if and only if for every ergodic μ supported on K, the smallest Lyapunov exponent of μ along F is strictly larger than the largest Lyapunov exponent of μ along E.
(d) Show that E is uniformly volume contracted if and only if for every ergodic μ supported on K, the sum of the Lyapunov exponents of μ along E is negative.

3. EXAMPLES

This section will present several examples of partially hyperbolic dynamics. It also serves as a way to show how partially hyperbolic dynamics arise in different areas of mathematics.

We recall that we are adopting the weak notion of partial hyperbolicity by assuming that there exists a filtration and such that every set of the filtration admits a partially hyperbolic splitting. However, many of the examples we shall present posses the much stronger property of being *globally* partially hyperbolic. We will also present *local* examples, by this we mean compact invariant sets admitting a partially hyperbolic splitting (without checking if it is possible to extend this to a filtration in a manifold).

Global partial hyperbolicity imposes (or at least one expects to impose) strong restrictions on the topology of the manifold and isotopy class in which one works. Local constructions can usually be embedded in every isotopy class of any manifold. The understanding of this mechanisms is not well understood, even for hyperbolic dynamics, and we therefore do not attempt here to present this kind of problems in detail (see [Fr_3] for the study of topological restrictions of hyperbolic dynamics).

Before we start giving a partial list of examples, let us mention that, being a robust property, all C^1-perturbations of our examples will also be examples of partially hyperbolic dynamics. In contrast with the hyperbolic setting, where perturbations are, up to topological change or coordinates, the same examples, in the partially hyperbolic setting, small perturbations might provide examples with quite different dynamical properties (even if the amount of hyperbolicity required for partial hyperbolicity will allow us to show later that some properties do persist after perturbation).

3.1. **Algebraic examples.**

By algebraic example we mean an example which arises via an algebraic construction. Typically, one looks for diffeomorphisms of Lie groups which preserve some algebraic structure and such that they descend to some compact quotient.

There are at least two kind of algebraic examples. Those arising from *automorphisms of the group*, and those arising from *translations*. Combining them, one obtains what it is sometimes called *affine examples*. We first explain the philosophy behind this type of construction and then present some concrete examples.

Let us first consider those examples arising from *automorphisms of a Lie group*. Consider a Lie group G (with associated Lie algebra $\mathfrak{g} \simeq T_eG$) which admits a compact quotient $G/_\Gamma$ by a closed subgroup $\Gamma < G$ (typically a cocompact lattice) and an automorphism $\varphi : G \to G$ with the following two properties:

- It preserves Γ, i.e. $\varphi(\Gamma) \subset \Gamma$ and $\varphi^{-1}(\Gamma) \subset \Gamma$.
- Its induced tangent map $\Phi := D_e\varphi$ (which can be seen as a Lie algebra automorphism $\Phi : \mathfrak{g} \to \mathfrak{g}$) is a partially hyperbolic linear map (i.e. it has at least one eigenvalue larger than 1 not contained in the Lie algebra[3] of Γ).

Notice that in many situations one can choose an invariant volume form which will be preserved by φ.

Now we present the candidate *translations* for examples of partially hyperbolic diffeomorphisms.

A typical Lie group G admits a one-sided invariant metric. We shall chose a right invariant one obtained by choosing a metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} = T_eG$ and transporting it by right

[3] In the most relevant case where Γ is discrete, its Lie algebra is trivial.

translations. In general, this metric will not be invariant under left translations, but there are many situations where one knows that the volume form defined by the metric is invariant[4] under left translations (these are called *unipotent* Lie groups). All the examples below have this property. The fact that the metric is not necessarily invariant under left translations is a good thing in our search for partially hyperbolic examples since it means that a left translation is a candidate for a partially hyperbolic element (if left translations were isometries there would be no hope that this happens).

Choose again a closed subgroup $\Gamma < G$ such that $G/_{\Gamma}$ is compact and an element $g \in G$. The map $L_g : G \to G$ defined as $x \mapsto g \cdot x$ is a diffeomorphism and preserves co-classes of Γ, this is, $L_g(x\Gamma) = g \cdot x\Gamma = gx\Gamma$ and thus induces a diffeomorphism ℓ_g of $G/_{\Gamma}$.

To see if it is partially hyperbolic, one can look at the derivative of L_g, but for doing this it is natural to look at the action in the Lie algebra. So, using the fact that right translations are isometries, one can look at the linear map in the tangent space of the identity given by $v \mapsto D_g R_{g^{-1}} \circ D_e L_g v$ (which is a linear automorphism of $\mathfrak{g}$), sometimes called the *adjoint map*[5] and denoted as $Ad(g) : \mathfrak{g} \to \mathfrak{g}$.

It follows that if $Ad(g)$ is partially hyperbolic then L_g defines a partially hyperbolic diffeomorphism on $G/_{\Gamma}$. Indeed, if $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is an invariant splitting under $Ad(g)$ where the eigenvalues of $Ad(g)$ along $\mathfrak{g}_1$ are strictly smaller than those of $Ad(g)$ along $\mathfrak{g}_2$ then one can transport the subspaces $\mathfrak{g}_i$ by right translations to all points in $G/_{\Gamma}$ and this will be an invariant splitting for DL_g. Moreover, since the right translations are isometries of the chosen metric, the contractions and expansions seen by $Ad(g)$ will be seen in every point in the same way.

Both examples can be included in a more general family, that of *affine maps*. This are diffeomorphisms $f : G/_{\Gamma} \to G/_{\Gamma}$ whose lift $\tilde{f}$ to G is of the form $\tilde{f} = L_g \circ \varphi$ where φ is an automorphism and L_g is a left translation. The condition is now that $Ad(g) \circ \Phi$ is partially hyperbolic.

Notice that in general, if $\Psi : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism, its eigenspaces need not be Lie subalgebras. However:

Exercise 19. Show that the subspace corresponding to eigenvalues of modulus larger than one (resp. smaller than one, one, larger or equal to one, smaller or equal to one) form a subalgebra of $\mathfrak{g}$. Show that the smallest subalgebra containing the eigenspaces corresponding to eigenvalues of modulus different from one forms an ideal[6] of $\mathfrak{g}$.

It is also interesting to remark that algebraic examples constructed this way are always *absolutely* partially hyperbolic.

In this subsection we shall produce some instances of this method to show both that several interesting examples appear this way, but also that verifying both conditions (partial hyperbolicity of the linear map, and preservation of the lattice) is not automatic and several difficulties arise. The study of algebraic examples alone comprises a large research topic which we do not attempt to survey completely as we present them only to have concrete examples of partially hyperbolic diffeomorphisms (see [KSS] and references therein for a more detailed account on homogeneous dynamics).

[4] This is always the case when there exists a cocompact lattice.

[5] It is the derivative of the automorphism of G induced by conjugation by g.

[6] This is important in the study of accessibility, mainly when one works with simple Lie algebras where the unique ideals are the whole Lie algebra.

3.1.1. *Torus automorphisms.* The simplest construction is obtained by considering an element $A \in \mathrm{SL}(d,\mathbb{Z})$: it induces a diffeomorphism f_A on $\mathbb{T}^d$. The diffeomorphism is partially hyperbolic if A has at least least one eigenvalue smaller than 1 in modulus (which automatically implies that there is one eigenvalue larger than 1 in modulus and therefore f_A is strongly partially hyperbolic).

For instance the following has two complex eigenvalues of modulus one and two real eigenvalues different from ± 1.

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$

This family of examples includes Anosov diffeomorphisms of tori, which are the ones for which there is no eigenvalue of modulus one. This examples can usually be seen as partially hyperbolic in many different ways according on how one chooses to group the eigenvalues and their respective eigenspaces.

In dimension 3, there are several possibilities which by algebraic reasons can be divided in the following cases:

- There is an eigenvalue of modulus one. In this case, this eigenvalue is unique and real (it is ± 1) and it follows that f_A preserves a foliation by compact circles corresponding to the projection of the eigenspace associated to this eigenvalue. The action of f_A in such circles is an isometry and this corresponds to the center direction.
- There are three different real eigenvalues all of them of modulus different from 1. In this case, one can see f_A as strongly partially hyperbolic in two different ways: One can consider that f_A has no center direction, or one can consider the middle eigenvalue as the center direction.
- There are two complex conjugate eigenvalues. In this case, f can only be regarded as strongly partially hyperbolic by considering the center direction to be trivial.

The possible ways to see the diffeomorphisms of the type f_A as strongly partially hyperbolic grow as one increases the dimension. The same happens if one allows to consider partially hyperbolic splittings in general (not necessarily strong).

We end this description by noticing that translations in $\mathbb{T}^d$ are isometries so that one can compose f_A with any translation to obtain an affine diffeomorphism which will also be partially hyperbolic.

3.1.2. *Nilmanifolds.* A generalization of the previous family of examples is to work in *nilmanifolds*, which can be seen as a generalization of tori from the point of view of Lie groups. Indeed, tori are compact quotients of simply connected abelian Lie groups and nilmanifolds are, by definition, compact quotients of nilpotent Lie groups.

The first important reason to consider this examples is the following result from Lie algebra theory which we shall not prove:

Proposition 3.1. *Let $\mathfrak{g}$ be a Lie algebra and $\Phi : \mathfrak{g} \to \mathfrak{g}$ a linear Lie algebra automorphism such that Φ has no eigenvalues which are roots of unity. Then, $\mathfrak{g}$ is nilpotent.*

Just to give a taste of the idea, consider eigenvectors x and y with eigenvalues λ and μ respectively, then $[x, y]$ is also an eigenvector, and if it is non-zero, then its eigenvalue is $\lambda\mu$. One deduces that $ad(x)^k y$ is either zero or an eigenvector of eigenvalue $\lambda^k\mu$. Since λ is not a root of unity, one deduces that eventually $ad(x)^k y$ vanishes. This implies that for every eigenvector $ad(x)$ is nilpotent. We refer the reader to [Sm, Section I.3] and references therein for more details on this. This proposition shows that in order to construct algebraic Anosov diffeomorphisms, the only hope is to work in nilmanifolds.

Borel-Smale examples (presented in [Sm, Section I.3] along with an account on how they were concieved) are examples of algebraic Anosov diffemorphisms in nilmanifolds. We refer the reader to [BuW] (and references therein) for a clear and detailed account on this examples. Borel-Smale examples live in 6-dimensional nilmanifolds. There is by now a good understanding of which nilpotent groups admit hyperbolic automorphisms which descend to compact quotients in low dimensions (up to dimension 8), but the general problem remains wide open, see [LW].

Here, we will present an easier set of examples, not of Anosov but of partially hyperbolic examples, which can be constructed in 3-dimensional nilmanifolds.

Exercise 20. Show that the only 3-dimensional nilpotent Lie algebras that admits hyperbolic automorphisms preserving a cocompact lattice is $\mathbb{R}^3$.

Consider $\mathcal{H}$ to be the *Heissenberg group*, this is, the group of 3×3-matrices of the form:

$$\mathcal{H} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \ : \ x,y,z \in \mathbb{R} \right\}$$

It is easy to check that it is closed under matrix products and therefore it is a Lie group (seen as a subgroup of $SL(3,\mathbb{R})$). Its Lie algebra $\mathfrak{h}$ is generated by the following three matrices:

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ , \ Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and as it can be easily checked one has that the only non-trivial bracket relation is $[X,Y] = XY - YX = Z$.

Consider the lattice $\mathcal{H}_k$:

$$\mathcal{H}_k = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \ : \ x,y \in \mathbb{Z} \ , \ z \in \frac{1}{k}\mathbb{Z} \right\}.$$

Exercise 21.
(1) Show that $\mathcal{H}_k$ is a subgroup of $\mathcal{H}$ but it is not normal.
(2) Show that $N_k = \mathcal{H}/_{\mathcal{H}_k}$ is compact.
(3) Show that N_k is a circle bundle over the torus whose Euler number is k.

To have an example of a partially hyperbolic diffeomorphism of N we need a partially hyperbolic Lie algebra automorphism which induces an automorphism of $\mathcal{H}$ and preserves some lattice $\mathcal{H}_k$. This is provided (for example) by the following automorphism:

$$(x,y,z) \mapsto (2x+y, x+y, z+x^2+\frac{y^2}{2}+xy)$$

which clearly preserves $\mathcal{H}_2$.

Exercise 22. Show that the automorphism defined above induces a partially hyperbolic diffeomorphism of N_2. Study which are the invariant bundles and show that the center direction integrates into a foliation by circles.

3.1.3. *Suspensions of Anosov automorphisms.* Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map with integer entries and determinant of modulus 1, that is, $A \in GL(d,\mathbb{Z})$. Let $A = PDP^{-1}$ the Jordan form of A and for $t \in \mathbb{R}$ we denote $A^t = PD^tP^{-1}$ (here D^t denotes the exponential at time t of the matrix $d \in \mathfrak{gl}(d,\mathbb{R})$ such that $e^d = D$).

Now, consider the following Lie group defined as the following crossed product:

$$G_A = \mathbb{R}^d \ltimes_A \mathbb{R}$$

We recall that this means that the product between elements (x,t) and (y,s) of G_A (with $x, y \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$) verifies:

$$(x,t)\cdot(y,s) = (A^s x + y, t+s)$$

A natural lattice in G_A is given by the subgroup $\Gamma_A := \mathbb{Z}^2 \ltimes_A \mathbb{Z}$ which consists of the elements of the form (m,n) with $m \in \mathbb{Z}^2$ and $n \in \mathbb{Z}$.

Consider $L_{(0,t)}$ the left translation by the element $(0,t)$. One has that $L_{(0,t)}(x,s) = (x, s+t)$. On the other hand, if one looks at $Ad((0,t)) : \mathfrak{g}_A \to \mathfrak{g}_A$ it is given by the derivative at $(0,0)$ of the map $R^{-1}_{(0,t)} L_{(0,t)}$ which maps $(x,s) \mapsto (A^t x, s)$. This map is partially hyperbolic if A is (it has the same eigenvalues plus the eigenvalue 1 associated to the tangent space of the curve $t \mapsto (0,t)$). When A is hyperbolic, the flow $\psi_t : G_A/_{\Gamma_A} \to G_A/_{\Gamma_A}$ is an *Anosov flow* as we shall see later.

The same construction can be made starting from automorphisms of nilmanifolds. Also, one can make crossed products with $\mathbb{R}^k$ (see for example [KKRH, Section 2.1]) or even more complicated Lie groups.

3.1.4. *Geodesic flows in surfaces of negative curvature.* Let $\mathbb{H}^2 = \{z \in \mathbb{C} : \Im(z) > 0\}$ be the hyperbolic plane with the usual metric given by $\|v\|_{\mathbb{H}^2} = \frac{\|v\|_{\mathbb{R}^2}}{\Im(z)}$ where $v \in T_z\mathbb{H}^2 \cong \mathbb{R}^2$.

It is standard to identify $T^1\mathbb{H}^2$ (the unit tangent bundle of the hyperbolic plane) with $PSL(2,\mathbb{R})$ via Möbius transformations. Indeed, given a point $z \in \mathbb{H}^2$ and a unit tangent vector v in $T_z\mathbb{H}^2$ one has that there exists a unique Möbius transformation which preserves $\mathbb{H}^2$ and sends the point i to z while the vertical vector is sent to v. This correspondence is smooth and it can be easily seen that Möbius transformations act as isometries of $\mathbb{H}^2$.

A *Fuchsian group* is a discrete subgroup of $PSL(2,\mathbb{R})$. A Fuchsian group Γ is called of *first type* if its limit set consists in the whole $\partial\mathbb{H}^2$, this means, that if one considers any point $x \in \mathbb{H}^2$, then its orbit $x \cdot \Gamma$ accumulates in the whole $\mathbb{R} \cup \{\infty\} = \partial\mathbb{H}^2$. A classical example of Fuchsian group of first type is the choice of a hyperbolic metric on a surface of genus ≥ 2. Other examples consist on branched quotients of such surfaces giving rise to hyperbolic orbifolds. It can be seen that even if the action in $\mathbb{H}^2$ may have torsion elements (and therefore it is not free) the action in $PSL(2,\mathbb{R}) \cong T^1\mathbb{H}^2$) is free and properly discontinuous (notice that if a Möbius transformation which is not the identity fixes a point in $\mathbb{H}^2$ then it must be a rotation around that point; since the group is discrete, the rotation is rational, and therefore it is free in the circle fiber of the fixed point). We now fix a Fuchsian group of first type Γ acting on $PSL(2,\mathbb{R})$ with quotient $M = PSL(2,\mathbb{R})/_\Gamma$.

We consider the action on $PSL(2,\mathbb{R})$ by left translations of the element

$$a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

defining the maps $\ell_{a_t} : M \to M$. To see that these are partially hyperbolic one has to compute the adjoint representation and it follows that in the Lie algebra one has the following 3-invariant subspaces with eigenvalues smaller, equal and larger to one respectively:

$$\mathfrak{g}^s = \langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rangle \ , \ \mathfrak{g}^c = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle \ , \ \mathfrak{g}^u = \langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle$$

Exercise 23. Compute the adjoint representation and check the claims made above.

One can check that the flow ℓ_{a_t} is *Anosov* (equivalently, that its time one map is partially hyperbolic) by more geometric and explicit calculations. This flow is in fact the geodesic flow of the hyperbolic metric induced by Γ when Γ is the fundamental group of a hyperbolic surface. See for example [KH, Sections 5.4 and 17.5].

3.1.5. *Other interesting algebraic examples.* Other instances of this construction are geodesic flows on rank 1 symmetric spaces (negatively curved), frame flows, and many others. We briefly explain the construction of the frame flow as an algebraic example since it is a genuinely partially hyperbolic flow whose study appears historically as one of the strong motivations for the study of partially hyperbolic systems. We refer the reader to [KH, Section 17.7] for an account on geodesic flows on rank 1 symmetric spaces and again to [KSS] and references therein for more examples of algebraic diffeomorphisms. We also recommend the notes [Q] where rank one symmetric spaces are presented with more detail.

Let $G = \mathrm{Isom}(\mathbb{H}^d)$ the Lie group of isometries of the hyperbolic space $\mathbb{H}^d$. It is possible to identify it with a well known matrix group. Consider the Lie group $\mathrm{SO}(1,d)$ consisting of matrices in $\mathrm{SL}(d+1,\mathbb{R})$ preserving the linear form q such that $q(x) = x_0^2 - x_1^2 - \ldots - x_d^2$ where $x = (x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1}$. It follows that every $g \in \mathrm{SO}(1,d)$ preserves both connected components of $x_0^2 - x_1^2 - \ldots - x_d^2 = 1$ since the determinant is positive. It is well known that $\mathbb{H}^d$, the d-dimensional hyperbolic space, is isometric to the connected component of $x_0^2 - x_1^2 - \ldots - x_d^2$ containing $(1,0,\ldots,0)$ with the metric induced by the quadratic form q, this way, one obtains the identification $\mathrm{Isom}(\mathbb{H}^d) \cong \mathrm{SO}(1,d)$. For details, see [Q, Section 2.2.1].

The stabilizer of a point consists on the matrices fixing $(1,0,\ldots,0)$ which can be identified with the group of rotations of $\mathbb{R}^d$, this is, $\mathrm{SO}(d)$, and the quotient $\mathrm{SO}(1,d)/_{\mathrm{SO}(d)} \simeq \mathbb{H}^d$ is the symmetric space, and so if Γ is a group of isometries of $\mathbb{H}^d$ which has a compact quotient then the quotient of $\mathrm{SO}(1,d)$ by Γ is also compact. Existence of such cocompact lattices is not immediate, a large such family can be obtained by algebraic methods (see [Ben, Chapter 2]).

