Monte Carlo method
and application to random processes

Lecture 4: Gaussian stochastic processes (13/3/2017)\(^1\)

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A stochastic processes $X$ is a collection of random variables

$$X = \{X(t) : t \in I\},$$

defined in a common probability space $(\Omega, \mathcal{F}, P)$, where $I$ is the index set. Examples of indexes are

- $I = \mathbb{N}$, giving a discrete time stochastic process,

- $I = [a, b]$, including $I = \mathbb{R}^+$ or $I = \mathbb{R}$, giving a continuous time stochastic process
A stochastic process can be seen as a function of two variables:

\[ X : \Omega \times I \rightarrow \mathbb{R}. \]

When we fix \( t \in I \) we obtain a random variable \( X(\omega, t) \), as it was defined. When we fix \( \omega \in \Omega \) we obtain a real function

\[ X : I \rightarrow \mathbb{R}. \]

If this function is almost sure continuous for \( \omega \in \Omega \), we have a process with *continuous trajectories*, or *continuous paths*. In other terms, it is the case when

\[ P(\{\omega : X(\omega, t) \text{ is a continuous function as } t \in I\}) = 1. \]

A process is determined by the distribution of its finite dimensional projections

\[ (X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \]

for all \( n = 1, 2, \ldots \) and all \((t_1, \ldots, t_n)\).
On the left we plot three trajectories of a process with continuous paths, on the right three trajectories of a process that has discontinuous paths (or \textit{jumps}).
Gaussian processes

When all the finite distributions are gaussian vectors, the process is called a *Gaussian process*. As a gaussian vector is determined by its mean and covariance matrix, a gaussian process is determined by the two functions

\[
\mu(t) = \mathbb{E}(X(t)), \quad \text{for all } t \in I
\]
\[
r(s, t) = \text{cov}(X(s), X(t)), \quad \text{for all } (s, t) \in I \times I.
\]

In particular, the variance of a coordinate is given by

\[
\text{var}(X(t)) = r(t, t).
\]

We are going to consider only *gaussian processes with continuous paths*. 
Simulation of a gaussian process

As we are able to simulate random gaussian vectors with the Cholesky method, to simulate a trajectory of a continuous gaussian process \( \{X(t): 0 \leq t \leq T\} \) with mean \( \mu(t) \) and covariance \( r(s, t) \), we proceed in the following way:

- Select \( n \) and time grid \( I_n = \{Ti/n: i = 0, \ldots, n\} \)
- Simulate a gaussian vector \((X_0, \ldots, X_n)\) with
  - mean \((\mu(Ti/n): i = 0, \ldots, n)\)
  - covariance matrix \(\Sigma_n = (r(Ti/n, Tj/n): i, j = 0, \ldots, n)\)

As the process is continuous, we can plot \( X(t) \) considering the polygonal chain defined at \( I_n \), by

\[
X(t_i) = X_i, \quad i = 0, \ldots, n.
\]
We now consider a continuous gaussian process, denoted \( W = \{ W(t) : t \geq 0 \} \), (i.e. \( I = [0, \infty) \)), with continuous trajectories, and

\[
\mu(t) = 0, \quad r(s, t) = \min(s, t).
\]

This process is called indistinctly *Brownian motion* and denoted \( B = \{ B(t) : t \geq 0 \} \), due to the botanist *Robert Brown*, who described the physical phenomena in 1827; and also *Wiener process*, and denoted \( W = \{ W(t) : t \geq 0 \} \) due to the mathematician *Norbert Wiener*, who in 1923 gave the first mathematical construction of the process.
Observe that $W(0) = 0$ a.s., because $E(W(0)) = 0$ and $\text{var}(W(0)) = 0$. Also, for $0 \leq s \leq t$,

$$E[(W(t) - W(s))W(s)] = E(W(t)W(s) - W(s)^2) = \min(s, t) - \min(s, s) = 0.$$ 

As they are gaussian random variables, 

$$(W(s), W(t) - W(s))$$ 

are independent random variables. 

and 

$$\text{var}(W(t) - W(s)) = E[(W(t) - W(s))^2]$$ 

$$= E(W(t)^2) - 2E(W(t)W(s)) + E(W(s)^2)$$ 

$$= t - 2\min(s, t) + s = t - s.$$
More in general, we can prove the following result.

**Proposition**

(a) The process start from zero, i.e.

\[ P(W(0) = 0) = 1. \]

(b) The increments of \( W \) are independent, i.e. for arbitrary \( n \) and arbitrary \( t_1, \ldots, t_n \), the random variables

\[ W(t_1) - W(0), \ldots, W(t_n) - W(t_{n-1}), \]

are gaussian, centered, independent, and with variances

\[ \text{var}[W(t_i) - W(t_{i-1})] = t_i - t_{i-1}. \]
Simulation of Brownian motion

The independence of the increments gives a second way of simulating Brownian motion (the first is the Cholesky method). To simulate $W$ over a time interval $[0, T]$, we use the same mesh $I_n = \{ Ti/n: i = 0, \ldots, n \}$ but simulate the increments of the process and accumulate them.

