

22. Fixed Income Finance

MA6622, Ernesto Mordecki, CityU, HK, 2006.

References for this Lecture:

Marco Avellaneda and Peter Laurence, *Quantitative Modelling of Derivative Securities*, Chapman&Hall (2000).

P. Willmot, *Derivatives*, Wiley (1998).

Hong Kong Monetary Authority (HKMA) Web Page:
<http://www.info.gov.hk/hkma/>

Plan of Lecture 22

(22a) Fixed Income

(22b) Fixed Income Securities

(22c) Bond Yields

(22d) Yields and Yields

22a. Fixed income

Fixed income refers to any type of investment that yields a regular or fixed payment:

- If you borrow money and have to pay interest once a month, you have issued a fixed income security.
- When this is done by the government of a country, or a company, we have a bond.
- The Hong Kong Government (through the HK Monetary Authority - HKMA) issues bonds, under the denomination of Exchange Fund Bills and Exchange Fund Notes.

22b. Fixed income securities

Fixed income securities can be contrasted with variable return securities: a bond is the typical fixed income security, whereas a stock is the typical variable return security.

A Zero Coupon Bonds is a contract paying a known fixed amount, the principal at some given date in the future, the maturity. For analysis we fix the principal to be 100 (one hundred points). Examples of zero coupon bonds are

- Treasury Bills or T-Bill issued by the US Government, commonly with maturity dates of 28 days (~ 1 month), 91 days (~ 3 months) and 182 days (~ 6 months).

- **Exchange Fund Bills** or **EFB**, issued by the Hong Kong Monetary Authority (HKMA) with maturities of
 - 91 days \sim three months, or **Quarter**,
 - 182 days \sim six months, or **Half year**,
 - 364 days \sim one **Year**.

A **Coupon Bearing Bond** is similar to the above, with the difference that, as well as paying the principal, they also pay smaller quantities, the **coupons**, at intervals up to and including the maturity date. Examples of Coupon Bearing Bonds are:

- **Treasury notes** or **T-Notes** issued by the US Government, that mature in two to ten years. They have a coupon payment every six months, and are commonly issued with maturities dates of 2, 3, 5 or 10 years.

- **Exchange Fund Notes** or **EFN** issued by the HKMA have maturities between 2 and 10 years, and pay a fixed coupon every six months.

Floating rate notes are bonds that have a **variable** coupon, equal to a money market reference rate, like LIBOR¹, plus a constant spread. A typical coupon would look like:

3 months USD LIBOR+0.20%.

For instance, if the payment coupon date was June 6, as the Libor in USD for 3 months² was 5.27000, the coupon will pay 7.27 points.

¹LIBOR (London Interbank Offered Rate) is the rate of interest at which banks borrow funds from each other in the London interbank market. LIBOR fixings are provided in ten currencies, including GBP, USD, Euro, and JPY.

²Taken from <http://www.bba.org.uk/>, the web site of the British Bankers' Association (BBA), that calculates the Libor rates.

22c. Bond Yields

In order to consistently compare the variety of fixed income products, we examine the [yield to maturity \(YTM\)](#).

The yield to maturity, denoted by y , is related to the [market price](#) of the financial instrument, and it measures the [percent earnings](#) that the instrument produces.

YTM of a zero coupon bond.

The simplest situation is a zero coupon bond. If the bond matures at time T , and the market price of the bond at time t (say, today) is $Z(t, T)$, then the YTM, denoted y , is defined by the relation:

$$Z(t, T) = \frac{100}{1 + (T - t)y}$$

where $T - t$ is the amount of days to maturity over 365. (this is the way of computing the YTM by the HKMA).

From this we obtain that

$$y = \frac{1}{T - t} \left[\frac{100}{Z(t, T)} - 1 \right].$$

We expect $Z(t, T)$ to be less than 100, i.e. we expect to have a positive yield.