Being rank-1 means that there is a distinguished 1-parameter subgroup $\{a_t\}_{t\in\mathbb{R}}$ which, when acting in the symmetric space of $\mathrm{SO}(1,d)$ represents the geodesic flow of this group, meaning that the projection of an orbit of the parameter subgroup to the symmetric space $\mathbb{H}^d$ is a (parametrized) geodesic of $\mathbb{H}^d$. It follows that when considering the action of $\{a_t\}_{t\in\mathbb{R}}$ in $\mathrm{SO}(1,d)/_{\Gamma}$ one obtains a lift of the geodesic flow to the fiber bundle over the manifold with structure group $\mathrm{SO}(d)$. The group $\mathrm{SO}(d)$ can be identified with the choice of an orthonormal basis in $T_x\mathbb{H}^d$. This flow is known as *frame flow* over $M = \mathbb{H}^d/_{\Gamma}$. It is not hard to see that this flow is partially hyperbolic and can be checked directly by computing the adjoint $Ad(a_1)$ of a_1.

3.2. **Skew-products.** Consider a partially hyperbolic diffeomorphism $f : M \to M$ with splitting of the form $TM = E^s \oplus E^c \oplus E^u$ and N a closed manifold. Now, consider a continuous map $g : M \to \mathrm{Diff}^1(N)$ verifying that for every $x \in M$ one has that:

$$\|Df|_{E^s(x)}\| < m(D_t g_x) \leq \|D_t g_x\| < m(Df|_{E^u(x)}) \qquad \forall t \in N \tag{3.2.1}$$

Then, if $p\colon M \times N \to M$ is the first projection and if one defines $F: M \times N \to M \times N$ as

$$F(x,t) = (f(x), g_x(t))$$

it is not hard to show that F is partially hyperbolic with splitting $T(M \times N) = \hat{E}^s \oplus \hat{E}^c \oplus \hat{E}^u$ with $\dim(\hat{E}^\sigma_x) = \dim(E^\sigma_{p(x)})$, $\sigma \in \{s,u\}$.

Exercise 24. Use the cone-field criteria to show that F is partially hyperbolic with the splitting announced above. Use uniqueness of the splitting to show that $\hat{E}^c_x = Dp_x^{-1}(E^c_{p(x)})$. (In particular $TN \subset \hat{E}^c$.) Construct examples showing that E^σ needs not be equal[7] to $\hat{E}^\sigma$ ($\sigma = s,u$) but show that the projection of $\hat{E}^\sigma$ to TM is E^σ.

This construction can be easily generalized to the case where the manifold is a fiber bundle instead of a product. Assume that $\mathscr{E}$ is a (C^1-)smooth fiber bundle over M with fiber N, this means that there exists a C^1-map $p: \mathscr{E} \to M$ such that for every $x \in M$ one has that $p^{-1}(\{x\})$ is diffeomorphic to N and these diffeomorphisms vary continuously. More precisely, for every $x \in M$ there is a neighborhood U of x such that $p^{-1}(U)$ is diffeomorphic to $U \times N$ via a diffeomorphism which sends $p^{-1}(\{y\})$ to $\{y\} \times N$ for every $y \in U$. We denote as $\mathscr{E}_x \simeq N$ the fiber $p^{-1}(\{x\})$.

Given a diffeomorphism $f: M \to M$ we say that $F: \mathscr{E} \to \mathscr{E}$ is a lift of f if one has that $p \circ F = f \circ p$. We denote as $F_x: \mathscr{E}_x \to \mathscr{E}_{f(x)}$ the diffeomorphism induced by restriction. One has the following general criterium generalizing equation 3.2.1:

Proposition 3.2. *Let $f: M \to M$ be a partially hyperbolic diffeomorphism with splitting of the form $TM = E^s \oplus E^c \oplus E^u$ and let F be a lift of f verifying that for every $x \in M$ and $t \in \mathscr{E}_x$ one has*

$$\|Df|_{E^s(x)}\| < m(D_t F_x) \leq \|D_t F_x\| < m(Df|_{E^u(x)})$$

Then, F is partially hyperbolic with splitting $T\mathscr{E} = \hat{E}^s \oplus \hat{E}^c \oplus \hat{E}^u$ where if $\xi \in \mathscr{E}$ one has that $T_\xi(\mathscr{E}_{p(\xi)}) \subset \hat{E}^c$ and $Dp(\hat{E}^\sigma) = E^\sigma$ where $\sigma = s,c,u$.

PROOF. This result is left as an exercise for the reader. □

Example 3.3 ([BoW])**.** Consider a non-trivial bundle $p: N \to \mathbb{T}^2$ over the torus. It is not hard to see that if you take U an open ball in $\mathbb{T}^2$ and V an open set such that $U \cup V = \mathbb{T}^2$ we can find differentiable charts of N (see the Exercise below) such that $\varphi_1: U \times S^1 \to N$ and $\varphi_2: V \times S^1 \to N$ verify the following properties:

- $\varphi_1(U \times S^1) \cup \varphi_2(V \times S^1) = N$. Moreover, $p(\varphi_1(x,t)) = x$ for every $x \in U$ and $p(\varphi_2(x,t)) = x$ for every $x \in V$.
- The change of coordinates is by a rotations in the fibers: This means, for $x \in U \cap V$, if $\pi_2: V \times S^1$ is the projection in the second coordinate we have that the map

$$\psi_x: S^1 \to S^1 \qquad \psi(t) = \pi_2 \varphi_2^{-1}(\varphi_1(x,t))$$

is a rigid rotation.

[7]Here we are considering E^s in $T(M \times N)$ to be the bundle $E^s \times \{0\}$ in the coordinates $T(M \times N) = TM \times TN$.

If $A:\mathbb{T}^2\to\mathbb{T}^2$ is an Anosov diffeomorphism then we can write $A=g_2\circ g_1$ where g_1 is the identity in V and g_2 the identity in U. For this, U and V must be properly be chosen (see the Exercise below).

We can thus define the following maps $G_1:N\to N$ is defined to be the identity in $V\times S^1$ and $\varphi_1\circ(g_1\times Id)\circ\varphi_1^{-1}$ in $U\times S^1$ and similarly $G_2:N\to N$ as the identity in $U\times S^1$ and $\varphi_2\circ(g_2\times Id)\circ\varphi_2^{-1}$ in $V\times S^1$. One can check that $F=G_2\circ G_1$ is a partially hyperbolic diffeomorphism.

Exercise 25.
(i) Show that every circle bundle over the torus can be decomposed as above. In particular, V can be chosen such that $U\cap V$ is an annulus and the bundle is determined up to homeomorphism by the degree of the map $x\mapsto\psi_x$ from $U\cap V$ to Homeo(S^1) (which is homotopy equivalent to a circle).
(ii) Show that one can choose two open sets U and V of $\mathbb{T}^2$ with U contractible such that $A=g_2\circ g_1$ as above.
(iii) Check that the diffeomorphism F defined above is (absolutely) partially hyperbolic.
(iv) Compare this example to the one of Exercise 22.

3.3. **Iterated function systems.** The skew product construction needs not be restricted to the whole manifold. If $f:M\to M$ is a diffeomorphism and $K\subset M$ is a compact f-invariant subset admitting a partially hyperbolic splitting of the form $T_KM=E^s\oplus E^c\oplus E^u$ and N is a closed manifold, one can construct a partially hyperbolic set of a diffeomorphism $F:M\times N\to M\times N$ straightforwardly as above from a map $g:M\to \text{Diff}^1(N)$ satisfying the properties of equation 3.2.1 for points in K. Of course, this can also be done for non-trivial fiber bundles, etc.

There are several relevant instances of this approach which can be unified by this point of view (the references below are very far from being exhaustive):

- the study of cocycles over hyperbolic dynamics (see [AV, KP, Wi$_3$] and references therein),
- the study of random dynamics (see [Ar, KL]),
- the study of iterated function systems (see [Fa] and also [GIKN, BBD] and references therein for connections with partial hyperbolicity).

Let us explain briefly how this kind of examples appear naturally in some contexts. Indeed, the typical model of a completely random (discrete) motion is the shift space with the Bernoulli measure, this gives rise to a choice of independent and identically distributed random variables with finite target.

Consider a finite set of diffeomorphisms $f_1,\dots,f_k\in\text{Diff}(N)$ (for simplicity we will assume that all the f_i's are isotopic to the identity) and choose a diffeomorphism g of a manifold M with the following property: there is an open set U such that the maximal invariant set $K\subset U$ is a hyperbolic set topologically conjugate to the shift in k-symbols $\sigma:\{1,\dots,k\}^{\mathbb{Z}}\to\{1,\dots,k\}^{\mathbb{Z}}$. We call the i'th leg of K to the set corresponding via the conjugacy to $\{1,\dots,k\}^{\mathbb{Z}}$ to sequences whose zeroth entry is i.

There are two relevant remarks for this:

- It is possible to construct such a diffeomorphism in any manifold of dimension ≥ 2. The construction is completely analogous to the horseshoe and it is left as an exercise.
- It is possible to do so in such a way that the contraction and expansion of the derivative of g in the bundles E^s and E^u over K are as strong as one wants. In particular, one can do in such a way that the weakest contraction of g along E^s

is smaller than the strongest contraction of any of the f_i's and that the weakest expansion of g along E^u is stronger than the strongest expansion of any of the f_i.

In this way, one can construct a diffeomorphism $F : U \times N \to U \times N$ which is of the form $(x, y) \mapsto (g(x), h(x, y))$ with the property that if x belongs to the i-th leg of K then $h(x, y) = f_i(y)$. This can be extended to a diffeomorphism of $M \times N$ which will contain a partially hyperbolic set of the form $K \times N$ and whose dynamics corresponds to the desired choice.

Exercise 26. Complete the details in the construction of F. In particular, show that one can construct the desired horseshoe with the expected contractions and expansions and show that the resulting diffeomorphism is partially hyperbolic.

Examples of this type, where the dynamics in the fiber N depends only on the zeroth position of the dynamics, are sometimes called *one-step skew products* to indicate that the dependence of the chosen dynamics does not depend on the future or past of the orbit. One can make more involved examples, with some Markov type dependence, or even an arbitrary dependence (such as in general skew-products).

3.4. **Hyperbolic flows and actions.** Given a flow $\varphi_t : M \to M$ of closed manifold, we say that a compact invariant subset $K \subset M$ is a *hyperbolic set* if (modulo a change of the metric) there exists a splitting $T_K M = E^s \oplus E^0 \oplus E^u$ into $D\varphi_t$-invariant bundles such that $\dim E^0 = 1$, one has $\|D\varphi_t|_{E^0}\| = 1$ and there exist constants $C > 0$ and $\lambda < 1$ such that:

$$\|D\varphi_t|_{E^s}\| < C\lambda^t \qquad ; \qquad \|D\varphi_{-t}|_{E^u}\| < C\lambda^t \quad \forall t \geq 0$$

As the reader can easily notice, the definition is very much related with the definition of partial hyperbolicity. In fact, it is very easy to see that if K is a hyperbolic set of a flow φ_t, then, K is an (absolutely) partially hyperbolic set for the diffeomorphism φ_1 when considering E^0 to be the center-direction. Moreover, it has been proved in [HPS] that there is a C^1-open neighborhood of φ_1 consisting on partially hyperbolic diffeomorphisms which fix a foliation homeomorphic to the foliation by the orbits of the flow φ_t (this will be explained later).

In a similar way, one can construct Anosov or hyperbolic actions of certain Lie-groups, implying that the transverse direction of the action is uniformly contracted or expanded in a stronger way than the direction of the action. This gives rise to further examples of partially hyperbolic sets. See [PSh$_1$, KS, GoSp].

3.4.1. *Anosov flows.* A particularly important family of examples are Anosov flows. An *Anosov flow* φ_t is a flow for which the whole manifold is a hyperbolic set. Some well known examples of Anosov flows are the following:

- The suspension of an Anosov diffeomorphism $f : M \to M$ is obtained as follows: Consider in $M \times [0,1]$ the constant vector field given by vectors tangent to the second coordinate (i.e. whose integral lines are of the form $\varphi_t((x,s)) = (x, t+s)$). Now, we can identify $M \times \{0\}$ with $M \times \{1\}$ via $(x,1) \sim (f(x), 0)$. The manifold one obtains by this process will be denoted as M_f and it is sometimes called the *mapping torus* of f. It is not hard to see that the flow φ_t is an Anosov flow (see the exercise below).
- Given a (closed) manifold M whose curvature is everywhere negative, it is a well known result that the geodesic flow on T^1M (the unit tangent bundle of M) is an Anosov flow (see for example [KH]).

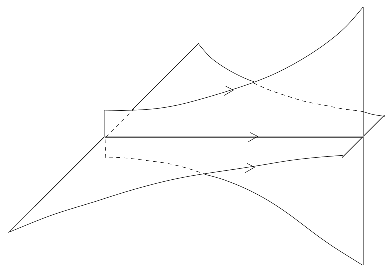

FIGURE 6. Local picture of a hyperbolic flow.

Exercise 27. Show that the suspension of an Anosov diffeomorphism is an Anosov flow.

Very little is known about general Anosov flows in higher dimensions. We refer the reader to the papers [PT, Ve, Ghy_1] for some references in the codimension one case, and to [BFL] for the contact case under assumptions on the regularity of the invariant foliations.

3.4.2. *Anosov flows in dimension 3.* No Anosov diffeomorphisms other than the algebraic ones are known to this moment (up to topological conjugacy). It is conjectured that these comprise all possible examples (see [BM, Man] and references therein). On the other hand, Anosov flows seem less rigid, and even in dimension 3, plenty of examples of non-algebraic Anosov flows have been constructed. The first ones were provided by Franks and Williams ([FW]) and were quite surprising since they were not transitive.

After those examples were made, plenty of examples appeared, one must mention in particular those of [HaTh, Goo]. These examples have the property of being constructed from the previous ones via *surgery*: One cuts a manifold admitting an Anosov flow along a certain tori in such a way that after doing a Dehn-twist along the torus and re-gluing the manifold the Anosov property will persist. These techniques allowed to construct plenty of new examples in different 3-manifolds.

Later, the technique of [FW], which consisted in gluing different manifolds admitting hyperbolic sets and being transverse to a torus was reinterpreted in [BL] and pushed much further recently in [BBY_1]. We refer the reader to the introduction of [BBY_2] for a nice account on the known examples of Anosov flows in dimension 3.

Still, this research subject is wide open. At the moment:

- We know *infinitely many 3-manifolds which do not admit Anosov flows.* The fundamental group must have exponential growth ([PT]), but the same obstruction which was pointed out in [PT] provides many other restrictions (there are no Reeb components, so the manifold must be irreducible, etc..). In particular, there is an infinite family of hyperbolic manifolds not admitting Anosov flows ([RSS]). We also known some other obstructions, of different nature ([Ghy_2, Ba]), on certain specific families of 3-manifolds.
- We know *infinitely many 3-manifolds which admit Anosov flows.* Other than finite lifts or quotients of geodesic flows in surfaces of negative curvature and suspensions of linear Anosov flows, there are all the examples mentioned above. In particular, let us mention that the construction of [Goo] allows one to construct examples in an infinite family of hyperbolic 3-manifolds.
- There are still *infinitely many 3-manifolds which we do not know if support Anosov flows.* In this case, insisting on the hyperbolic manifold case, "most" hyperbolic 3-manifolds fall in this category.

Let us also remark that the constructions in [BBY_1] show that even in manifolds which support Anosov flows the panorama is not clear; the same manifold may admit plenty of topologically inequivalent Anosov flows.

3.5. **Deformations.** Partially hyperbolic diffeomorphisms are C^1-open. This is because there existence of cone-fields which are sent by Df inside their interior. When one is able to control the global (or local) position of cone-fields, it is possible to push this argument in order to consider different kind of deformations, or even compositions with large diffeomorphisms.

3.5.1. *Local modifications.* Let f be a partially hyperbolic diffeomorphism with a splitting $TM = E^s \oplus E^c \oplus E^u$ and let $\mathscr{C}^s$ and $\mathscr{C}^u$ be cones associated to the bundles E^s and E^u and such that:

- the cone $\mathscr{C}^u$ is contracted by Df, and moreover non-zero vectors $u \in \mathscr{C}^u$ are contracted by Df^{-1}, i.e. $\|Df^{-1}u\| < \frac{1}{2}\|u\|$,
- the cone $\mathscr{C}^s$ is contracted by Df^{-1}, and moreover vector $u \in \mathscr{C}^s$ are contracted by Df, i.e. $\|Dfu\| < \frac{1}{2}\|u\|$.

Then, one can consider deformations of f through isotopies among diffeomorphisms which still have these properties, hence are partially hyperbolic by the cone field criterion.

Derived from Anosov examples.

As an example of system obtained by a local modification, let us consider a linear Anosov automorphism of $\mathbb{T}^3$ with three real eigenvalues $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3$. For instance given by

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

The map can thus be locally written $(x, y, z) \mapsto (\lambda_1.x, \lambda_2.y, \lambda_3.z)$. Let us deform it in a small ball, for $a \in (0,1)$ as a diffeomorphism f_a of the form:

$$f_a \colon (x, y, z) \mapsto (\lambda_1.x, a g_{x,z}(a^{-1}y), \lambda_3.z), \text{ such that } \lambda_1 < \|Dg_{x,z}\| < \lambda_3.$$

Notice that the derivative of $ag_{x,z}(a^{-1}y)$ with respect to x and z can be chosen as small as desired by considering small $a > 0$.

Let us consider the cone $\mathscr{C}^{cu}$ of vectors (v_x, v_y, v_z) such that $\|v_x\| \leq \alpha\|v_y + v_z\|$.
The image (v'_x, v'_y, v'_z) satisfies

$$\|v'_x\| = \lambda_1\|v_x\|,$$
$$\|v'_y + v'_z\| \geq \min_{x,z}(\|Dg_{x,z}\|)\|v_y + v_z\| - |\frac{\partial g_{x,z}}{\partial x} v_x| - |\frac{\partial g_{x,z}}{\partial x} v_z|$$

If one chooses a small enough, noticing that $|v_x| \leq \alpha\|v_y + v_z\|$ and $\|v_z\| \leq \|v_y + v_z\|$ one has that the cone-field is invariant. Also, vectors inside the cone $\mathscr{C}^{cu,*}$ are contracted by Df_a so that the invariant bundle inside $\mathscr{C}^{cu,*}$ is uniformly contracted.

This proves that there exists a dominated splitting $E^s \oplus F_1$, where $\dim(E^s) = 1$. Arguing symmetrically, one gets a splitting $TM = E^s \oplus E^c \oplus E^u$, as required. Notice also that if one chooses a small enough, one can consider the cone-field to be as narrow as desired.

The new system can be non-hyperbolic: it is enough to deform near a fixed point in order to introduce new fixed points with different stable dimension[8], see figure 7.

[8]This requires an argument. In principle, it could be that the new saddle is in a different chain-recurrence class than the rest of the points, however, as we shall see in section 5, this is not the case.

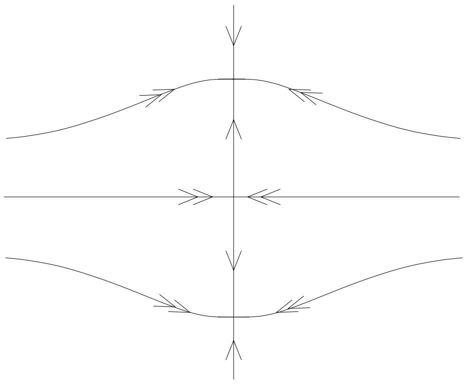

FIGURE 7. Deformation inside the stable manifold of a fixed point.

The technique of local perturbations can be applied to any partially hyperbolic system in order to change the index of periodic points. We just refer here to the example by Bonatti and Viana ([BV]) which provides an example of a robustly transitive (and stably ergodic) diffeomorphism without uniform bundles and we refer the reader to [BDV, Section 7.1] or [Cr_2, Section 5.10] for a survey on this type of constructions.