- Set $W(0) = 0$, choose $n$
- for $i = 1$ in 1 : n simulate $\Delta = \mathcal{N}(0, T/n)$ and set $W(iT/n) = W((i - 1)T/n) + \Delta$.

In this way we obtain a discretization $\{ W(iT/n): i = 0, \ldots, n \}$ of a trajectory of a Brownian motion. In R, we can use the command `cumsum`:

```
n<-1e4
t<-1
bm<-c(0, cumsum(rnorm(n,0,sqrt(t/n))))
```
Quadratic variation of a Brownian motion

Given \( \{ W(t) : 0 \leq t \leq T \} \), we consider the variation of order 2, given by:

\[
V_n^2 = \sum_{k=1}^{n} [W(t_{i+1}^n) - W(t_i^n)]^2.
\]

where \( t_i^n = Ti/n \). We have

\[
E(V_n^2) = \sum_{k=1}^{n} E[W(t_{i+1}^n) - W(t_i^n)]^2 = \sum_{k=1}^{n} (t_{i+1}^n - t_i^n) = T.
\]

It can be proved that

\[
V_n^2 \to T \text{ a.s.}
\]
Application: the Geometric Brownian Motion

Given a Brownian motion $\mathcal{W}$, the process

$$S(t) = S(0)e^{\sigma \mathcal{W}(t)+\mu t}, \quad S(0) \text{ a positive constant.}$$

is named a Geometric Brownian motion (GBM). It is used in Black-Scholes model of a financial market, that has to investment possibilities:

- A deterministic savings account, that evolves according to
  $$B(t) = B(0)e^{rt}$$

- A stock, that has a random evolution, according to
  $$S(t) = S(0)e^{\sigma \mathcal{W}(t)+\mu t}.$$
Asian options

An Asian option is an asset that pays, at time $T$, the average price of a stock over the time interval $[0, T]$, in case this average is larger than a prescribed strike price $K$. Its price is

$$A(r, \sigma, T, K) = e^{-rT} \mathbb{E} \left( \frac{1}{T} \int_0^T S(r)dr - K \right)^+$$

Here $\{S(r) : 0 \leq r \leq T\}$ is a GBM with $\mu = r - \sigma^2/2$. A variant of this option is the multiplicative or geometric asian option. The price of this option is

$$\tilde{A}(r, \sigma, T, K) = e^{-rT} \mathbb{E} \left( S_0 e^{\frac{1}{T} \int_0^T (\sigma W(r) + \mu r)dr} - K \right)^+$$

The payoff of $\tilde{A}$ is used to estimate by MC the value $A$ using control variates.
Brownian bridge

A Brownian bridge \( \{ R(t) : 0 \leq t \leq 1 \} \) is a continuous gaussian stochastic process defined by

\[
\mu(t) = 0, \quad r(t,s) = \min(s,t) - st.
\]

Observe that, as \( \text{var}(R(0)) = \text{var}(R(1)) = 0 \), we have

\[
P(R(0) = R(1) = 0).
\]

A key property follows. If \( \{ W(t) : 0 \leq t \leq 1 \} \) is a Brownian motion\(^2\) in \([0,1]\), then

\[
R(t) = W(t) - tW(1)
\]

is a Brownian bridge.

\(^2\)Brownian motion in intervals \([a,b]\) is defined similarly.
This property gives a way of simulating trajectories of the Brownian bridge:

On the left we plot a trajectory of a Brownian motion and also the line $tW(1)$. On the left we plot the difference $W(t) - tW(1)$, that is a trajectory of a Brownian bridge.
A key role in statistics is played by the random variable

\[ K = \max_{0 \leq t \leq 1} |R(t)| \]

To explain this result, let us define the empirical distribution function of a sample of iid rvs \(X_1, \ldots, X_n\) as

\[ F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1\{X_i \leq x\} \]

i.e. we count how many values \(X_i\) are smaller or equal than \(x\) (and divide by \(n\)).
Theorem

Assume that the random sample of iid rv has continuous distribution $F$. Then

$$K_n := \sqrt{n} \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \xrightarrow{d} K.$$ 

The distribution of $K$ is explicit, through the formula

$$P(K \geq x) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2k^2 x^2}$$

and notably, does not depend on $F$. 
If we suspect that our number generator does not generate, for instance, uniform random variables, we perform a hypothesis test in the following way.

- We produce a sample $(X_1, \ldots, X_n)$ and compute $K_n$.
- We construct a confidence interval for $K$ with confidence $1 - \alpha$, of the form

$$P(K \geq k_0) = \alpha.$$

- If our computed value $K_n$ is larger than $k_0$, we reject the hypothesis that the random variables are uniform.

Observe that what is tested is the distribution, but the independence is assumed to hold.