Remember that the principal is assumed to be $P = 100$: in case of a principal P the YTM is defined by

$$Z(t, T) = \frac{P}{1 + (T - t)y}.$$

Example The Exchange Fund Bill Issue No. Y593, with maturity on 23/08/2006 has a quoted yield of 3.91%, on July 4, 2006. This means that its market price was

$$Z = \frac{100}{1 + 0.0391 \times (63/365)} = 99.33,$$

as we have 63 calendar days between quotation date and expiration.

It is possible to use the continuous compounding, to define the **continuous compounded YTM** through the relation

$$Z(t, T) = 100e^{-y(T-t)}.$$

Yield of a coupon bearing bond.

Assume that a coupon bearing bond pays a principal P at maturity, and coupons C at specified dates

$$t_1 < \dots < t_n = T,$$

as usually the last coupon payment date t_n coincides with the maturity T .

In this case, if the market value of the bond is $V(t, T)$, the yield to maturity YTM, denoted by y is defined by the relation

$$\begin{aligned} V(t, T) &= \frac{P}{(1+y)^{T-t}} + \frac{C}{(1+y)^{t_1-t}} + \dots + \frac{C}{(1+y)^{t_n-t}} \\ &= \frac{P+C}{(1+y)^{T-t}} + C \times \sum_{k=1}^{n-1} \frac{1}{(1+y)^{t_k-t}}. \end{aligned}$$

Example Let us analyze the information contained in the Press Release

Tender of 5-Year EFN to be held on 9/6/06

(see the web page of the Course, we present the relevant information for our calculations)

The HKMA has issued an EFN (Coupons Bearing Bond) under the following characteristics:

- Issue date: Monday 12 June 2006
- Maturity: Five years.
- Maturity date: 13 June 2011.
- Interest rate: 4.57% p.a. (per annum)

- Interest Payment dates (10 coupons):

$t_1 = 12 \text{ Dec } 2006$, $t_2 = 12 \text{ Jun } 2007$, $t_3 = 12 \text{ Dec } 2007$,
 $t_4 = 12 \text{ Jun } 2008$, $t_5 = 12 \text{ Dec } 2008$, $t_6 = 12 \text{ Jun } 2009$,
 $t_7 = 14 \text{ Dec } 2009$, $t_8 = 14 \text{ Jun } 2010$, $t_9 = 13 \text{ Dec } 2010$,
 $t_{10} = T = 13 \text{ Jun } 2011$.

As we have two payments and a rate of 4.57%, each payments has amount of $4.57/2 = 2.285$ points.

The HKMA includes in the press release, for reference, the following Price/Yield table:

Yield TM	Price	Yield TM	Price	Yield TM	Price
3.57	104.69	3.62	104.46	3.67	104.23
3.72	104.01	3.77	103.78	3.82	103.55
3.87	103.33	3.92	103.10	3.97	102.88
4.02	102.65	4.07	102.43	4.12	102.20
4.17	101.98	4.22	101.76	4.27	101.54
4.32	101.32	4.37	101.10	4.42	100.88
4.47	100.66	4.52	100.44	4.57	100.23
4.62	100.01	4.67	99.79	4.72	99.58
4.77	99.36	4.82	99.15	4.87	98.93
4.92	98.72	4.97	98.51	5.02	98.30
5.07	98.09	5.12	97.87	5.17	97.66
5.22	97.45	5.27	97.25	5.32	97.04
5.37	96.83	5.42	96.62	5.47	96.41
5.52	96.21	5.57	96.00		

To compute the yields in this example:

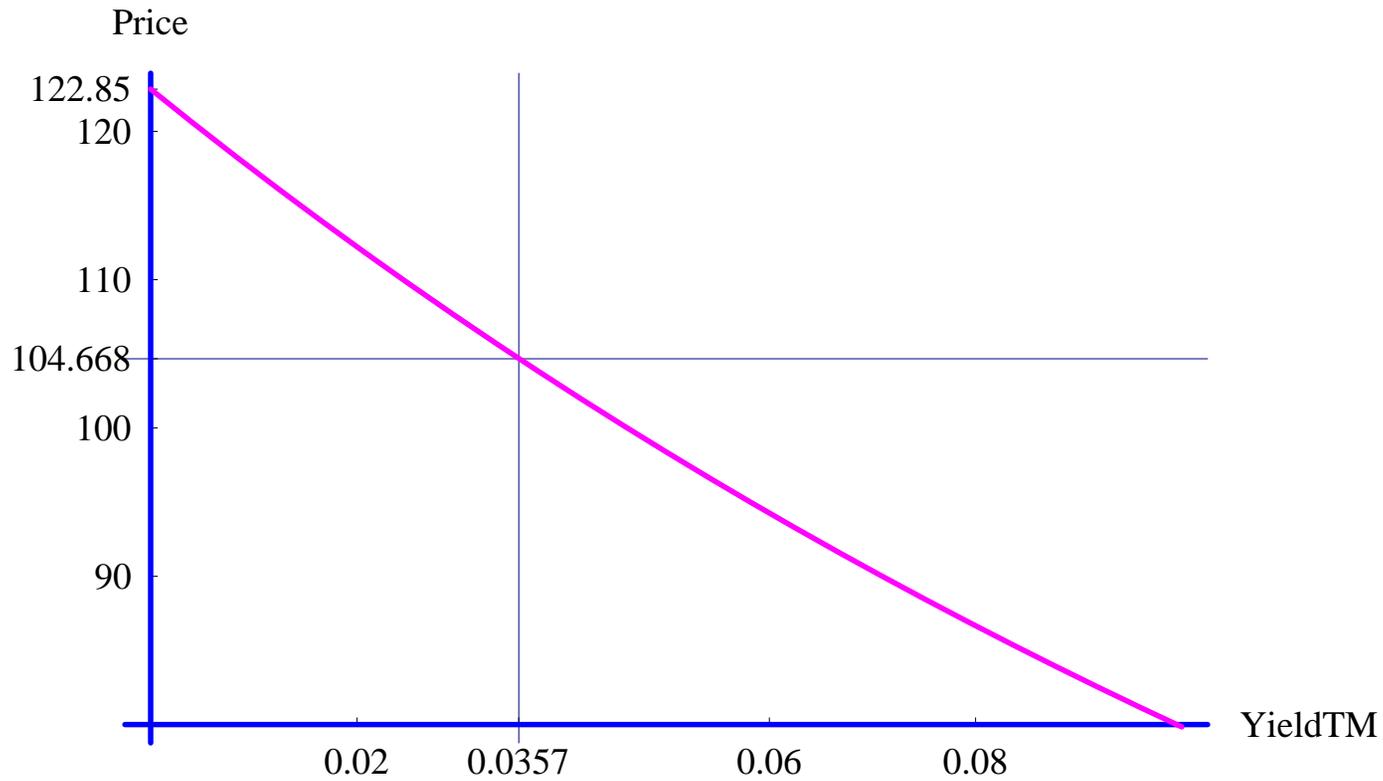
Assume that $P = 100$ points.

As it pays an annual interest rate of 4.57%, in two coupon payments, it pays $4.57/2 = 2.285$ points on each payment date. The payment dates are every six, months, i.e. every $1/2$ a year. So

$$V(y) = \frac{102.285}{(1+y)^5} + \frac{2.285}{(1+y)^{0.5}} + \frac{2.285}{(1+y)^1} + \dots + \frac{2.285}{(1+y)^{4.5}}.$$

This formula for $y = 0.0357$ (the 3.57%) gives $V = 104.688$ (and in the table we see 104.69)

It must be noticed that the total amount of points to be paid by the bond is 122.85 equal to the principal plus the coupons. This value is obtained with $y = 0$ in the above formula, i.e. $V(0) = 122.85$.



The graphic show the Price of the Bond as a function of the Yield to Maturity.

22d. Yields and Yields.