Exercise 28. (1) Use a local modification to construct a globally partially hyperbolic diffeomorphism for which the domination is not absolute.
(2) Use a local modification to construct a diffeomorphism isotopic to the same linear Anosov diffeomorphism as above whose finest dominated splitting is of the form $T\mathbb{T}^3 = E^{cs} \oplus E^u$ and such that E^{cs} is not uniformly contracted but it contracts volume uniformly. (Hint: See figure 1 of [BV],page 181.)

3.5.2. *Global deformations.* The following criteria has been used in [BPP, BGP, BGHP] to construct new examples of partially hyperbolic diffeomorphisms in dimension 3. The point is that one can compose the time T map of an Anosov flow (for some large T) by a diffeomorphism which is not isotopic to the identity and preserve partial hyperbolicity. Notice that this Proposition was already essentially used in section 3.5.1.

Proposition 3.4. *Let $f : M \to M$ be a diffeomorphism with a dominated splitting $TM = E \oplus F$. Let $h : M \to M$ be a diffeomorphism such that for every $x \in M$ one has:*

$$Dh(E(x)) \pitchfork F(h(x)) \tag{3.5.1}$$

then, there exists $n > 0$ such that $f^n \circ h$ admits a dominated splitting.

This proof essentially follows from the classical cone-field criteria. Let us briefly give a sketch of the proof:

Sketch of the proof: Let us first show that there exists n such that $f^n \circ h$ preserves an unstable cone-field.

To do this, consider first a given cone-field $\mathscr{C}$ for the bundle F of f, that is, $Df(\overline{\mathscr{C}(x)}) \subset \mathscr{C}(f(x))$. Notice that by considering the cone-fields $Df^k(\mathscr{C})$ one can assume that $\mathscr{C}$ is as narrow as one wishes.

By compactness and the fact that $Dh(F) \pitchfork E$ one can assume that $Dh(\overline{\mathscr{C}(x)}) \pitchfork E(h(x))$. In particular, there exists $n > 0$ such that $Df^n(Dh(\overline{\mathscr{C}})) \subset \mathscr{C}$. This concludes. □

Just to give a taste on the applications of the Proposition, let us briefly explain an example from [BGP].

Consider a hyperbolic surface S with Riemannian metric g and its geodesic flow $\varphi_t : T^1S \to T^1S$. It is well known that if the curvature of g is everywhere negative, then φ_t is an Anosov flow.

Let $f : S \to S$ be any smooth diffeomorphism. Consider its projectivization $Pf : T^1S \to T^1S$ defined as $Pf(v) = \frac{Dfv}{\|Dfv\|}$ which conjugates the geodesic flow of the metric g with the one obtained by the pullback f_*g. That is, if ψ_t denotes the geodesic flow of f_*g one has that:

$$Pf \circ \varphi_t = \psi_t \circ Pf$$

In particular, since the conjugacy is smooth, it follows that the derivative of Pf sends the invariant bundles of φ_t onto the ones of ψ_t. So the key remark is:

Proposition 3.5. *It is possible to construct $f : S \to S$ not isotopic to the identity in S so that f_*g and g are very close to each other (in the C^∞-topology) and with g being* uniformly Anosov[9].

This is done in [BGP, Section 2] via considering a hyperbolic metric of constant curvature -1 with a very short geodesic and choosing f to be a Dehn-twist along this curve, since the geodesic is short, it follows that, in the universal cover, the lift of the diffeomorphism f is very close to the identity.

It follows that the diffeomorphism $Pf \circ \varphi_N$ for large N will be partially hyperbolic thanks to Proposition 3.4 and not isotopic to the identity because Pf isn't. Notice that Pf is dissipative along the fibers, but it is not hard to correct it in order that a small perturbation of Pf (which still guarantees condition (3.5.1) for all the involved bundles) preserves the Liouville measure.

3.6. **Attractors.** Attractors play an important role in this notes. An *attractor* for a diffeomorphism $f : M \to M$ is a compact f-invariant set K such that it admits an open neighborhood U such that $f(\overline{U}) \subset U$ and $K = \bigcap_{n>0} f^n(U)$. The importance of considering attractors, other than the fact that one expects points to approach them (and contain the *physical measures*) is that their structure allows one to have more tools to study them (in particular, they are saturated by unstable manifolds). Notice that we are not assuming here dynamical indecomposability of the attractor, so that the whole manifold M is always an attractor. However, if one looks for *partially hyperbolic attractors* it might be important to restrict to some subsets of the manifold.

3.6.1. *Hyperbolic attractors.* Hyperbolic attractors are still far from being completely understood. A hyperbolic attractor is a compact f-invariant set Λ with a hyperbolic splitting $T_\Lambda M = E^s \oplus E^u$ such that it admits a neighborhood U such that $f(\overline{U}) \subset U$ and $\Lambda = \bigcap_{n>0} f^n(U)$. It follows that Λ is saturated by unstable manifolds (c.f. Section 4). The Plykin attractor explained in the Introduction is one such example.

When Λ is a hyperbolic attractor, it follows that its neighborhood U is foliated by local stable manifolds. It seems reasonable to quotient down by the stable manifolds which "escape" the attractor and one obtains a hyperbolic map from a *branched manifold*. This can be formalized when the attractor is *expanding*, meaning that its topological dimension equals $\dim E^u$. These have been studied in detail (see [Wil]).

Exercise 29. An expanding map $f : M \to M$ is a C^1-map such that $\|D_x f v\| > \|v\|$ for every $x \in M$ and $v \in T_xM \setminus \{0\}$. Show that if M is compact and $f : M \to M$ is an expanding map then:

[9]We shall not define this here. This is implied for example if the curvature of g is bounded away from 0 which implies that the strength of the dominations, contractions and expansions is uniform.

(1) There exists $\lambda > 1$ such that $\|D_x f v\| \geq \lambda \|v\|$ for every $x \in M$ and $v \in T_x M$.
(2) If $\tilde{M}$ denotes the universal cover of M with the lifted Riemannian metric, show that there exists a polynomial P such that the volume of a ball of radius R in $\tilde{M}$ is smaller than $P(R)$. (Hint: Show that the volume of a ball of radius R centered at a fixed point verifies the following inequality: $\mathrm{Vol}(B(x,\lambda R)) \leq C\,\mathrm{Vol}(B(x,R))$ for some $\lambda > 1$ and $C > 0$. Show that this inequality implies polynomial growth.)

In general, if one has a C^1-map $f : M \to M$ which is hyperbolic yet not necessarily invertible (for example, a hyperbolic $d \times d$ matrix with integer coeficients but determinant different from one acting on $\mathbb{T}^d$) it is possible to consider its *inverse limit* and one will obtain a homeomorphism of a *solenoid* which can be embedded as a hyperbolic attractor in a larger dimensional manifold. When f is invertible, one has just to embed M in a larger dimensional manifold as an attracting manifold.

In [Bro] the topology of hyperbolic attractors in 3-dimensional manifolds was studied and a classification is provided. The examples as the one mentioned above are also studied in detail.

3.6.2. *Partially hyperbolic attractors.* We explain here some mechanisms to construct partially hyperbolic attractors.

- The first consists in generalizing the one explained in the hyperbolic case. One considers a partially hyperbolic map (not necessarily invertible) and constructs its solenoid by taking its inverse limit and then one embeds the example in a larger dimensional manifold. This will provide an example of partially hyperbolic attractor.
- Other way to construct examples is to start with hyperbolic (or partially hyperbolic) attractors and make its suspension, obtaining a partially hyperbolic attractor of a flow; the time one map of this flow (and its small perturbations) will have a partially hyperbolic attractor. Similarly, one can do skew-products over (partially) hyperbolic attractors. A different, and to this moment quite unexplored example of this type is to consider perturbations of the time one map of singularly hyperbolic attractors[10], such as the *Lorenz attractor*.
- A third mechanism to construct examples is to start with previous examples and perform *local modifications* as explained in section 3.5.1. This way, one creates new partially hyperbolic sets for which one can eliminate some of the dominations or create genuine center bundles instead of the original uniform bundles (for example, start with a uniformly contracting direction and create a periodic point which has an eigenvalue of modulus larger than 1 in that direction as in section 3.5.1).

Let us explain a bit further on this last mechanism.

In [Carv] an attractor is constructed by starting with a linear Anosov automorphism of $\mathbb{T}^3$ with $\dim E^s = 2$ and performing a Hopf bifurcation respecting the conditions of section 3.5.1. This provides an example of an attractor with splitting $T_\Lambda \mathbb{T}^3 = E^{cs} \oplus E^u$ on which E^{cs} is not decomposable into further subbundles. This attractor is the maximal invariant set outside a neighborhood of a hyperbolic source p. Of course, since it is done with the technique of section 3.5.1 the example admits a global splitting of the form $T_\Lambda \mathbb{T}^3 = E^{cs} \oplus E^u$, but the advantage of considering the attracting set Λ is that on Λ one can guarantee that the jacobian is uniformly contracted, this is good to understand

[10]Some progress has been reported by C. Bonatti and Y. Shi who showed that it is possible to perturb this time one map to get a robustly transitive attractor.

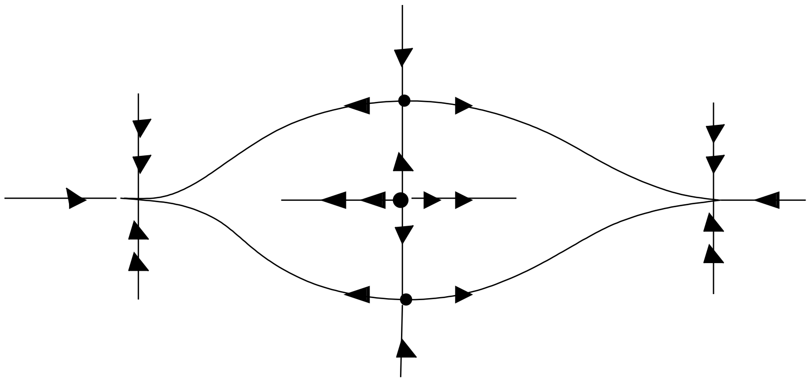

FIGURE 8. Deformation inside the stable manifold of a fixed point.

properties such as transitivity and further statistical properties. It can be shown that this attractor Λ is transitive and that its topology is transversally a Sierpinsky carpet. Nikolaz Gourmelon has remarked that one can do the example by a standard bifurcation (instead of a Hopf one) and obtain a diffeomorphism such that $T\mathbb{T}^3 = E_1 \oplus E_2 \oplus E^u$ by bifurcating a fixed point into a source (in the center-stable direction two saddles and two sinks appear, see figure 8). It follows that the attractor Λ obtained as the maximal invariant set outside a neighborhood of the source is partially hyperbolic with splitting $T_\Lambda \mathbb{T}^3 = E^s \oplus E^c \oplus E^u$ with all bundles one-dimensional.

This kind of constructions can be done starting with any partially hyperbolic attractor. In general, the key difficulty relies on ensuring that the new periodic points that one has created belong to the same chain-recurrence class of the attractor (for example, this is never the case when one creates a source). An important tool to create examples with this property are *blenders* (see [BD$_1$]).

This ideas have been used in [BLY] to construct examples of generic diffeomorphisms without attractors. See also [Pot$_1$] for a construction in the lines of Carvalho's example and [BoS] for a volume hyperbolic example.

3.7. **Other examples and questions.** Many examples have been neglected in this section. Just to name a few important ones:

- Frame flows on general manifolds of negative curvature are also globally partially hyperbolic. Historically their importance lies in that they are one of the first genuine partially hyperbolic (not Anosov) examples on which the question of ergodicity came up and was solved in [BG] (see also [Br$_1$]).
- Skew products can be done the other way around; by choosing the fibers to have the strong contractions and expansions. The advantage of this point of view is that the topology of a fiber bundle is at least as complicated as the base topology, but can be less complicated than the fiber[11]. This approach has allowed Gogolev-Ontaneda-Rodriguez Hertz to obtain partially hyperbolic diffeomorphisms in simply connected manifolds ([GORH]). See also the work of Farrel-Gogolev ([FG]) for a systematic study of this approach.

Exercise 30. (1) Construct examples of Anosov diffeomorphisms in non-trivial torus bundles over a torus via a skew-product construction.

[11]Think for example at the Hopf fibration of S^3 which is a circle bundle over the sphere. The total space is simply connected (as the base); however, the fiber is the circle which is not simply connected.

(2) (∗) Show that the only way to obtain an Anosov diffeomorphism as a skew product over a linear Anosov diffeomorphism on a torus gives a nilmanifold.

Classification in dimension 3. We end this section by very briefly discussing the problem of classification of partially hyperbolic diffeomorphisms in dimension 3. It is not the purpose of this notes to describe this problem in detail, so we refer the reader to [CRHRHU] or the forthcoming [HPo_3] for more information.

It seems natural to try to understand globally (strong) partially hyperbolic dynamics in low dimensions and hope for a classification, at least transitive or volume preserving ones. In dimension 3, strongly partially hyperbolic diffeomorphisms have all three bundles of dimension one so it seems a reasonable class to work with. Anosov flows in dimension 3 are far from being classified and provide examples of strong partially hyperbolic dynamics and so the problem seems hopeless. However, Pujals has proposed to classify partially hyperbolic systems *modulo* Anosov systems and this has opened the field for many works in the direction of this classification. Let us point some of the main results:

- In [BoW] the problem suggested by Pujals was formalized and many results were obtained under assumptions on the structure of the center-foliations (in particular, it assumes the existence of such foliation),
- In [BI] (building on the previous work [BBI_1]) the first topological obstructions to admitting partially hyperbolic systems were found. They are related to the existence of Reebless foliations in the manifolds. This was pushed in [Par] to obtain a characterization of 3-manifolds with polynomial growth of volume which admit globally partially hyperbolic diffeomorphisms.
- In [Ham] a new notion of classification was implicitely proposed, this is leaf conjugacy and it is related to the notion of structural stability of normally hyperbolic foliations of [HPS].
- The examples in [$RHRHU_3$] clarified the requirement of transitivity in the "conjectures" and introduced the notion of *Anosov tori*. See [CRHRHU].
- In [HPo_1, HPo_2] the classification was achieved for 3-manifolds with (virtually) solvable fundamental group.
- Very recently, several new examples were produced ([BPP, BGP, BGHP]) and the main task now is to understand how to make these new examples fit into a classification program.

We end by mentioning that recently, in a joint work of the second author with A. Hammerlindl and M. Shannon, the obstructions related to the work [Ghy_2] on 3-manifolds which are circle bundles over higher genus surfaces were shown to be obstructions for the existence of partially hyperbolic diffeomorhisms too.

In higher dimensions, the classification of globally partially hyperbolic diffeomorphisms is much less developed. However, the idea of pursuing a classification modulo Anosov systems as suggested by Pujals and [BoW] is for the moment the main guiding force. See for example [Bonh, Carr, Go, Ham, Pot_3] for partial results in this direction.

4. Invariant manifolds

We discuss in this section the existence and smoothness of invariant manifolds which extend the classical stable manifold for hyperbolic sets.

4.1. **Strong manifolds and laminations.** Consider $f \in \mathrm{Diff}^r(M)$ with $r \geq 1$ and K a compact f-invariant set admitting a partially hyperbolic splitting of the form $T_K M = E^s \oplus E^c$ with $\dim E^s \geq 1$.

Definition 4.1. For $\varepsilon > 0$ small enough, one defines at each $x \in K$ its *strong stable set*:

$$W^{ss}(x) = \left\{ y \in M,\ \exists c > 0,\ \forall n \geq 0, d(f^n(y), f^n(x)) < c e^{-\varepsilon n} \min\{m(Df^n|_{E^c(x)})\ , 1\} \right\} \quad (4.1.1)$$

In other terms $W^{ss}(x)$ is the set of points whose orbit converge to the orbit of x faster than the contractions $Df^n|_{E^c(x)}$.

In the case there is a partially hyperbolic splitting $E^c \oplus E^u$ we define symmetrically the *strong unstable set* $W^{uu}(x)$ as the strong stable set of x for f^{-1}.

Remark 4.2. (i) The strong stable set does not depend on the metric.
(ii) We have $f(W^{ss}(x)) = W^{ss}(x)$.
(iii) There could exist a different choice for the partially hyperbolic splitting, leading to a different stable manifold (of smaller or larger dimensions). However, thanks to Proposition 2.2 once the dimension of the stable bundle is fixed, the strong stable manifold is unique (see below).

Theorem 4.3 (Stable Manifold Theorem [HPS])**.** *Let $f \in \mathrm{Diff}^r(M)$ and $K \subset M$ a compact f-invariant set with a partially hyperbolic splitting of the form $T_K M = E^s \oplus E^c$ and E^s uniformly contracted.*

(a) *For any $x \in K$, the strong stable set $W^{ss}(x)$ is an injectively immersed C^r-submanifold diffeomorphic to $\mathbb{R}^{\dim(E^s)}$, which is tangent to $E^s(x)$ at x.*
(b) *The strong stable set does not depend on the choice of ε in equation* (4.1.1) *as long as it is small enough.*
(c) *For any $x, y \in K$, the strong stable sets $W^{ss}(x), W^{ss}(y)$ are either disjoint or coincide.*
(d) *For $\eta > 0$ small, the ball $D^s_\eta(x)$ in $W^{ss}(x)$ centered at x of radius η depends continuously on x and f for the C^r-topology.*

The sets $W^{ss}(x)$, $W^{uu}(x)$ are called *strong stable* and *strong unstable manifolds* of x.

Remark 4.4. 1. In the case $K = M$, the manifold $W^{ss}(x)$ is tangent to E^s at each of its points. The coherence argument form the section 4.3 below proves that conversely any C^1-manifold tangent to E^s at each of its points is contained in a manifold $W^{ss}(x)$.

2. The regularity r can take non integral values: f is C^r means that it admits $[r]$ derivatives and that its $[r]^{\text{th}}$ derivative is $r - [r]$-Hölder. One will see below that submanifolds with intermediate C^r-regularity, $r \in (1,2)$, may appear naturally in dynamics, even if one considers smooth systems.

3. As a consequence of the previous result, one deduces that the collection of strong stable (resp. strong unstable) sets form a *lamination* which we sometimes denote as $\mathcal{W}^{ss}$ (resp. $\mathcal{W}^{uu}$). When the whole manifold M is partially hyperbolic, this lamination is indeed a foliation (notice that even if leaves are C^r, the foliation is, in principle, just continuous).

4. If g is C^r-close to f and y is a point close to $x \in K$ and whose orbit remains close to K, then the ball $D^s_\eta(y)$ in $W^{ss}(y)$ for g is C^r-close to $D^s_\eta(x)$.

Exercise 31. Let $f : M \to M$ partially hyperbolic with splitting $TM = E^s \oplus E^c$ at every point and such that $\dim E^s = 1$.

(1) Use Peano's existence theorem for differential equations to show that through every point $x \in M$ there exists a curve η_x tangent to E^s such that for every compact arc $I_x \subset \eta_x$ the length of $f^n(I_x)$ converges to zero exponentially fast.
(2) Show that the bundle E^s is uniquely integrable into a foliation tangent to E^s.

4.2. **Plaque families – The graph transform argument.** Theorem 4.3 for the C^1-topology will be a consequence of the following local version.

Theorem 4.5 (Plaque Families [HPS]). *Let $f : M \to M$ be a C^1-diffeomorphism and $K \subset M$ a compact f-invariant subset admitting a dominated splitting of the form $T_K M = E \oplus F$. Then, for every $x \in K$ there exists a C^1-embedding $\mathcal{D}_E(x) : B(0,1) \subset E(x) \to M$ with the following properties:*

- *(**Tangency:**) for every $x \in K$ one has that $\mathcal{D}_E(x)(0) = x$ and the image of $\mathcal{D}_E(x)$ is tangent to $E(x)$ at x.*
- *(**Continuity:**) the embeddings $\mathcal{D}_E(x)$ depend continuously on the C^1-topology on $x \in K$.*
- *(**Local invariance:**) there is $\delta_0 < 1$ such that for $x \in \Lambda$ one has that $f(\mathcal{D}_E(x)(B(0,\delta_0))) \subset \mathcal{D}_E(f(x))(B(0,1))$.*

The setting for this theorem is more general than in the strong stable manifold theorem. For that reason, in general the plaques $\mathcal{D}_E(x)(B(0,1))$ are not uniquely defined, they are a tangent to F only at their center and the union of two plaques is not necessarily a submanifold. The plaques are not C^2 in general, even if f is C^r, r large.

PROOF OF THE PLAQUE FAMILY THEOREM. This can be obtained by a graph transform argument (Hadamard's method). Let us explain its principle.

a. *Lifted dynamics.* Consider $x \in K$. One can assume that the exponential map identifies the ball $B(0,1)$ in T_xM with a neighborhood of x in M. One can thus lift f as a C^k local diffeomorphism $f_x := \exp_{f(x)} \circ f \circ \exp_x^{-1}$ from the ball $B(0,\alpha)$ in T_xM to a neighborhood of 0 in $T_{f(x)}M$. It can be glued with the restriction of D_xf to the complement of $B(0,\alpha)$ by a bump function. In this way one obtains a diffeomorphism $\widehat{f}_x \colon T_xM \to T_{f(x)}M$. This diffeomorphism is C^1-close to D_xf if α is small enough (see Lemma 4.6 below).

b. *Lipschitz graphs.* Each tangent space T_xM has a splitting $E_x \oplus F_x$. Section 2.2 gives a cone field $\mathscr{C}$ along the direction F and which is contracted by f and a cone field $\mathscr{C}^*$ along the direction E and which is contracted by f^{-1}. One obtains in each tangent space T_xM a constant cone field which coincides with $\mathscr{C}^*_x$. The map $\widehat{f}_x^{-1}$ contracts the cone field $\mathscr{C}^*_{f(x)}$ into the cone field $\mathscr{C}^*_x$.

Let us consider the family L_x of Lipschitz graphs tangent to $\mathscr{C}^*_x$ containing 0, that is graphs of Lipschitz functions $\psi\colon E_x \to F_x$ such that $\psi(0) = 0$ and for each $u, u' \in E_x$, the vector $(u - u', \psi(u) - \psi(u'))$ (for the decomposition $E_x \oplus F_x$) is tangent to $\mathscr{C}^*_x$. The cone contraction implies (this requires an argument, see Lemma 4.7 below) that the image by $\widehat{f}_x^{-1}$ of each $\psi \in L_{f(x)}$ is a graph in L_x.

The space L_x is complete for the distance

$$d(\psi_1,\psi_2) = \max_{u \in F_x} \frac{d(\psi_1(u),\psi_2(u))}{\|u\|},$$

which is bounded since the graphs are uniformly Lipschitz.

c. *Contraction.* Let us fix n large. We will show that the distance on the spaces L_x is contracted in the past by large iterates $\widehat{F} = \widehat{f}_{f^{n-1}(x)} \circ \cdots \circ \widehat{f}_x$. Indeed, let ψ'_1, ψ'_2 be the graphs images by $\widehat{F}^{-1}$ of $\psi_1, \psi_2 \in L_{f^n(x)}$ and fix $u \in F_x$. Let us consider $(u, \psi'_1(u))$ and $(u, \psi'_2(u))$. They are the image by $\widehat{F}^{-1}$ of two points $(v, \psi_1(v))$, $(w, \psi_2(w))$. In order to simplify, one will assume that $v = w$.

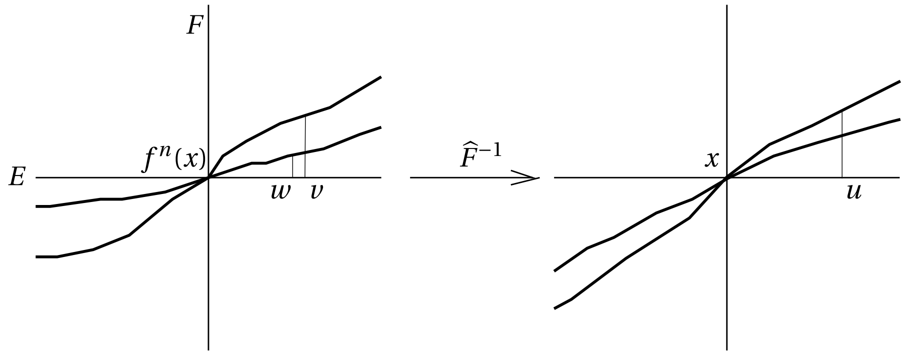

FIGURE 9. Contraction by the graph transform.

The point $(v, \psi_1(v))$ is mapped on $(u, \psi'_1(u))$. These two vectors are tangent to the cones $\mathscr{C}^*_{f^n(x)}$ and $\mathscr{C}^*_x$ and the maps $\widehat{f}_y$ are close to the tangent maps $D_y f$. Hence

$$\|u\| \le \|Df^n(x)_{|E}\| e^{\varepsilon n} \|v\|.$$

One the other hand, $(0, \psi'_1(u) - \psi'_2(u))$ is tangent to the cone $\mathscr{C}_x$, which is contracted by $\widehat{F}$. This gives

$$d(\psi_1(v), \psi_2(v)) \ge m(Df^n_{|F}(x)) e^{-\varepsilon n} d(\psi'_1(u), \psi'_2(u)).$$

One deduces:

$$\frac{d(\psi'_1(u), \psi'_2(u))}{\|u\|} \le e^{2\varepsilon n} \frac{\|Df^n_E(x)\|}{m(Df^n(x)_{|F})} \frac{d(\psi_1(v), \psi_2(v))}{\|v\|}.$$

With the domination this gives

$$d(\psi'_1, \psi'_2) \le c\lambda^n e^{2\varepsilon n} d(\psi_1, \psi_2),$$

hence the uniform contraction.

d. *Construction of the embedding.* Let $\mathscr{L}_x$ be the product space $\prod_{n\in\mathbb{Z}} L_{f^n(x)}$ with the distance given by the supremum of the distances on each $L_{f^n(x)}$. The product map $(\widehat{f}_{f^n(x)})$ acts and its inverse contracts this complete space. Consequently, there exists a fixed point $(\psi_{f^n(x)})$. The embedding $\mathscr{D}_E(x)$ is the map

$$\mathscr{D}_E(x)\colon u \mapsto \exp_x(u, \psi_x(u)).$$

Since $\widehat{f}$ coincides with $f_x := \exp_{f(x)} \circ f \circ \exp_x^{-1}$ on $B(0,\alpha)$, and since $\widehat{f}_x$ sends the graph ψ_x on the graph $\psi_{f(x)}$, for $\eta > 0$ small enough the disc $\mathscr{D}_E(x)(B(0,\eta))$ is mapped inside the disc $\mathscr{D}_E(f(x))(B(0,1))$ by f.

e. *C^1-smoothness.* The properties we obtained in Chapter 2 still hold for the maps $\widehat{f}_x$, in particular since the cones $\mathscr{C}_x$ are contracted, there exists at each $u \in T_xM$ a splitting $\widehat{E}_u \oplus \widehat{F}_u$ with the property that E_u is the collection of vectors ζ tangent at u in T_xM such that its iterates by $(\widehat{f}_{f^n(x)})$ remain in the cones $\mathscr{C}^*_{f^n(x)}$ and F_u is the collection of vectors ζ tangent at u in T_xM such that its iterates by $(\widehat{f}_{f^n(x)})$ remain in the cones $\mathscr{C}_{f^n(x)}$.

As Lipschitz graph, ψ_x is differentiable at almost every point u, hence has a tangent space at u whose iterates remain in the cones $\mathscr{C}^*_{f^n(x)}$. Consequently the tangent space is

$\widehat{E}_u$. Since $\widehat{E}_u$ varies continuously with u, one deduces that ψ is C^1 and tangent to $\widehat{E}_u$ at each of its points. In particular it is tangent to E_x at 0.

f. *Continity with respect to x.* By construction, the graph ψ_x that we built is close to

$$\widehat{f}_x^{-1} \circ \widehat{f}_{f(x)}^{-1} \circ \cdots \circ \widehat{f}_{f^{n-1}(x)}^{-1}(\psi')$$

for arbitrary graphs $\psi' \in \mathscr{L}_{f^n(x)}$ and n large. If one fixes n and considers $\psi' = \psi_{f^{-n}(x')}$ for x' close to x, it implies that ψ_x and ψ'_x are close on the ball $B(0,1)$.

□

It remains to prove the intermediate lemmas.

Lemma 4.6. *Given $\varepsilon > 0$ there exists α such that $d_{C^1}(\hat{f}_x, D_x f) < \varepsilon$.*

PROOF. Consider a smooth bump function $\rho : T_x M \to [0,1]$ with the following properties:

- $\rho(v) = 1$ if $\|v\| \le \frac{\alpha}{2}$.
- $\rho(v) = 0$ if $\|v\| \ge \alpha$
- $\|\nabla\rho(v)\| \le \frac{4}{\alpha}$ for every α.

We consider then the function $\hat{f}_x : T_x M \to T_{f(x)} M$ defined as $\hat{f}_x = \rho f_x + (1-\rho) D_x f$.

The C^1-distance of $\hat{f}_x$ and $D_x f$ is the C^1 size of $\rho(f_x - D_x f)$ which is of the order of the C^0 size of $\nabla\rho(f_x - D_x f) + \rho D(f_x - D_x f)$.

Both terms can be seen to go to zero when $\alpha \to 0$. The first one is small: since f is C^1, the quantity $\|f_x - D_x f\|/|\alpha|$ goes to 0 as $\alpha \to 0$. The second term goes to zero with α since ρ is bounded and the derivative of f_x is continuous and equal to $D_x f$ in 0.

□

Lemma 4.7. *The image by $\hat{f}_x^{-1}$ of a function $\psi \in L_{f(x)}$ is contained in L_x.*

PROOF. Let us prove that the projection on E_x is injective on the image of the graph. Consider two points $(v, \psi(v))$ and $(w, \psi(w))$ whose images have the same projection u. The difference between the two images is tangent to $\mathscr{C}$. Since the cone $\mathscr{C}$ is contracted by $\widehat{f}$, one deduces that $(v - w, \psi(v) - \psi(w))$ is tangent to $D\hat{f}(\mathscr{C}) \subset \mathscr{C}$. This is a contradiction (since ψ is tangent to $\mathscr{C}^*$), unless the two points are the same.

The image of the graph of ψ is thus a graph over a subset of E_x. This set is homeomorphic to E_x (by invariance of the domain) and proper, hence it is E_x (see the exercise below).

The fact that the graph is Lipschitz with the same constants is direct from the fact that the cone-field is contracted by $D\hat{f}$.

□

Exercise 32. Show that if $f : \mathbb{R}^d \to \mathbb{R}^d$ is continuous, injective and proper, then it is a homeomorphism. (Hint: consider the one point compactification of $\mathbb{R}^d$, since f is proper, it extends. Then use the fact that continuous and injective implies that f is of degree 1 or −1.)

4.3. **Proof of the stable manifold theorem (C^1-version) – The coherence argument.** The following is a direct consequence of Theorem 4.5.

Corollary 4.8. *Let us consider a partially hyperbolic splitting $T_K M = E^s \oplus E^c$, the embeddings $\mathscr{D}_{E^s}(x)$ given by Theorem 4.5 associated to the bundle E^s and $N \ge 1$ large. For $\eta > 0$ small, one defines $W^{ss}_{loc}(x) = \mathscr{D}_{E^s}(x)(B(0,\eta))$.*

(a) *For any* $x \in K$ *and* $n \geq 0$*, one has* $\operatorname{diam}(f^n(W^{ss}_{loc}(x))) \leq e^{n\varepsilon} \prod_{i=0}^{[n/N]} \|Df^{iN}|_{E^s(x)}\|$
(b) *For any* $x \in K$ *and* $n \geq N$*,* $f^n(W^{ss}_{loc}(x)) \subset W^{ss}_{loc}(f^n(x))$.
(c) *For any* $x \in K$*,* $W^{ss}(x) = \bigcup_{n\geq 0} f^{-n}(W^{ss}_{loc}(f^n(x)))$.

PROOF. Covering with charts, one works in $\mathbb{R}^d$. By compactness, for $\eta > 0$ small enough, the embedded disc $W^{ss}_{loc}(x)$ is almost linear and the action of f^n is close to $Df^n|_{E^s(x)}$. One gets (a) inductively checking by the local invariance that $f^n(W^{ss}_{loc}(x)) \subset \mathcal{D}_{E^s}(f^n(x))(B(0,1))$. In particular item (b) follows. The (a) and the domination give also the inclusion $\supset$ in (c).

For proving the inclusion $\subset$ in (c), one applies the following argument.

The coherence argument. Considers a point $z \in W^{ss}(x)$ and assume (up to replace by iterates) that the forward orbits of x and z remain at distance $\ll \eta$. Consider a small disk D containing z and a point y in $W^{ss}_{loc}(x)$ and tangent to a contracted cone field $\mathscr{C}$. By forward iterations the disk remains tangent to $\mathscr{C}$. By domination, the distance between forward iterates of y and z in D decays slower than $e^{\varepsilon n}\|Df^n|_{E^s(x)}\|$ and since all iterates remain in a small neighborhood of the orbit of x where the bundles are almost constant, this distance is comparable to the distance in the manifold. Hence this contradicts $x \in W^{ss}(x)$ unless $z = y$.

□

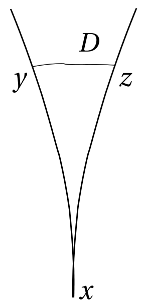

FIGURE 10. The coherence argument.

Remark 4.9. In the coherence argument, it is essential that the manifolds one works with have some dynamical properties (the design of figure 10 remains at small scale for every future iterate). This fact is essential for two important reasons:

- it allows to compare derivatives in different points in order to compare lengths of the iterates with derivatives at some fixed point,
- it allows to control the geometry of the figure; at small scales the bundles are more or less constant and one can compare lengths and distances.

In [Br$_3$] this argument is pushed to an extreme and he shows unique integrability of certain invariant bundles by assuming both absolute domination (to deal with the first issue) and quasi-isometry of the foliations (to deal with the second).

Remark 4.10. Notice that the graph transform argument also provides a uniqueness result. In general it is not satisfying since it is unique once one chooses the extension of f_x to the whole T_xM which is non-canonical. However, for points that remain close to the orbit of x, the choice of the extension is irrelevant and therefore those points must be inside the plaques no matter which extension we consider.

PROOF OF THE STABLE MANIFOLD THEOREM (THEOREM 4.3) FOR C^1-DIFFEOMORPHISMS. From (b) and (c) of Corollary 4.8, $W^{ss}(x)$ is an increasing union of the submanifolds

$f^{-nN}(W^{ss}_{loc}(f^{nN}(x)))$, $n \geq 0$. Hence it is an injectively immersed submanifold diffeomorphic to $\mathbb{R}^{\dim(E^s)}$ proving (a) of Theorem 4.3 (only providing C^1-regularity). From (c) of Corollary 4.8, one deduces that $W^{ss}(x)$ does not depend on ε, that is (b) of Theorem 4.3.

If one assumes that $W^{ss}(x)$ and $W^{ss}(y)$ intersect, up to replace x, y, by forward iterates, the distance $d(f^n(x), f^n(y))$ is small for each $n \geq 0$. In particular $\prod_{i=0}^{[n/N]} \|Df^{iN}|_{E^s(y)}\| \leq e^{n\varepsilon} \prod_{i=0}^{[n/N]} \|Df^{iN}|_{E^s(x)}\|$. So by (a) of Corollary 4.8, the distance $d(f^n(x), f^n(y))$ is smaller than $2e^{2\varepsilon n} \prod_{i=0}^{[n/N]} \|Df^{iN}|_{E^s(x)}\|$. Similarly, for any point z in $W^{ss}(y)$, there exists c_z such that $d(f^n(y), f^n(z))$ is smaller than $c_z e^{2\varepsilon n} \prod_{i=0}^{[n/N]} \|Df^{iN}|_{E^s(x)}\|$. These two estimate together and the partial hyperbolicity imply that the distance $d(f^n(x), f^n(z))$ is smaller than $ce^{-n\varepsilon} \min\{m(Df^n|_{E^c(x)}), 1\}$. Hence $z \in W^{ss}(x)$. We have shown $W^{ss}(y) \subset W^{ss}(x)$. The reverse inclusion holds similarly. This gives item (c).

Property (d) (again for the C^1-topology) comes from the dependance of $\mathcal{D}_E(x)$ in Theorem 4.5.

□

4.4. **Hölder continuity of the bundles.** Let M be a smooth manifold, $\Lambda \subset M$ a compact subset and $E \subset T_\Lambda M$ a continuous subbundle. We say that E is θ-Hölder if there exists $c > 0$ such that $d(E(x), E(y)) \leq c d(x, y)^\theta$. Here, the distance we consider is any distance in the Grasmannian bundle $G_k(M)$ over M of dimension $k = \dim E$ which is smoothly equivalent to the one given by the Riemannian metric in M. The next result should go back to Anosov for Anosov systems, see for instance [Br2].

Theorem 4.11. *Let $f : M \to M$ be a C^2-diffeomorphism and $\Lambda \subset M$ a compact f-invariant set admitting a dominated splitting of the form $T_\Lambda M = E \oplus F$. Then, the bundles E is θ-Hölder continuous for some $\theta \in (0, 1]$.*

Remark 4.12. a. Even if f is smooth, the bundles E, F are in general not Lipschitz.

b. One can take any exponent θ such that for $N \geq 1$ large and for any $x \in K$, the following *pinching condition* is satisfied

$$\|Df^N|_{E(x)}\| \, \|Df^N|_{F(x)}\|^\theta < m(Df^N|_{F(x)}).$$

In particular, it is sometimes possible to get a Lipschitz regularity; under a *bunching condition*: $\|Df|_{E(x)}\| < m(Df|_{F(x)}) / \|Df|_{F(x)}\|$ (see also Section 4.7).

c. If f is just C^1, then one can construct examples of diffeomorphisms with a dominated splitting (even Anosov) on which the regularity of the bundles is no better than continuous. To do this, start with a linear Anosov diffeomorphism of $\mathbb{T}^2$ and consider a sequence of periodic orbits $\mathcal{O}_n \to p$ where p is a fixed point. Using Franks Lemma (see for example [Cr2, Section 2.3]) one can modify the invariant bundles of $\mathcal{O}_n$ about the size of any function which goes to zero with $d(\mathcal{O}_n, p)$ and therefore break all possible modulus of continuity better than plain continuity.

PROOF OF THEOREM 4.11. Let us assume that the metric is adapted to the domination and to the θ-pinching condition: It satisfies at any point $x \in K$:

$$\lambda(x) := \max_{x \in K} \|Df|_{E(x)}\| \,/\, m(Df|_{F(x)}) < 1 \text{ and } \|Df|_{E(x)}\| \, \|Df_{|}F(x)\|^\theta < m(Df|_{F(x)}).$$

By working in charts, one has for some constant $c > 0$,

$$\begin{aligned} d(E_x, E_y) &\leq d(Df^{-1}_{f(x)}(E_{f(x)}), Df^{-1}_{f(x)}(E_{f(y)})) + d(Df^{-1}_{f(x)}(E_{f(y)}), Df^{-1}_{f(y)}(E_{f(y)})) \\ &\leq \lambda(x) d(E_{f(x)}, E_{f(y)}) + c d(f(x), f(y))^\theta. \end{aligned}$$

The first part of the estimate comes from the contraction in the Grassmanian bundle, see Exercise 11. The second from the C^2 regularity.

By induction one gets *for any* $k \geq 1$,

$$d(E_x, E_y) \leq c \sum_{j=0}^{k-1} \Big(\lambda(f^j(x)) \dots \lambda(x)\Big) d(f^{j+1}(x), f^{j+1}(y))^{\theta} +$$

$$+ \Big(\lambda(f^k(x)) \dots \lambda(x)\Big) d(E_{f^k(x)}, E_{f^k(y)}).$$

For $\varepsilon > 0$ small, one can bound $d(f^j(x), f^j(y))$ by $e^{\varepsilon j}.\|Df^j|_{F(x)}\|.d(x,y)$, provided x, y are close enough. Since $\lambda(x)\|Df|_{F(x)}\|^{\theta}$ is smaller than some constant < 1, this gives

$$d(E_x, E_y) \leq c(d(x,y)^{\theta} + \lambda^k).$$

By choosing k large enough, one gets the estimate.

□

The proof of the previous theorem also implies the following.

Lemma 4.13. *Let $f : M \to M$ be a C^2-diffeomorphism and $\Lambda \subset M$ a compact f-invariant set admitting a dominated splitting of the form $T_{\Lambda}M = E \oplus F$ such that for N large enough: $\|Df^N|_{E(x)}\|^2 < m(Df^N|_{F(x)})$. Then there exists $c, \delta > 0$ with the following property.*

Consider any two points $x, y \in K$ whose iterates satisfy:

$$d(f^i(x), f^i(y)) < \delta \text{ for any } 0 \leq i \leq n \quad \Rightarrow \quad f^n(y) - f^n(x) \in \mathscr{C}^*(x).$$

Then, $d(E(x), E(y)) \leq c d(x,y)$.

Proof. By our assumptions on x, y, one bounds $d(f^j(x), f^j(y))$ by $e^{\varepsilon j}.\|Df^j|_{E(x)}\|.d(x,y)$. The condition $\|Df^N|_{E(x)}\|^2 < m(Df^N|_{F(x)})$ gives $\lambda(x)\|Df|_{E(x)}\| < 1$ which ensures the uniform convergence of the series in the previous argument.

□

4.5. **Smoothness of the leaves – r-domination.** We now discuss the smoothness of the stable manifolds and of the plaques.

Definition 4.14. The bundle E in invariant splitting $T_K M = E \oplus F$ is *r-dominated* with $r \geq 1$, if there exists $c > 0$ and $\lambda \in (0,1)$ such that for any $x \in K$, $u \in E_x \setminus \{0\}$, $v \in F_x \setminus \{0\}$ and $n \geq 0$,

$$\max\left\{ \frac{\|D_x f^n u\|}{\|u\|}, \left(\frac{\|D_x f^n u\|}{\|u\|}\right)^r \right\} \leq c\lambda^n \frac{\|D_x f^n v\|}{\|v\|} \tag{4.5.1}$$

Exercise 33. If E is r-dominated, it is ℓ-dominated for any $1 \leq \ell \leq r$.

We state a C^r-version of the plaque family theorem.

Theorem 4.15 (C^r-plaque family [HPS])**.** *If f is C^r, if K admits a dominated splitting $T_K M = E \oplus F$ and if E is* r-dominated, $r \geq 1$, *then one can choose the embeddings $\mathscr{D}_E(x)$ defining a plaque family tangent to E to be C^r and to depend continuously on x for the C^r-topology.*

It allows to end the proof of the stable manifold theorem.

Proof of the stable manifold theorem (C^r version). From Corollary 4.8, the (local) strong stable manifold coincides locally with the plaques of any plaque family tangent to E^s. It remains to prove that one can take these plaques to be C^r, i.e. that theorem 4.15 applies.

Note that in the case the bundle E is uniformly contracted $\left(\frac{\|D_x f^n u\|}{\|u\|}\right)^r \leq \frac{\|D_x f^n u\|}{\|u\|}$ provided n is large enough and for any $u \in E$. Hence $E = E^s$ is r-contracted for any r, in particular, for the r such that $f \in \mathrm{Diff}^r(M)$ and the conclusion then follows.

□

PROOF OF THE C^r-PLAQUE FAMILY THEOREM (THEOREM 4.15). One will prove it inductively on the smoothness r.

Induction argument. Let k be the dimension of the fibers of the bundle E. Let $p\colon G_k(M) \to M$ be the Grassmannian bundle, that is, for each $x \in M$, $(G_k(M))_x$ denotes the space of k-dimensional vector planes of T_xM. This defines a smooth compact manifold without boundary. Assuming that $r \geq 2$, the tangent map $\varphi := Df$ acts as a C^{r-1} diffeomorphism on this space.

Moreover the bundle E_K defines a compact invariant set $\Lambda \subset G_k(M)$ which projects homeomorphically by p on K.

Lemma 4.16. *If $r \geq 2$ and if E is r-dominated, then Λ admits a dominated splitting $\mathscr{E} \oplus \mathscr{F}$ which is $(r-1)$-dominated. The map $Dp\colon \mathscr{E}_\Lambda \to E_K$ is a diffeomorphism and $\mathscr{F}_x$ contains* $\ker(D_x p)$ *for each $x \in \Lambda$.*

Idea of the proof. We will assume that the metric is adapted. Note that the tangent spaces $T\mathscr{P}$ to the fibers $\mathscr{P}(x) = p^{-1}(x)$ of $p\colon G_k(M) \to M$ are $D\varphi$-invariant.

The dominated splitting $\mathscr{E} \oplus \mathscr{F}$ is obtained with the cone field criterion. Instead to check it in detail, we just observe that how $D\varphi$ expands in the different directions at points of Λ:

- transversally to the fibers, $D\varphi$ acts as Df on TM and reproduces the decomposition $E \oplus F$,
- along the fibers, $D\varphi$ acts as an expansion, bounded from below by the quantity $m(Df(x)_{|F})/\|Df(x)_{|E}\|$, which is larger than 1 by the domination.

In order to check this second point, consider a plane $\Delta \subset T_xM$ close to E_x and let us see how close is the image $Df(x).\Delta$ to $E_{f(x)}$. Let us take a vector $u \in \Delta$ such that $u = u_0 + \tilde{u}$ where $u_0 \in E$ is a unit vector and $\tilde{u} \in F$ is small. Its image after normalization by $\|D_x f u_0\|^{-1}$ is $v = v_0 + \tilde{v}$, where

- $v_0 = \frac{D_x f u_0}{\|D_x f u_0\|}$ belongs to $E(f(x))$ and is a unit vector
- $\tilde{v} = \frac{D_x f \tilde{v}}{\|D_x f u_0\|}$ belongs to $F(f(x))$ and is still small.

We have
$$\|\tilde{v}\| = \frac{\|D_x f \tilde{u}\|}{\|D_x f u_0\|} \geq \frac{m(Df|_{F(x)})}{\|Df|_{E(x)}\|} \|\tilde{u}\|,$$
which gives the announced expansion.

The 1-dominated splitting: Since E is 2-dominated, one has
$$\frac{m(Df|_{F(x)})}{\|Df|_{E(x)}\|} > \|Df|_{E(x)}\|,$$
hence the expansion along the fibers is stronger than along E.

This allows to obtain a dominated splitting $\mathscr{E}^c \oplus \mathscr{F}$, where $\mathscr{E}^c$ lifts E and $\mathscr{F} = Dp^{-1}(F)$.

The $r-1$-domination: It remains to check that $\mathscr{E}$ is $r-1$-dominated, i.e. that
$$\frac{m(Df|_{F(x)})}{\|Df|_{E(x)}\|} > \|Df|_{E(x)}\|^{r-1},$$
which is direct from the fact that E is r-dominated. This ends the proof of the lemma. □

Let us continue with the proof of Theorem 4.15. As in the proof of the plaque family theorem, one introduces a fibred diffeomorphism $\hat{f}$ on TM that is tangent to Df at the 0-section. It is close to Df, hence there exists a splitting $\widehat{E}\oplus\widehat{F}$ at each point of each space T_xM, $x\in K$, which extends $E\oplus F$ and such that $\widehat{E}$ is still r-dominated. The plaque family theorem gives a family of C^1-graphs $\psi_x\subset T_xM$ for each $x\in K$. By invariance, the graphs are tangent to $\widehat{E}$. Since $\widehat{E}$ is 2-dominated and $\hat{f}$ is C^2 on each space T_xM, the Lemma 4.13 shows that the bundle $\widehat{E}$ is Lipschitz along each graph ψ_x.

The map $\hat{\varphi}:=D\hat{f}$ is a C^{r-1}-diffeomorphism acts on the Grassmanian bundle of TM. The Lemma 4.16 applied to $\hat{f}$ shows that $\hat{\varphi}$ also preserves a dominated splitting $\mathscr{E}\oplus\mathscr{F}$ and that $\mathscr{E}$ is $(r-1)$-dominated. The bundle $\widehat{E}$ in restriction to the graphs ψ_x defines a collection of Lipschitz graphs Ψ in $(G_k(M))$ which is invariant by $\hat{\varphi}$ and transverse to the bundle $\mathscr{F}$. As in the proof of Theorem 4.5, one deduces that the graphs Ψ are C^1 and tangent to $\mathscr{E}$. This means that E is C^1 on the graphs ψ_x, i.e. that these graphs are C^2.

If $r\geq 3$, one can repeat this argument and prove that the graphs Ψ are C^2, i.e. that the graphs ψ are C^3. By induction, one concludes that the plaques are C^r. □

4.6. **Reduction of the dimension and normally hyperbolic manifolds.** Possibly, the most important information given by the existence of a dominated splitting or of the existence of a partially hyperbolic splitting comes with the fact that Theorem 4.5 allows one to "reduce the dimension" of the study. In general, if one has a strong partially hyperbolic splitting, one can use Theorem 4.5 to reduce the situation to a kind of skew-product over a hyperbolic set, at least, one can think the skew-product over a hyperbolic set as a *toy model* for the general situation. This approach has been pursued when $\dim E^c=1$ (see [Cr$_1$]).

However, there are some cases where the reduction of dimension is even more drastic: instead of obtaining a sequence of maps of a lower dimensional manifolds, one can in some cases deal with a unique one. This is the case when the dynamics one is interested in lives in a normally hyperbolic submanifold.

Theorem 4.17 (Bonatti-Crovisier [BC$_2$])**.** *Let K be a compact f-invariant set admitting a partially hyperbolic splitting of the form $T_KM=E^c\oplus E^u$. Assume moreover that for every $x\in K$ one has that $W^{uu}(x)\cap K=\{x\}$.*

Then, there exists a C^1-submanifold $\Sigma\subset M$ containing Λ and tangent to E^c at every point of K such that it is locally invariant (i.e. $f(\Sigma)\cap\Sigma$ is a neighborhood of K inside Σ).

If E^c is r-dominated and f is C^r, the submanifold Σ can be chosen C^r also.

We do not prove this result, but mention two ingredients:

- The assumption $W^{uu}(x)\cap K=\{x\}$ allows to show that near each point $x\in K$, the set K is "tangent" to E^c_x. Whitney's extension theorem then provides a submanifold S that contains K and is tangent to E^c_x at each point of K.
- We then define an iteration scheme which allows to iterate backward S and obtain a sequence of submanifolds having uniform sizes. The graph transform technique shows that it converges to a locally invariant submanifold Σ.

Exercise 34. Show the converse: if K is partially hyperbolic and it is contained in such a submanifold, then one has that $W^{uu}(x)\cap K=\{x\}$ for every $x\in K$.

Definition 4.18. An compact C^1-manifold $N\subset M$ is r-*normally hyperbolic* if there exists a partially hyperbolic splitting $T_NM=E^s\oplus E^c\oplus E^u$ with $E^c=TN$ such that $(E^s\oplus E^c)$ is r-dominated by E^u for f and $(E^c\oplus E^u)$ is r-dominated by E^s for f^{-1}.

Corollary 4.19 ([HPS]). *If f is a C^r diffeomorphism and N is a r-normally hyperbolic manifold, then N is C^r.*

Proof. Indeed by the previous theorem N is contained in a C^r-submanifold Σ. It is tangent to $E^c = TN$, so $N = \Sigma$. □

4.7. **Transverse smoothness of the laminations – Bunching.** Let us now consider a diffeomorphism f which preserves a partially hyperbolic splitting $TM = E^c \oplus E^u$ over the whole manifold M (i.e. a globally partially hyperbolic diffeomorphism). In this case the strong unstable manifold induces a foliation of M.

Since the local leaves vary continuously for the C^1-topology, this is a $C^{0,1}$-foliation: each point $x \in M$ has a neighborhood U and a continuous chart $\varphi\colon U \to \mathbb{R}^d$ which is C^1 along unstable manifolds and sends each local leaf to a horizontal plane $\mathbb{R}^k \times \{0\}^{d-k}$ where $d = \dim M$ and $k = \dim E^u$. In general this foliation (i.e. the chart φ) is not even differentiable. More precisely let us consider two discs $\mathscr{D}_1$, $\mathscr{D}_2$ transverse to a same leaf $W^{uu}(x)$ at some points x_1, x_2; the *holonomy* along the unstable foliation induces a homeomorphism h between a neighborhood of x_1 in $\mathscr{D}_1$ to a neighborhood of x_2 in $\mathscr{D}_2$. This holonomy is in general not even Lipschitz. Similarly to Theorem 4.11, if the diffeomorphism f is C^2, the holonomies are θ-Hölder for some $\theta \in (0,1]$ (see [PSW]).

Definition 4.20. If K has a partially hyperbolic splitting $T_K M = E^c \oplus E^u$, the bundle E^c is *bunched* if there exists $N \geq 1$ satisfying for each $x \in K$

$$\frac{\|Df^N_{|E^c}(x)\|}{m(Df^N_{|E^c}(x))} < m(Df^N_{|E^u}(x)).$$

The following result admits stronger and more precise formulations with sharper estimates and different regularities depending on the type of bunching. Here we present only this statement to give a taste on the type of results.

Theorem 4.21 (Pugh-Shub-Wilkinson [PSW]). *Let f be a globally partially hyperbolic C^2-diffeomorphism preserving a splitting $TM = E^c \oplus E^u$ which is center-bunched.*

Then the strong unstable foliation is C^1.

PROOF. It is enough to prove that the bundle E^u is C^1. Indeed, one can then define locally a decomposition $E^u = \mathbb{R}.X_1 \oplus \cdots \oplus \mathbb{R}.X_k$ by C^1 non-singular vector fields. (Considering coordinates $(x_1, \dots, x_n)$ such that the projection of E^u to $x_1, \dots, x_d$ is an isomorphism, X_i is the unit vector whose projection to x_j vanishes, for all $j \neq i$.) By construction, the integral curves to $\mathbb{R}.X_1$ are contained in the strong unstable leaves. Hence the Lie brackets $[X_i, X_j]$ all vanish. One then apply Frobenius theorem which asserts that the plane field E^u uniquely integrate as a foliation by C^1-leaves and that this foliation is C^1.

As in the proof of Theorem 4.15, one considers the action φ induced by Df on the Grassmannian bundle $p\colon G_k(M) \to M$, where k denotes the dimension of the unstable spaces $E^u(x)$. Since f is C^2, the map φ is C^1. The unstable bundle E^u defines an invariant compact set Λ of φ and the map $p\colon \Lambda \to M$ is a homeomorphism.

Lemma 4.22. *If E^c is bunched, then $\Lambda \subset G_k(M)$ admits a partially hyperbolic splitting $TG_k(M)_{|\Lambda} = \mathscr{E}^s \oplus \mathscr{E}^c$, where $\mathscr{E}^s$ is the tangent space to the fibers of the bundle $G_k(M) \to M$.*

Idea of the proof. The tangent space $T\mathscr{P}$ to the fibers $\mathscr{P}(x) = p^{-1}(x)$ of p are invariant and are a candidate for $\mathscr{E}^s$. The bundle $\mathscr{E}^c$ will be transverse to the fibers, hence $Dp\colon \mathscr{E}^c \to TM$ will be a diffeomorphism which conjugates $D\varphi_{|\mathscr{E}^c}$ to Df. The partial hyperbolicity should be checked with the cone field criterion, but the main points are:

– the fiber $\mathscr{P}(x)$ is contracted in a neighborhood of $E^u(x) \in \Lambda$,
– the contraction is stronger than the contraction by Df on TM.

Consider a plane $F \subset T_xM$ close to $E^u(x)$ and let us see how close is the image $D_xf(F)$ to $E^u(f(x))$. We will assume that the metric is adapted. Let us take a vector $u \in F$ such that $u = u_0 + \tilde{u}$ where $u \in E^u$ is a unit vector and $\tilde{u} \in E^c$ is small.

Its image after normalization by $\|D_xfu_0\|^{-1}$ is $v = v_0 + \tilde{v}$, where

– $v_0 = \frac{D_xfu_0}{\|D_xfu_0\|}$ belongs to $E^u(f(x))$ is a unit vector
– $\tilde{v} = \frac{D_xf\tilde{u}}{\|D_xfu_0\|}$ belongs to $E^c(f(x))$ and is still small.

Note that
$$\|\tilde{v}\| = \frac{\|D_xf\tilde{u}\|}{\|D_xfu_0\|} \leq \frac{\|Df|_{E^c(x)}\|}{m(Df|_{E^u(x)})}\|\tilde{u}\|.$$
This gives a contraction by $\frac{\|Df|_{E^c(x)}\|}{m(Df|_{E^u(x)})}$, which is smaller than 1 by the domination.

The bunching gives
$$\frac{\|Df|_{E^c(x)}\|}{m(Df|_{E^u(x)})} < m(D_xf) = m(Df|_{E^c(x)}).$$
which gives the second required property. □

Since the fibers of p are invariant by φ and tangent to $\mathscr{E}^s$, they contain the strong stable manifolds of Λ. Each fiber meets Λ in a unique point. In particular $W^{ss}(p) \cap \Lambda = \{p\}$ for each $p \in \Lambda$. The assumptions of Theorem 4.17 are thus satisfied. One deduces that Λ is contained in a locally invariant C^1-submanifold Σ tangent to $\mathscr{E}^s$. This manifold is transverse to the fibers, and since $p\colon \Lambda \to M$ is a homeomorphism, Σ coincides with Λ. We have thus shown that the plane field E^u is C^1, as required. □

Exercise 35. a) Prove that for a C^2 Anosov diffeomorphism such that E^u is one-dimensional, the stable foliation is C^1.

b) For a C^2 Anosov flow such that E^u is one-dimensional, the center-stable foliation is C^1.

c) For an Anosov C^3-diffeomorphism of $\mathbb{T}^2$ which preserves the volume, the stable foliation is $C^{1+\alpha}$ for any $\alpha \in (0,1)$.

See [Hass, HuK] for more results on regularity of the splitting for Anosov systems.

4.8. **Dynamical coherence.** Let us consider a globally partially hyperbolic diffeomorphism f with a splitting $TM = E^s \oplus E^c \oplus E^u$. In general one can not find sub manifolds tangent to E^c. Indeed there is an algebraic example [Wi$_1$] such that E^c is smooth but the Frobenius integrability condition fails (see also [BuW]).

Definition 4.23. One says that a globally partially hyperbolic diffeomorphism f is *dynamically coherent* if there exists two foliations $\mathscr{W}^{cs}$, $\mathscr{W}^{cu}$, that are invariant by f, whose leaves are C^1 and tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively.

Notice that we do not ask for uniqueness of the foliation. In principle there might be many foliations tangent to $E^s \oplus E^c$ (or $E^c \oplus E^u$), even many invariant foliations. See [BuW] for discussion of different types of integrability of the bundles.

Exercise 36. Show that if f is dynamically coherent, then there exists an f-invariant foliation $\mathscr{W}^c$ tangent to E^c.

Dynamical coherence is interesting since it allows locally to project transversally to the center. These systems behave to some extent as fibred examples and finer properties may sometimes be obtained (such as their statistical description).

When the center is one-dimensional, there exist curves tangent to E^c by Peano's theorem. But an invariant foliation tangent to E^c does not necessarily exist.

Example 4.24 (Non-dynamically coherent examples [RHRHU$_3$])**.** We will construct a globally partially hyperbolic diffeomorphism of $\mathbb{T}^3$ with one-dimensional center bundle which is not dynamically coherent.

Consider $A \in SL(2,\mathbb{Z})$ a hyperbolic matrix, we denote as $\lambda \in (0,1)$ the stable eigenvalue of A and v^s, v^u denote unit vectors in the eigenspaces associated respectively to λ and λ^{-1}. We further consider $\psi : S^1 \to S^1$ a diffeomorphism of S^1 which we view as $[-1,1]/_{(-1)\sim 1}$ such that it verifies the following properties:

- $\psi(0) = 0$ and $\psi(1) = 1$ are the only fixed points of ψ.
- $0 < \psi'(0) < \lambda < 1 < \psi'(1) < \lambda^{-1}$.

Define a diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ viewed as $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$ and defined by:

$$f(x,t) = (Ax + \varphi(t)\mathrm{v}^s, \psi(t))$$

where $\varphi : S^1 \to \mathbb{R}$ will be defined later. We get that

$$f^{-1}(x,t) = (A^{-1}x - \varphi(\psi^{-1}(t))\mathrm{v}^s, \psi(t))$$

We denote v^c to be a vector tangent to $\{x\} \times S^1$. So, at each point we have a basis for the tangent space given by the vectors $\mathrm{v}^s, \mathrm{v}^c, \mathrm{v}^u$, we denote as $\langle B \rangle$ to the vector space spanned by a subset B in a vector space. For such an f we have the following properties:

(P1) There exists a normally attracting (see section 4.9) torus $T^{cu} = \mathbb{T}^2 \times \{0\}$. Moreover, $f|_{T^{cu}}$ is partially hyperbolic with splitting given by $E^s = \langle \mathrm{v}^c \rangle$, $E^c = \langle \mathrm{v}^s \rangle$ and $E^u = \langle \mathrm{v}^u \rangle$.

(P2) There exists a repelling (but not normally repelling) torus $T^{su} = \mathbb{T}^2 \times \{1\}$. Moreover, $f|_{T^{su}}$ is partially hyperbolic with splitting given by $E^s = \langle \mathrm{v}^s \rangle$, $E^c = \langle \mathrm{v}^c \rangle$ and $E^u = \langle \mathrm{v}^u \rangle$.

We must choose $\varphi : S^1 \to \mathbb{R}$ in such a way that f is globally partially hyperbolic[12]. We will use the cone criterium (Theorem 2.6). Let us first write the derivative of f:

$$Df_{(x,t)}(u\mathrm{v}^u + s\mathrm{v}^s + c\mathrm{v}^c) = (\lambda^{-1}u)\mathrm{v}^u + (\lambda s + \varphi'(t)c)\mathrm{v}^s + (\psi'(t)c)\mathrm{v}^c$$

This implies that the subspaces $\langle \mathrm{v}^u \rangle$ and $\langle \mathrm{v}^s, \mathrm{v}^c \rangle$ are invariant and a small cone field around $\langle \mathrm{v}^s, \mathrm{v}^c \rangle$ is contracted by Df^{-1} while vectors outside this cone field are expanded by Df. This implies that we must only concentrate on constructing the cone field $\mathscr{C}^{cu}$ for a suitable choice of φ. From the invariance seen above, it suffices to work in the plane $\langle \mathrm{v}^s, \mathrm{v}^c \rangle$ and define the cones in that plane.

An easy calculation shows that:

$$Df^n_{(x,t)}(s\mathrm{v}^s + c\mathrm{v}^c) = \left(\lambda^n s + \sum_{j=1}^{n} \lambda^{j-1}\varphi'(\psi^{n-j}(t))(\psi^{n-j})'(t)c\right)\mathrm{v}^s + ((\psi^n)'(t)c)\mathrm{v}^c$$

We will demand φ to verify the following.

[12]Since the non-wandering set is trivially partially hyperbolic we already know that f is partially hyperbolic. However, the splitting might not extend to the whole manifold.

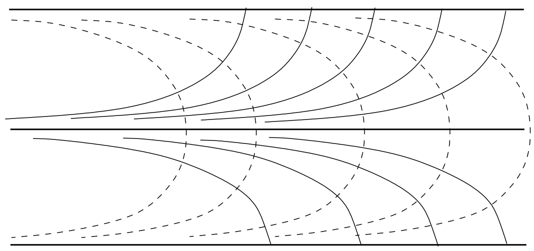

FIGURE 11. The center and stable directions inside a center-stable manifold.

(F1) φ' are different from zero and have constant sign in $(0,1)$ and $(-1,0)$.
(F2) $\varphi'(0) = \varphi'(1) = 0$ and $\varphi''(0)$, $\varphi''(1)$ are non-zero.

To fix ideas, we can assume that φ' is positive in $(0,1)$ and negative in $(-1,0)$. We claim that this is enough to guarantee that f is partially hyperbolic. We will work only in $(0,1)$ the other side is symmetric.

We have to define $\mathscr{C}^{cu}$ in $\mathbb{T}^3$. We will just define $\mathscr{C}^{cu} \cap \langle \mathrm{v}^s, \mathrm{v}^c \rangle$ which as we mentioned is enough. First we define it in a neighborhood of T^{su}. There, it must contain the center direction of $f|_{T^{su}}$ so we choose a very narrow cone field in a small neighborhood of T^{su} of vectors of the form $v = a\mathrm{v}^s + b\mathrm{v}^c$ with $|a| \le \varepsilon |b|$. In a small neighborhood of T^{su}, since $\psi'(t)$ is larger than λ we know that the cone field is invariant.

Now, we will propagate this cone field which we have defined in $\mathbb{T}^2 \times [t_0, 1]$ (with t_0 very close to 1) by iterating[13] it by Df^n which defines a cone field in $\mathbb{T}^2 \times (0,1]$. Since φ' is positive in all of $(0,1)$ we can see that after some iterates, the cone field gets twisted into the quadrant of vectors $v = a\mathrm{v}^s + b\mathrm{v}^c$ with $ab > 0$ (or possibly $a = b = 0$). This is crucial for the construction of the cone-field, and it is where the need to add φ becomes clear (notice that if $\varphi = 0$ the diffeomorphism cannot be partially hyperbolic).

Once the points arrive to the region $(0, t_1)$ where ψ' is much smaller than λ one gets that this cone field starts getting thinner and closer to the subspace $\langle \mathrm{v}^s \rangle$. Since in $\mathbb{T}^2 \times [0, t_1)$ one can consider the cone field of vectors of the form $v = a\mathrm{v}^s + b\mathrm{v}^c$ with $|b| \le \varepsilon |a|$ which is also Df-invariant one gets that one can glue both cone-fields in order to get a well defined global cone-field $\mathscr{C}^{cu}$ which is Df-invariant. It is easy to check that vectors outside $\mathscr{C}^{cu}$ when iterated by Df^{-n} get expanded uniformly so that the stable bundle is uniformly contracted. The same argument can be done in $\mathbb{T}^2 \times [-1, 0]$ which implies partial hyperbolicity.

To show that E^c is not integrable, we will use a simple argument we learned from C. Bonatti[14]. Assume by contradiction that f is dynamically coherent. Choose a fixed point $p \in T^{su}$ and consider a small arc I in the center foliation for which p is one extreme point. Since in p the center direction is expanding, this arc I gets expanded by forward iterations. Call I_n to the arc $f^n(I)$, it follows by invariance of the center foliation that $I_n \subset I_{n+1}$. After enough iterations, one gets that I_n intersects a small neighborhood of T^{cu}. Since the center direction gets tilted towards the center direction in T^{cu} which is exponentially contracted, one obtains that the length of $f^n(I)$ is bounded by a geometric sum and therefore, the union of all I_n has finite length. Since T^{cu} is attracting, this implies that the curve must "land" in T^{cu} but T^{cu} is foliated by center curves. This implies that E^c is not uniquely integrable. Using the fact that the same happens from both sides (and

[13]One must consider its iterate and then thicken it a little in order to have that the closure of the cone gets mapped into the interior of its image.

[14]Who attributes it to Grines, Levchenko, Medvedeev and Pochinka.

that the curves land with the same direction because of property (F2), see figure 11) one contradicts dynamical coherence[15].

$\diamondsuit$

It is an open problem to decide if a transitive partially hyperbolic diffeomorphism with one-dimensional center must be dynamically coherent. In dimension 3, the work of Burago-Ivanov ([BI]) is quite relevant and an important partial answer. It allows one to define *branching foliations* which are sometimes useful to understand dynamical coherence (one must notice however that this branching foliations exist in the example of [RHRHU_3]).

In [BBI_2] branching foliations as well as a criteria of Brin (see remark 4.9) is used to obtain dynamical coherence for absolutely partially hyperbolic diffeomorphisms of $\mathbb{T}^3$.

In the series of works [Pot_2, HPo_1, HPo_2], among other things, dynamical coherence is established for strong partially hyperbolic diffeomorphisms in 3-manifolds with (virtually) solvable fundamental group, unless the diffeomorphism admits a torus tangent to either $E^s \oplus E^c$ or $E^c \oplus E^u$. In the setting of 3-manifolds, it is a conjecture of [RHRHU_3] that the existence of such tori is the unique obstruction for dynamical coherence (see [CRHRHU] for a more detailed account on this problem).

We wish to remark that when the center-dimension is higher, less is known. There are the local stability results ([HPS]) which we shall review in the next section, but very few "global stability results" are known. We refer the reader to [Pot_2, FPS] and reference therein for some results in this direction under some strong assumptions on the underlying topology of the manifold and the isotopy class of the diffeomorphism. Techniques for studying the involutivity of bundles under dynamical assumptions are being developed by Luzzatto, Turelli and War [LTW], but still many questions remain.

Exercise 37 ([BI] Proposition 3.1)**.** Let $f : M \to M$ be a partially hyperbolic diffeomorphism with splitting $TM = E^s \oplus E^c \oplus E^u$ with $\dim E^c = 1$ and γ be a closed arc tangent to E^c. Show that the saturation of γ by local strong stable manifolds is an embedded submanifold tangent to $E^s \oplus E^c$.

Notice that this does not imply that the saturation of a center curve by strong stable manifolds is a complete submanifold of M. Indeed, in the example of [RHRHU_3] center-stable leaves are not complete. The problem of completeness of center-stable manifolds seems to be at the heart of difficulties concerning the understanding of dynamical coherence in dimension 3 (see [BoW, BI, HPo_2]).

4.9. **Structural stability of normally hyperbolic laminations.** It is sometimes useful to perform the graph transform method globally. This is the case in the proof of persistence of normally hyperbolic submanifolds or foliations. We refer the reader to [HPS] or [Ber] for detailed proofs.

Consider $f : M \to M$ a C^1-diffeomorphism and $\Lambda \subset M$ a compact f-invariant set. We shall assume that Λ is *laminated* by an f-invariant lamination $\mathscr{L}$. This means that for each $x \in \Lambda$ there exists a C^1-injectively immersed submanifold $\mathscr{L}(x) \subset \Lambda$ with the following properties:

- if $\mathscr{L}(x) \cap \mathscr{L}(y) \neq \emptyset$ then $\mathscr{L}(x) = \mathscr{L}(y)$,

[15] If one chooses φ differently one can construct a dynamically coherent example for which E^c is not uniquely integrable, see [RHRHU_3].

- if $x_n \to x$ then $\mathscr{L}(x_n)$ converges to $\mathscr{L}(x)$ uniformly in the C^1-topology in compact subsets, (in particular the map $x \mapsto T_x\mathscr{L}(x) \subset T_xM$ defines a continuous distribution),
- $f(\mathscr{L}(x)) = \mathscr{L}(f(x))$ for each $x \in \Lambda$.

We say that the lamination $\mathscr{L}$ is *normally hyperbolic* if f admits a partially hyperbolic splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ where $E^c(x) = T_x\mathscr{L}(x)$ for every $x \in \Lambda$. Moreover, we say it is *normally expanded* (resp. *normally contracted*) if $E^s = \{0\}$ (resp. $E^u = \{0\}$). One defines similarly (see Definition 4.18) an r-normally hyperbolic lamination.

Remark 4.25. Notice that if $\mathscr{L}$ is a lamination by points, normal hyperbolicity of $\mathscr{L}$ is equivalent to have that uniform hyperbolicity of Λ.

Whenever there is a normally hyperbolic lamination, one has the following persistence result:

Theorem 4.26 (Stability of normally hyperbolic laminations)**.** *Let $f : M \to M$ be a C^1-diffeomorphism, leaving invariant a normally hyperbolic lamination $\mathscr{L}$ on a compact set Λ. Then, a C^1-neighborhood $\mathscr{U}$ of f exists such that for every $g \in \mathscr{U}$ there exists a continuous map $i^g : \Lambda \to M$ with the following properties:*

- *(**Invariance:**) $g(i^g_x(\mathscr{L}(x))) = i^g_{f(x)}(\mathscr{L}(f(x)))$ (and hence $\Lambda_g := i^g(\Lambda)$ is invariant).*
- *(**Continuation of leaves:**) the map i^g induces an immersion $i^g_x : \mathscr{L}(x) \to M$ (possibly no longer injective) whose image is everywhere tangent to E^c_g (the continuation of the bundle E^c of f for g on Λ_g).*
  As g gets C^1-close to f, the maps i^g_x get C^1-close to the inclusions $i_x : \mathscr{L}(x) \hookrightarrow M$.
- *(**Continuity:**) The leaves $L^x_g = i^g_x(\mathscr{L}(x))$ with $x \in \Lambda$ vary continuously for the C^1-topology on compact subsets.*
- *(**Uniqueness:**) The (image of the) immersion i^g_x is unique with the condition of being C^0-close to i_x.*

The idea of the proof is to perform a graph transform argument in an entire neighborhood of the immersion. This involves unwrapping the immersion to an abstract immersion into a neighborhood of the leaf which depends on the point and then applying arguments very similar to the ones we have already done albeit more technical.

This result is not completely satisfactory since in principle leaves of the new "lamination" could merge. One sometimes calls this *branching laminations* (see [BI] for use of this notion).

Under a technical condition (which is always satisfied in case the lamination can be extended to a neighborhood into a C^1-foliation) it is possible to improve Theorem 4.26 to have a true lamination for diffeomorphisms close to f. This condition is known by the name of *plaque-expansiveness* (we refer the reader to [HPS] and [Ber] for more information about it). In subsection 4.10, we give some indications on how to use this property to show that leaves do not merge.

We make the following remarks on Theorem 4.26 since we shall not enter in the details of its proof. The first remark is that to be able to perform a global graph transform one uses strongly the fact that the dynamics are C^0-close (and not only that the invariant bundles are close). The other remark is that even in the simplest case of a closed submanifold $N \subset M$ which is normally hyperbolic, the graph transform must be performed with some care since does not have a priori a fixed point on which to "center" the graph transform argument. We refer the reader to [BerB] for a short proof in this particular and easier case and propose the following:

Exercise 38. Show the persistence of a normally attracting circle in a surface.

Exercise 39. Give an example of an integrable distribution $E \subset TM$ and of an integrable perturbation such that the tangent foliations are very "different" (for example, leaves get arbitrarily separated in the universal cover). Compare with the stability ensured by Theorem 4.26.

4.10. **Plaque expansivity.** For simplicity, we will work with globally partially hyperbolic diffeomorphisms. Let $f : M \to M$ be a globally partially hyperbolic diffeomorphism with a splitting $TM = E^s \oplus E^c \oplus E^u$ leaving invariant a foliation $\mathscr{F}^c$ tangent to E^c.

Definition 4.27. f is *plaque-expansive* if there exists $\varepsilon > 0$ such that if $(x_n)_{n\in\mathbb{Z}}$ and $(y_n)_{n\in\mathbb{Z}}$ are two sequences sastifying for each $n \in \mathbb{Z}$:

- $f(x_n) \in \mathscr{F}^c_\varepsilon(x_{n+1})$ and $f(y_n) \in \mathscr{F}^c_\varepsilon(y_{n+1})$ (i.e. x_n and y_n are ε-pseudo orbits with jumps in center plaques),
- $d(x_n, y_n) < \varepsilon$ for every $n \in \mathbb{Z}$,

then, $y_0 \in \mathscr{F}^c_\varepsilon(x_0)$.

The following result is contained in [HPS]:

Theorem 4.28. *Let f be a globally partially hyperbolic diffeomorphism preserving a foliation $\mathscr{F}^c$ tangent to E^c which is plaque expansive. Then, for any C^1-small perturbation g of f, there exists a foliation $\mathscr{F}^c_g$ tangent to the center bundle of g and a homeomorphism $h : M \to M$ such that $h \circ f(\mathscr{F}^c(x)) = g(\mathscr{F}^c_g(h(x)))$.*

PROOF. We apply Theorem 4.26. Define the map $h = i^g$. Assume by contradiction that there are points x, y such that $y \notin \mathscr{F}^c_\varepsilon(x)$ and $i^g(x) = i^g(y) = z$.

We consider two sequences: $(x_n)_{n\in\mathbb{Z}}$, $(y_n)_{n\in\mathbb{Z}}$ defined by

$$x_n = (i^g_{f^n x})^{-1}(g^n(z)) \in \mathscr{F}^c(f^n(x)) \text{ and } y_n = (i^g_{f^n y})^{-1}(g^n(z)) \in \mathscr{F}^c(f^n(y)).$$

Since g and f are C^0-close, they are ε-pseudo-orbits with jumps in center plaques. Moreover, since the maps are close to the inclusion and one takes the pre image of the same point, one gets that $d(x_n, y_n) \leq \varepsilon$ for every n. Using plaque-expansiveness one deduces that $y \in \mathscr{F}^c_\varepsilon(x)$ a contradiction. This proves that $h = i^g$ is injective. Since h is close to the identity, it has degree 1, hence is surjective. Hence h is a homeomorphism. We refer the reader to [HPS, Chapter 7] for more details.

□

Exercise 40 ([HPS])**.** Show that if $\mathscr{F}$ is a normally hyperbolic C^1-foliation then it is plaque expansive.

Corollary 4.29 ([HPS])**.** *There exist C^1 open sets of globally partially hyperbolic diffeomorphisms such that all of them are dynamically coherent.*

It is an open problem to determine whether every foliation tangent to the center bundle of a partially hyperbolic diffeomorphism is plaque expansive.

5. ROBUST TRANSITIVITY

In the last parts of these notes, we discuss how to generalize Smale's spectral decomposition theorem to the partially hyperbolic setting. In this section we discuss the case where the dynamics is undecomposable.

We will focus on the class PH^r of C^r-diffeomorphisms such that M is endowed with a global partially hyperbolic structure $TM = E^s \oplus E^c \oplus E^u$ and both E^s and E^u non-trivial. We will mainly concentrate on the subclass $\mathsf{PH}^r_{c=1}$ where the center is one dimensional.

Clearly, the dynamics decomposes when there exists a non-trivial trapping region:

Trapping region: there exists a proper non-empty open set U such that $f(\overline{U}) \subset U$.

In this case there exists at least two chain-recurrence classes (see Section 1.1.2 above). When this does not happen, the whole system is chain-transitive: for any $\varepsilon > 0$, there exists a dense forward ε-pseudo-orbit. But one usually considers stronger forms of recurrence (or undecomposability):

Transitivity: f is transitive if it admits a dense forward orbit.

In the C^1-topology, the following dichotomy holds.

Theorem 5.1 (Bonatti-Crovisier [BC$_1$])**.** *There exist two disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathrm{Diff}^1(M)$ whose union is dense:*

- *the diffeomorphisms in a dense G_δ subset of $\mathcal{O}_1$ are transitive,*
- *the diffeomorphisms in $\mathcal{O}_2$ have a trapping region.*

In this statement, the transitivity only occurs for a dense set of diffeomorphisms in $\mathrm{Diff}^1(M)\setminus\overline{\mathcal{O}}_2$. One would like to improve this result by removing the genericity and getting a robust property, i.e. replace the G_δ set by an open set. This lead to the following notion.

Robust transitivity: A diffeomorphism is robustly transitive if any diffeomorphism C^1-close is still transitive.

This section introduces several geometrical properties of the strong stable/unstable laminations which have dynamical consequences and in particular allows – in some cases – to prove the robust transitivity. Our ultimate goal is to give the following improvement of the previous theorem, which is a combination of results established in [ACP] and [CPoS].

Theorem 5.2 (Abdenur-Crovisier-Potrie-Sambarino)**.** *There exists two disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathsf{PH}^1_{c=1}$ whose union is dense:*

- *the diffeomorphisms in $\mathcal{O}_1$ are (robustly) transitive,*
- *the diffeomorphisms in $\mathcal{O}_2$ have a trapping region.*

Exercise 41. Consider an homeomorphism f on a compact metric space X. Prove that: The homeomorphism f is transitive if and only if for any non-empty open sets U, V, there exists $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. Assuming that X has no isolated points, f is transitive if and only if for any non-empty open sets U, V, there exists $n \in \mathbb{Z}$ such that $f^n(U) \cap V \neq \emptyset$.

5.1. **Accessibility.**
Here we do not necessarily assume that the center is one-dimensional, but the partial hyperbolicity holds on the whole M.

Definition 5.3. A diffeomorphism f which preserves a partially hyperbolic splitting $TM = E^s \oplus E^c \oplus E^u$ is *accessible* if for any $x, y \in M$, there exists a finite collection of points $x_0 = x, x_1, \dots, x_n = y$ such that each pair x_i, x_{i+1} belong to a same strong stable manifold or a same strong unstable manifold. (In particular M has to be connected.)

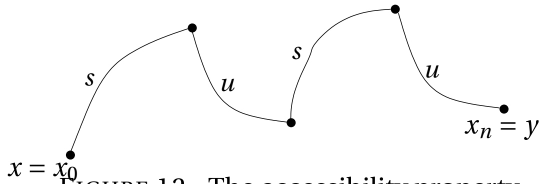

FIGURE 12. The accessibility property.

In $W^\sigma(x)$ the leaf of $\mathcal{W}^\sigma$ through x one has a well defined intrinsic metric induced by the restriction of the riemannian metric to the leaf $W^\sigma(x)$ (for $\sigma = ss, uu$). We say that a subset $U \subset W^\sigma(x)$ has *internal radius* $\geq R$ if it contains a disk of radius R with this metric.

Theorem 5.4 (Brin's argument [Br$_1$])**.** *If f is accessible and $\Omega(f) = M$, then f is transitive.*

In particular if f is accessible and preserves a volume form on M, it must be transitive.

PROOF. Consider two open sets U and $V \subset M$. Choose $x \in U$ and $y \in V$ and consider a sequence $x_0 = x, x_1, \dots, x_n = y$ such that each pair x_i, x_{i+1} belongs to the same stable or unstable manifold as in the definition of accessibility.

Claim. *Given neighborhoods B_i, B_{i+1} of x_i, x_{i+1} and a sequence of open sets $W_k \subset B_i$ accumulating on x_i, there exists a sequence (n_k) in $\mathbb{Z}$ such that $f^{n_k}(W_k)$ accumulates to x_{i+1}.*

Proof. Choose an arbitrary $\varepsilon > 0$. We can assume it is so small that $B(x_{i+1}, \varepsilon) \subset B_{i+1}$. Continuity of the unstable manifold (Theorem 4.3) gives δ and $R > 0$ such that if a point z is at distance smaller than δ from x_i then its unstable manifold of size R intersects $B(x_{i+1}, \varepsilon)$.

Now, choose k large enough so that $W'_k := W_k \cap B(x_i, \delta) \neq \emptyset$. Using the fact that $\Omega(f) = M$ one can choose a points $z_k \in W'_k$ and such that $f^{n_k}(z_k) \in W'_k$ with n_i arbitrarily large. It follows that $f^{n_k}(W^{uu}(z_k) \cap W'_k)$ has arbitrarily large internal radius and so, for large k, intersects $B(x_{i+1}, \varepsilon)$. Since ε was arbitrary small, this concludes the proof of the claim. □

Now, one can conclude the proof of the theorem by induction on i, finding iterates of U which intersect small neighborhoods of x_i's until they intersect a small neighborhood of y and therefore intersect V as desired. The criterion for the transitivity stated in exercise 41 is thus satisfied. □

Accessibility holds for many diffeomorphisms:

Theorem 5.5 (Dolgopyat-Wilkinson [DW])**.** *Let us assume M connected.*

There exists an open and dense subset $\mathcal{U} \subset \mathsf{PH}^1$ (for the C^1-topology) such that the diffeomorphisms in $\mathcal{U}$ are accessible.

The space of volume preserving globally partially hyperbolic C^1-diffeomorphisms contains an open and dense subset $\mathcal{U} \subset \mathsf{PH}^1_{vol}$ whose elements are accessible.

Corollary 5.6. *If M is connected, then the set of transitive diffeomorphisms contains a dense open set of PH^1_{vol}.*

In the case the center is one-dimensional, stronger results hold:

- accessibility is open (for the C^1-topology) in $\mathsf{PH}^1_{c=1}$ (Didier [Di]),
- accessibility is dense in $\mathsf{PH}^k_{c=1}$ for any $k \geq 1$ (F. Rodriguez-Hertz - M.A. Rodriguez-Hertz - Ures [RHRHU$_2$]).

5.2. **Criteria for robust transitivity.** Different arguments for producing robust transitivity have been introduced:

– Shub, Mañé, Bonatti-Viana,... constructed examples of robustly transitive systems by deformation of Anosov diffeomorphisms,
– Bonatti-Díaz and Bonatti-Diaz-Viana have given a geometrical mechanism, which uses the construction of large hyperbolic sets (blenders).

We refer the reader to [BDV, Chapter 7] for an introduction to these mechanisms with plenty of references. We present here another mechanism, quite related to Bonatti-Diaz-Viana's criterion (but which does not use blenders) or to Pujals-Sambarino [PS_2].

a. Dynamically minimal strong foliations.

Definition 5.7. Consider a diffeomorphism $f \in \mathsf{PH}$. Its strong stable foliation $\mathcal{W}^{ss}$ is *dynamically minimal* if any f-invariant compact set, saturated by strong stable leaves is either empty or equal to M.

We will mainly work with dynamically minimal strong stable foliations but symmetric statements hold for strong unstable foliations.

Proposition 5.8. *If $f \in \mathsf{PH}$ has a dynamically minimal strong stable foliation, then it is transitive.*

Exercise 42. (1) Show that the strong stable foliation is dynamically minimal if and only if for every disk D in a strong stable leaf verifies that $\overline{\bigcup_{n\leq 0} f^n(D)} = M$.
(2) Prove Proposition 5.8.
(3) Show that if the strong stable foliation is dynamically minimal, then for every $\varepsilon > 0$ there exists $R > 0$ and $n_0 > 0$ such that every disk D in a strong stable manifold with internal radius $\geq R$ verifies that $D \cup f^{-1}(D) \cup \ldots \cup f^{-n_0}(D)$ intersects every ball of radius ε in M (i.e. it is ε-dense). Conclude that if the strong stable foliation is dynamically minimal, then for every $\varepsilon > 0$ there is a C^1-open neighborhood of f such that every g in the neighborhood verifies that its strong stable foliation is dynamically ε-minimal.
(4) Give an example of $f \in \mathsf{PH}$ whose strong stable foliation is dynamically minimal but not minimal. (Hint: Consider a product example.)

However the dynamical minimality is in general not a robust property as can be seen by considering the diffeomorphism $f : \mathbb{T}^2 \times S^1 \to \mathbb{T}^2 \times S^1$ given as $(x, t) \mapsto (Ax, t+\alpha)$ where $A \in \mathrm{SL}(2, \mathbb{R})$ is hyperbolic and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. However, one has the following:

Proposition 5.9. *Let $f \in \mathsf{PH}$ whose strong stable foliation is dynamically minimal, then, for every $\varepsilon > 0$ there exists $\mathcal{U}$ a C^1-neighborhood of f such that the strong stable foliation of any $g \in \mathcal{U}$ is dynamically ε-minimal. More precisely, there exist $n_0 > 0$ and $R > 0$ (independent of g) such that if D is a disk in a strong stable manifold of g of internal radius larger than R then $D \cup g^{-1}(D) \cup \ldots \cup g^{-n_0}(D)$ is ε-dense in M.*

Proof. This follows from property (3) in Exercise 42 and the continuous variation of strong manifolds with respect to the diffeomorphism. □

b. Property SH ("some hyperbolicity"). The following property was introduced in [PS_2] to obtain a criteria for having robustly minimal foliations:

Definition 5.10. Let $f \in \mathsf{PH}$. We say that the strong unstable foliation has the *property SH* if there exists $N \geq 1$ and $\varepsilon > 0$ such that any a strong unstable disk D^u of internal radius $> \varepsilon$ contains a closed subdisk $D' \subset D^u$ such that:

- $f^N(D')$ has internal radius $> \varepsilon$,
- for any $x \in D'$ one has that $m(Df^N|_{E^c(x)}) > 2$.

Remark 5.11. This property can be defined also for invariant strong compact sets which are union of strong unstable leaves, see Section 5.3 below.

Exercise 43. a– An equivalent definition of SH: there exists $N \geq 1$ and $\varepsilon > 0$ such that any any a strong unstable disk D^u of internal radius $> \varepsilon$ contains a point x which satisfies:

$$\prod_{k=0}^{n-1} m(Df^N|_{E^c}(f^{Nk}(x))) > 2^n. \tag{5.2.1}$$

b– The property SH is open: If the strong unstable foliation of $f \in \mathrm{PH}$ has the property SH, then for any g in a C^1-neighborhood of f the strong unstable foliation has the property SH (with the same constants N, ε).

To show that how property SH implies robust transitivity, let us state the following result, close to [PS_2].

Proposition 5.12. *Let $f \in \mathrm{PH}$ whose strong stable foliation is dynamically minimal and whose strong unstable foliation has the property SH. Then for any diffeomorphism C^1-close to f the strong stable foliation is dynamically minimal.*

PROOF. Since property SH is open, one can choose $\mathcal{U}_0$, a C^1-neighborhood of f such that for every $g \in \mathcal{U}_0$ and any strong unstable disk D^u of internal radius larger than ε has a point x satisfying equation (5.2.1). This implies that there exists $\delta > 0$ such that any center-unstable disk around x has a forward iterate with internal radius larger than δ.

Choose $\mathcal{U}_1 \subset \mathcal{U}_0$ a smaller neighborhood of f such that if $g \in \mathcal{U}_1$ one has that the strong stable foliation is dynamically δ-minimal (c.f. exercise 42).

Choose $g \in \mathcal{U}_1$, a strong stable disk D^s for g and an arbitrary open set $U \subset M$. We must show that there exists $n > 0$ such that $g^{-n}(D^s) \cap U \neq \emptyset$ to apply again exercise 42 and conclude.

Consider D^u an unstable disk contained in U and after a finite forward iterate there is a point $x \in g^{n_1}(D^u)$ such that equation (5.2.1) holds. Choose an arbitrary center-unstable disk D^{cu} containing $g^{-n_1}(x)$ and contained in U. It follows that it eventually attains an internal radius larger than δ; i.e. there exists n_2 such that $g^{n_1+n_2}(D^{cu})$ contains a disk of radius δ tangent to the center-unstable bundle. Now, since the strong stable foliation is dynamically δ-minimal it follows that there exists $n_3 > 0$ such that $g^{-n_3}(D^s)$ intersects $g^{n_1+n_2}(D^{cu})$. It follows that $g^{-n_1-n_2-n_3}(D^s) \cap U \neq \emptyset$ as desired. □

Remark 5.13. Notice that for the proof it is not essential that the time it takes to the center-unstable disk to grow is controlled. Moreover, it is enough to have a property "like" property SH in a dense set of unstable leafs, since we have chosen any unstable disk inside U but we could have chosen a different one without a problem.

In the next subsections, we will discuss how to check the property SH for a certain partially hyperbolic diffeomorphism. See also [HeTe] for related results.

c. Mañé's example of a robustly transitve non-hyperbolic dynamics. We show here how to construct a derived from Anosov in $\mathbb{T}^3$ which is robustly transitive yet not hyperbolic, as proposed in [Ma_1]. The proof uses the deformation introduced in Section 3.5.1 and then the criterion of Proposition 5.12 as in [PS_2].

We start by a linear Anosov diffeomorphism $f_A : \mathbb{T}^3 \to \mathbb{T}^3$ where $A \in SL(3,\mathbb{Z})$ is a hyperbolic matrix with eigenvalues $0 < \lambda_1 < 1 < 2 < \lambda_2 < \lambda_3$. In a small neighborhood U of a fixed point p of f_A we perform an isotopy f_t with $t \in [0,1]$ (and $f_0 = f_A$) of the same form as the one made in subsection 3.5.1, which in local coordinates looks, for $t < 1$, like

$$f_t : (x, y, z) \mapsto (\lambda_1.x, g^t_{x,z}(y), \lambda_3.z), \text{ such that } 1 < \|Dg^t_{x,z}\| < \lambda_3.$$

As in subsection 3.5.1 we require that cone-fields around the initial bundles are preserved by Df_t for every t, in particular f_t is partially hyperbolic with splitting $T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$ (and if $t < 1$ the bundle E^c is uniformly expanded and f_t is Anosov).

By choosing g^t correctly we can ensure that for $t = 1$ the point p is no longer hyperbolic (has eigenvalue equal to 1 in the y direction) while at any other point it continues to hold that $1 < \|Dg^1_{x,z}\| < \lambda_3$. It follows that f_1 is no longer Anosov, however:

Exercise 44. Show that the strong stable foliation of f_1 is minimal. (Hint: f_1 is still topologically conjugate to f_A.)

Now, we must check that the strong unstable foliation $\mathcal{W}^{uu}$ of f_1 satisfies the SH property. But this is quite direct since we have performed the perturbation in a small ball around p, and therefore every unstable arc of length larger than one has at least one point outside the perturbation region. Therefore, the derivative of f_1 along that point expands by at least $\lambda_2 > 2$ establishing property SH.

Using Proposition 5.12 we know that f_1 is robustly transitive. Moreover, one can perturb f_1 to a diffeomorphism g which is C^1-close and for which there is at least one periodic point whose stable dimension is 2; in particular, it is robustly non-Anosov.

d. Another criterion for robust transitivity. We give a variant of the previous criterion which does not use the minimality of the strong stable foliation.

Proposition 5.14. *Let $f \in \mathsf{PH}$ transitive whose both strong stable and strong unstable foliations have the property SH. Then f is robustly transitive.*

PROOF. As in the proof of Proposition 5.12 one has $\delta > 0$ and a neighborhood $\mathcal{U}_0$ of f such that if $g \in \mathcal{U}_0$ and D^u is any unstable disk, then there is a point $x \in D^u$ such that for any center-unstable disk D^{cu} containing x has a forward iterate by g containing a center-unstable disk of internal radius $\geq \delta$. Since property SH also holds for the stable foliation one can assume that the same δ and $\mathcal{U}_0$ work for strong stable disks and backward iterates.

Now, choose $\varepsilon \ll \delta$ and pick a neighborhood $\mathcal{U}_1 \subset \mathcal{U}_0$ of f such that for every $g \in \mathcal{U}_1$ there is a point y whose forward and backward orbits are ε-dense.

Let $g \in \mathcal{U}_1$ and open sets $U, V \subset M$. Since $g \in \mathcal{U}_0$ there exists n_1 such that $g^{n_1}(U)$ contains a center-unstable disk D^{cu} of internal radius $\geq \delta$. Similarly, there exists n_2 such that $g^{-n_2}(V)$ contains a center-stable disk D^{cs} of internal radius $\geq \delta$. Now, consider the point y with ε-dense backward and forward orbit, so one has that there exists $n_3, n_4 > 0$ such that $g^{-n_3}(y)$ is very close to D^{cu} and $g^{n_4}(y)$ is very close to D^{cs}. It follows that $W^{ss}_\varepsilon(g^{-n_3}(y)) \cap D^{cu} \neq \emptyset$ and $W^{uu}_\varepsilon(g^{n_4}(y)) \cap D^{cs} \neq \emptyset$.

Let $z \in W^{ss}_\varepsilon(g^{-n_3}(y)) \cap D^{cu}$ and consider $g^{n_3+n_4}(W^{uu}_\varepsilon(z)) \subset g^{n_3+n_4}(D^{cu})$ which is very close to a compact part of $W^{uu}_{loc}(g^{n_4}(y))$ and therefore intersects D^{cs}. One has proved that $g^{n_1+n_2+n_3+n_4}(U) \cap V \neq \emptyset$ showing transitivity of g. $\square$

5.3. **Minimal strong unstable laminations.** Let $f \in \mathsf{PH}$. In general its strong foliations may not be dynamically minimal and one introduces the following notions.

Definition 5.15. A compact set Λ is a *strong unstable lamination* if it consists of entire strong unstable leaves (i.e. if $x \in \Lambda$ then $W^{uu}(x) \subset \Lambda$).

A set Λ will be called a *minimal strong unstable lamination* if

- it is a non-empty f-invariant strong unstable lamination,
- any subset of Λ satisfying the same properties coincides with Λ.

Exercise 45. 1. Any invariant strong unstable lamination contains a minimal strong unstable laminations.

2. On any minimal strong unstable lamination the dynamics is transitive.

Minimal strong unstable laminations will re-appear in the next section, but in this one we are interested in such sets mainly because of the following property from [ACP]:

Theorem 5.16 (Abdenur-Crovisier-Potrie)**.** *There exists a dense G_δ subset $\mathcal{G} \subset \mathsf{PH}^1_{c=1}$ such that any $f \in \mathcal{G}$ and any minimal strong unstable lamination Λ has the following property: Either Λ has a basis of U neighborhoods satisfying $f(\overline{U}) \subset U$, or Λ satisfies the property SH.*

Corollary 5.17. *For any non-Anosov transitive diffeomorphism f in a dense G_δ set of $\mathsf{PH}^1_{c=1}$, the property SH holds for any minimal strong stable lamination and any minimal strong unstable lamination.*

In fact, from the Theorem one can show that every strict strong stable (resp. strong unstable) lamination has property SH. But being non-Anosov and using the fact that we can choose f in a dense G_δ set one knows that if the strong stable (resp. strong unstable) lamination is minimal then it must cross a *blender* and therefore will also have property SH. See [BD_2].

5.4. **Transversality.** In this section we discuss a geometric property of the strong unstable foliation. This will be discussed further and with more details in the following section. The definition only makes sense when the center direction is one-dimensional (so that the strong-unstable manifolds are one-codimensional inside the center-unstable discs).

Definition 5.18. Let $f \in \mathsf{PH}^1_{c=1}$. We say that the strong unstable foliation $\mathcal{W}^{uu}$ (or a minimal strong unstable lamination Λ) is *transverse* if there exists $R > 0$ large and $\varepsilon > 0$ small so that inside any unstable disk D of internal radius $> R$ there exist points $x \neq y$ with:

- $d(x, y) < \varepsilon$ and $y \in W^{ss}_{loc}(y)$,
- using the holonomy Π^{ss} along the strong stable leaves onto a center unstable disc $W^{cu}_{loc}(x)$ centered at x, the projection of the local unstable manifold $W^{uu}_{loc}(y)$ intersects both connected components of $W^{cu}_{loc}(x) \setminus W^{uu}_{loc}(x)$. See figure 13.

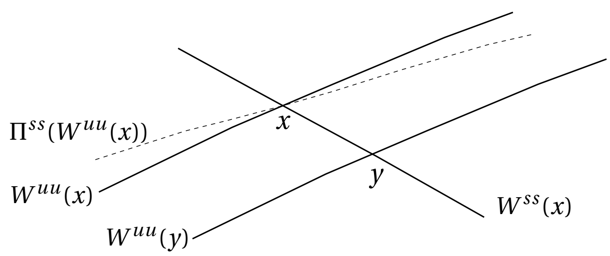

FIGURE 13. Transversality of the unstable lamination.

Exercise 46. a) Let $f \in \mathsf{PH}_{c=1}$. If the transversality holds in restriction to each minimal strong unstable lamination, then the strong unstable foliation is transverse.

b) The transversality of the strong unstable foliation is a C^1 open property in $\mathsf{PH}_{c=1}$.

c) If the strong unstable foliation of $f \in \mathsf{PH}_{c=1}$ is transverse, then there exist at most finitely many minimal strong unstable laminations.

d) If f is generic in $\mathsf{PH}^r_{c=1}$ and has a transverse strong unstable foliation, the the number of minimal strong unstable lamination is constant on a neighborhood of f. Moreover for g close to f, the minimal strong unstable laminations of g are contained in small neighborhoods of the minimal strong unstable laminations of f.

The following improves the Theorem 6.7 which will be discussed in the next section.

Theorem 5.19 ([ACP])**.** *There exists a dense G_δ subset $\mathscr{G} \subset \mathsf{PH}^1_{c=1}$ such that any $f \in \mathscr{G}$ and any minimal strong unstable lamination Λ has the following property:*

- *Either there exists a basis of neighborhoods U of Λ satisfying $f(\overline{U}) \subset U$,*
- *or Λ is transverse.*

Corollary 5.20. *For any transitive diffeomorphism f in a dense G_δ subset of $\mathsf{PH}^1_{c=1}$ the strong stable and strong unstable foliations are transverse.*

5.5. **A dichotomy: robust transitivity *versus* trapping region.** We now explain shortly how to conclude the proof of Theorem 5.2. It is enough to consider a transitive diffeomorphism $f \in \mathsf{PH}^1$ which belong to the dense G_δ subsets provided by Corollaries 5.17 and 5.20 and to show that f is robustly transitive.

1. Corollary 5.20 implies that the strong stable and unstable foliations are transverse, this is a robust property.

2. The exercise 46 implies that the minimal strong unstable laminations are finite and remain close for diffeomorphisms g close to f.

3. Corollary 5.17 holds on any minimal strong stable lamination and any minimal strong unstable lamination of f. From item (2), this is still satisfied for g close to f (with the same constants N, ε, see also exercise 43).

4. We then conclude by a variation of Proposition 5.14 (see also remark 5.13) that f is robustly transitive:

Proposition 5.21. *Let $f \in \mathsf{PH}$ transitive whose both strong stable and strong unstable foliations are transverse and such that SH holds for any minimal strong stable or unstable lamination of any diffeomorphism close. Then f is robustly transitive.*

6. Attractors and unstable laminations

In order to generalize Smale's spectral decomposition (that holds for hyperbolic diffeomorphisms), we would like to describe how the dynamics may be decomposed.

a. Finiteness of the decomposition. Following the discussion of section 1.1.2, one asks:

Question 1. *How many (non-trivial) levels can we obtain in a filtration?*
That is: can one find filtrations $U_0 \subset U_1 \subset \cdots \subset U_N$ with N arbitrarily large such that the maximal invariant set in $U_{i+1} \setminus U_i$ is non-empty?
Equivalently, is the set of chain-recurrence classes infinite?

Inside the whole space of diffeomorphisms we have seen that there are large classes of diffeomorphisms which possess infinitely many chain-recurrence classes (for instance these diffeomorphisms may have infinitely many sinks, as in the Newhouse phenomenon): all known examples occur close to diffeomorphisms exhibiting homoclinic tangencies. In the class of partially hyperbolic diffeomorphisms whose center bundle splits into one-dimensional subbundles (as discussed in Section 1.2.1) we expect that it is not the case:

Finiteness conjecture. (Bonatti [Bon]) *For f in a dense open subset of $\mathsf{PH}^1_{c=1}$, the number of chain-recurrence classes is finite.*

As in Section 5, $\mathsf{PH}^1_{c=1}$ denotes the space of globally partially hyperbolic C^1-diffeomorphisms with one-dimensional center. One can also consider the more general class $\widehat{\mathsf{PH}}^1_{c=1}$ of C^1-diffeomorphisms whose chain-recurrence classes have a partially hyperbolic decomposition $T_\Lambda M = E^s \oplus E^c \oplus E^u$, such that $\dim(E^c) \leq 1$, and E^s or E^u can be trivial.

b. Finiteness of attractors. We will restrict the problem of finiteness to specific chain-recurrence classes, called quasi-attractors, which include the classical notion of transitive attractor. We will present some parts of [CPoS] where the following result is obtained.

Theorem 6.1 (Crovisier-Potrie-Sambarino)**.** *For f is a dense open subset of $\mathsf{PH}^1_{c=1}$ or more generally of $\widehat{\mathsf{PH}}^1_{c=1}$, there exist at most finitely many (quasi-)attractors.*

For hyperbolic diffeomorphisms, the finiteness of the spectral decomposition holds as a simple consequence of the uniform size of stable and unstable manifolds. For partially hyperbolic diffeomorphisms stable and unstable sets are small or degenerate in the center direction and this argument is no longer available. Instead, we use the fact that quasi-attractors are saturated by unstable manifolds and we obtain the result as a consequence of a geometric property satisfied by unstable laminations which, in a nutshell, implies that each quasi-attractor must occupy a uniform space in M. This study was initiated in [CPu] where the case of diffeomorphisms far from homoclinic tangencies and heterodimensional cycles was treated. The problem for classes which are not attractors is much harder and, for the moment, there is no known technique to approach the problem of finiteness of chain-recurrence classes in this general setting.

Together with Theorem 1.6 (which uses completely different techniques) our main result gives as a consequence the following one which is a step towards the understanding of dynamics far from homoclinic tangencies. It also improves (in dimension 3) a result announced[16] in [BGLY] (though their result holds in any dimension).

[16] In [BGLY] they show that there exists a dense G_δ subset $\mathscr{G}$ of diffeomorphisms far away from homoclinic tangencies such that if $f \in \mathscr{G}$ then all quasi-attractors of f are isolated from each other (but might in principle

Corollary 6.2. *Let M be a 3-dimensional manifold. Then, there exists an open and dense subset $\mathcal{U}$ of* $\text{Diff}^1(M)$ *such that if $f \in \mathcal{U}$ then:*

- *either f has robustly finitely many quasi-attractors,*
- *or f can be C^1-approximated by a diffeomorphism g which exhibits a homoclinic tangency.*

c. Recurrence inside the classes. It is possible to pose some questions about the internal dynamics of each of the chain-recurrence classes, particularly when they are isolated:

> **Question 2.** *How recurrent is the dynamics in each level?*
> *What happens when one perturbs the dynamics?*

Let us consider a level of a filtration defined by a filtration pair $U_i \subset U_{i+1}$ and which cannot be split further: there is no open set V such that $\emptyset \subset U_i \subset V \subset U_{i+1} \subset M$ is a filtration and the maximal invariant sets in $V \setminus U_i$ and $U_{i+1} \setminus V$ are non-empty.

Does it mean that the dynamics on the maximal invariant in $U_{i+1} \setminus U_i$ is transitive?

In this case Is it still the case for the diffeomorphisms close?

Again, it may make sense to separate this study depending on further properties of the chain-recurrence classes (for example, as we did in the last section, the case where the whole manifold is a chain recurrence class). We refer the reader to the introduction of [BCGP] for a more detailed presentation of this problem and we also mention the work [No] for some progress in the case of attractors.

6.1. **Attracting sets and quasi-attractors.**

a. *General definition.* Let f be a homeomorphism of a compact metric space M.

Definition 6.3. We say that a compact invariant set Λ is an *attractor* if

- there exists an open set $U \subset M$ such that $f(\overline{U}) \subset U$ and $\Lambda = \bigcap_{n>0} f^n(U)$,
- the restriction of f to Λ is transitive.

An attractor for f^{-1} is called a *repeller.*

In general, a diffeomorphism may not have any attractor, however, it will always have what we call *quasi-attractors.*

Definition 6.4. A compact invariant set Q is a *quasi-attractor* for f if

- there exist arbitrarily small neighborhoods $U \subset M$ of Q such that $f(\overline{U}) \subset U$,
- Λ is a chain-recurrence class.

A quasi-attractor for f^{-1} is called a *quasi-repeller.*

Exercise 47. Show that every homeomorphism $f : M \to M$ has a quasi-attractor.

In contrast, large sets of diffeomorphisms may present no attractors at all ([BLY]).

accumulate in a set which is not a quasi-attractor). They call *essential attractors* such quasi-attractors since it can be shown that their basin contains a dense G_δ subset of a neighborhood.

b. *Partially hyperbolic quasi-attractors.* Now f is a diffeomorphism. Consider a filtrating pair defined by open sets $V \subset U \subset M$ satisfying $f(\overline{U}) \subset U$ and $f(\overline{V}) \subset V$ and let us assume that the maximal invariant set $K = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U} \setminus V)$ has a partially hyperbolic structure $T_\Lambda M = E^s \oplus E^c \oplus E^u$ where both E^s and E^u are non-trivial and such that $\dim E^c = 1$.

Exercise 48. Show that any quasi-attractor $Q \subset U \setminus \overline{V}$ is saturated by strong unstable manifolds.

Exercise 49. Show that if $\{Q_n\}$ is a sequence of quasi-attractors in K converging to Λ in the Hausdorff topology, then $\Lambda \subset K$ is $\mathcal{W}^{uu}$-saturated.

c. Minimal strong unstable laminations. If Q and Q' are two different quasi-attractors then $Q \cap Q' = \emptyset$. Therefore, there are fewer quasi-attractors in K than there are minimal strong unstable laminations (recall Definition 5.15). The main result on [CPoS] is the following. Clearly it implies Theorem 6.1.

Theorem 6.5 ([CPoS])**.** *For any diffeomorphism f in a dense G_δ subset of* $\mathrm{Diff}^1(M)$*, an compact f-invariant set K admitting a strong partially hyperbolic splitting $T_K M = E^s \oplus E^c \oplus E^u$ with* $\dim E^c = 1$ *then K contains at most finitely many minimal strong unstable laminations.*

Proof of Theorem 6.1. It is enough to consider a diffeomorphism in a dense G_δ subset of $\widehat{\mathrm{PH}}^1_{c=1}$, since the number of quasi-attractor varies semi-continuously with respect to f.

Let us assume by contradiction that there exists an infinite sequence of quasi-attractors, hence of minimal strong unstable laminations (L_n). Each of them is contained in a chain-recurrence class C_n. Up to consider a sub-sequence, C_n converges to a set Λ contained in a chain-recurrence class C (prove it!). Since $f \in \widehat{\mathrm{PH}}^1_{c=1}$, the class C has a partially hyperbolic structure with one-dimensional center. Moreover, there exists a filtrating pair $V \subset U$ such that the maximal invariant set K in $U \setminus V$ is a small neighborhood of C, hence is also partially hyperbolic. For n large L_n is contained in $U \setminus V$, hence in K. This contradicts Theorem 6.5. □

The proof of Theorem 6.5 has two stages which will be described in the next sections:

- obtain by perturbation a geometric property of strong unstable laminations verified in an open and dense subset of $\mathrm{Diff}^1(M)$,
- deduce from this geometric property the finiteness of minimal strong unstable laminations.

6.2. **Non joint integrability inside unstable laminations.** Let $f \in \mathrm{Diff}^1(M)$. As before we consider the geometry of strong unstable laminations contained in a partially hyperbolic set with a one-dimensional center.

Definition 6.6. We say that a strong unstable lamination Λ is *non-jointly integrable* (NJI) if for every $r, r'\, t > 0$ sufficiently small, there exists $\delta > 0$ with the following property.

If $x, y \in \Lambda'$ satisfy $y \in W^{ss}_{loc}(x)$ and $d_s(x,y) \in (r, r')$, then there is $z \in W^{uu}_t(y)$ such that:

$$d(W^{ss}_{loc}(z), W^{uu}_{loc}(x)) > \delta$$

By d_s we refer to the distance inside $\mathcal{W}^{ss}$ and by $W^\sigma_\varepsilon(x)$ ($\sigma = ss, uu$) we denote the ε-ball around x in $W^\sigma(x)$ with the intrinsic metric.

This property is weaker than the transversality, but is a strong form of non-joint integrability: the joint integrability between E^{ss} and E^{uu} fails for *any* pair of strong unstable manifolds connected by a strong stable leaf.

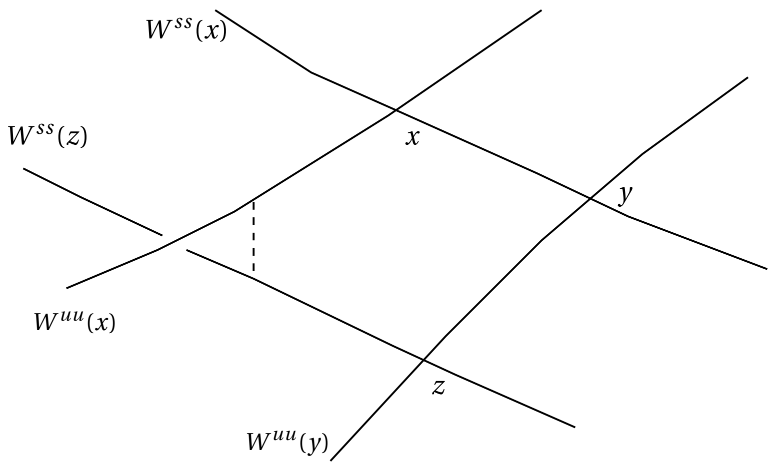

FIGURE 14. The non-joint integrability.

Theorem 6.7 ([CPoS])**.** *For any diffeomorphism f in a G_δ-dense subset $\mathcal{G}$ of* $\text{Diff}^1(M)$ *and inside any partially hyperbolic set with one-dimensional center bundle, each strong unstable lamination Λ is NJI.*

Remark 6.8. Using the continuity of the strong manifolds with respect to the diffeomorphism (see Remark 4.4), one sees that at a given scale (i.e. if one fixes the values of r, r', t), this property holds for small perturbations of $f \in \mathcal{G}$ and unstable laminations close to Λ.

This allows to get a uniform lack of joint integrability on a dense open subset of PH^1 which is a starting point for proving the transversality in Theorem 5.19.

6.3. **Finiteness of minimal strong unstable laminations.** Now, we use Theorem 6.7 and the results in the previous sections of this notes to conclude the proof of Theorem 6.5.

Let f be a diffeomorphism in the G_δ subset provided by Theorem 6.7 and let K be a partially hyperbolic set with one-dimensional center. We must show that K contains at most finitely many minimal strong unstable laminations.

Exercise 50. Show that one can reduce to the case K itself is a strong unstable lamination. (In particular it is NJI.)

The following remark will be important in the proof.

Exercise 51. Show that if $\Lambda_1, \Lambda_2 \subset K$ are different minimal strong unstable laminations, then their stable manifolds are disjoint.

1. First case: minimal laminations with no strong stable connection.

Proposition 6.9. *The set K contains at most finitely many minimal strong unstable laminations Λ with the property that $W^{ss}(x) \cap \Lambda = \{x\}$ for every $x \in \Lambda$.*

PROOF. Assume by contradiction that there are infinitely many such subsets and denote them as $\{\Lambda_n\}_n$. From Theorem 4.17 , each Λ_n is contained in a locally invariant submanifold Σ_n tangent to $E^c \oplus E^u$ at each point of Λ_n.

Notice moreover that there exists $h > 0$ such that $h_{top}(f|_{\Lambda_n}) > h$ for every n. This follows from the following argument: consider a finite covering of Λ_n by balls of radius ε where ε is small enough (independent on n) so that any disk tangent to a small cone around E^u of diameter 1 contains at least two disks of radius ε contained in different balls of the covering. Given any disk tangent to a small cone around E^u its iterates grow so that the internal radius multiplies by an uniform amount (independent of n). We can choose such a disk D to be contained in Λ_n (since it is $\mathscr{W}^{uu}$-saturated). We get that for some k_0 (independent of n), the image $f^{k_0}(D)$ contains two such disks. Therefore, inside D one has that in k_0 iterates we duplicate the number of "different" orbits and therefore the entropy of f in Λ_n is larger than $\frac{1}{k_0}\log 2$ (independent of n).

Using the variational principle and Ruelle's inequality for f^{-1} (see [Ma$_4$]) we obtain that Λ_n has a measure μ_n whose Lyapunov exponent for f^{-1} along E^c (recall that Λ_n "lives" in Σ_n) is larger than h. This means (see the exercise 52 below) that Λ_n contains a point x_n whose stable manifold has uniform size along $E^s \oplus E^c$.

Now one can conclude as in the uniformly hyperbolic case: from exercise 51, for different n, m the points x_n, x_m are separated by a uniform distance. This is impossible if there are infinitely many Λ_n.

□

Exercise 52. Let f be a C^1-diffeomorphism and K be a partially hyperbolic set with one-dimension center. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if μ is an ergodic measure supported on K whose Lyapunov exponent along E^c is smaller than $-\varepsilon$, then there is a point in K whose stable manifold along E^c is of size δ. (See [ABC$_2$] for more results in this direction.)

2. Second case: minimal laminations with strong stable connection.

Proposition 6.10. *The set K contains at most finitely many minimal strong unstable laminations Λ with the property that $W^{ss}(x) \cap \Lambda \neq \{x\}$ for some $x \in \Lambda$.*

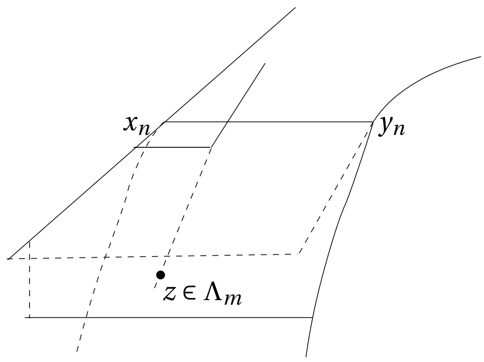

FIGURE 15. The stable manifolds of the minimal sets must intersect.

PROOF. Assume by contradiction that there are infinitely many such laminations $\{\Lambda_n\}_n$ in K. For every n there exists $x_n \neq y_n \in \Lambda_n$ such that $y_n \in W^{ss}(x_n)$. By iteration, we can assume that $y \in W^{ss}_{loc}(x)$ and $d_s(x_n, y_n) \in (r, r')$, where r, r' are chosen small and such that $r' > \max_x\{\|D_x f^{\pm 1}\|\} r$.

Then, these pairs of points converge to points $x, y \in K$ which still satisfy $y \in W^{ss}_{loc}(x)$ and $d_s(x, y) \in (r, r')$. Since the strong unstable manifolds get separated by projection by stable holonomy, it is possible to show (see Figure 15) that this configuration forces the strong stable manifold of one of the Λ_n to intersect some other Λ_m for n, m large, contradicting the fact that the minimal sets where different. This concludes.

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