It must be noticed that the way of computing the yield to maturity of bonds is not uniform over the literature, and also, not uniform in practice.

All methods give **approximate** the same yields, but as the face values of the Bonds are usually large amounts of money, each EFN is faced 50,000 HKD, so **one point** is 500 HKD, and small difference in percents are important.

Different ways of computing the YTM of a paying coupon bond with market price V , paying coupons C at times t_k : include:

- The continuous compounded way, defines y through the equation

$$V = 100 e^{y(T-t)} + C \times \sum_k e^{y(t_k-t)}.$$

as in “Derivatives” by Willmot.

- Avellaneda and Laurence denote by ω the number of coupons per year, the [frequency](#) of coupon payments (for EFN we have $\omega = 2$), and define the YTM y as the solution to the equation

$$V = \frac{100}{(1 + y/\omega)^{\omega(T-t)}} + C \times \sum_k \frac{1}{(1 + y/\omega)^{\omega(t_k-t)}}$$

- In our computations we have used the formula

$$V = \frac{100}{(1 + y)^{(T-t)}} + C \times \sum_k \frac{1}{(1 + y)^{t_k - t}}$$

as it seems to give best approximations to the quoted prices and yields in the releases of the HKMA.

- Here is is important to be **consistent**, meaning to use the same calculation method in order to compare different bond yields, and to know wich method has been used in each case.

23. Yields, Duration and Yield Curves

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Plan of Lecture 23

(23a) Yield to maturity of a EFN

(23b) Bond Duration

(23c) Yield Curve

(23d) Cash Flows

(23e) Other Fixed Income securities

23a. Yield to maturity of a EFN

The yield to maturity is also computed during the life of the bond.

Example We take for reference the Exchange Fund Note 5709. in the “Reference Prices and Yields for Exchange Fund Notes, from 11:00 a.m. July 4, 2006” issued by the HKMA. Relevant bond characteristics information for us are:

- EFN Issue Nr 1801,
- Maturity: Jan 21, 2008,
- Coupon interest rate: 9.89%,
- Price: 108.06,
- Yield: 4.515.

In fact, we want to understand [how](#) to compute the yield with the market information (the YTM in this case was computed by Reuters).

Other relevant characteristics about the EFM 1801 (from HKMA web site) are:

- 1801 EFM is a 10 year bond
- It was issued on Tuesday 20 January 1998,
- Remaining coupon payments dates are: $t_1 = 20$ Jul 2006, $t_2 = 22$ Jan 2007, $t_3 = 20$ Jul 2007, and $t_4 = T = 21$ Jan 2008.

In order to compute the YTM, the first step is to compute $t_k - t$.

We have $t = 4$ Jul 2006 and $t_1 = 20$ Jul 2006 giving

$$\Delta = t_1 - t = 16/365 = 0.0438.$$

The rest of the values are:

	Jul 20, 06	Jan 22, 07	Jul 20, 07	Jan 21, 08
$t_k - t$	Δ	$0.5 + \Delta$	$1 + \Delta$	$1.5 + \Delta$

As the interest rate is 9.89%, we have $C = 9.89/2 = 4.945$ points payed at each t_k .

With this information the yield y is such that

$$\begin{aligned} Clean = & \frac{104.945}{(1 + y)^{1.5 + \Delta}} \\ & + 4.945 \left[\frac{1}{(1 + y)^\Delta} + \frac{1}{(1 + y)^{0.5 + \Delta}} + \frac{1}{(1 + y)^{1 + \Delta}} \right] \end{aligned}$$

Here *Clean* is not the market value of the Bond, as from the 4.945 points of the current inter-coupon payments, a part corresponds to the seller.

This part f is assumed to be proportional to the time elapsed since the last coupon payment date and the current date t . In our case

$$f = \frac{182 - 16}{182} = 0.912$$

The market price is obtained as

$$108.06 = \textit{Clean} - 4.945 \times 0.912.$$

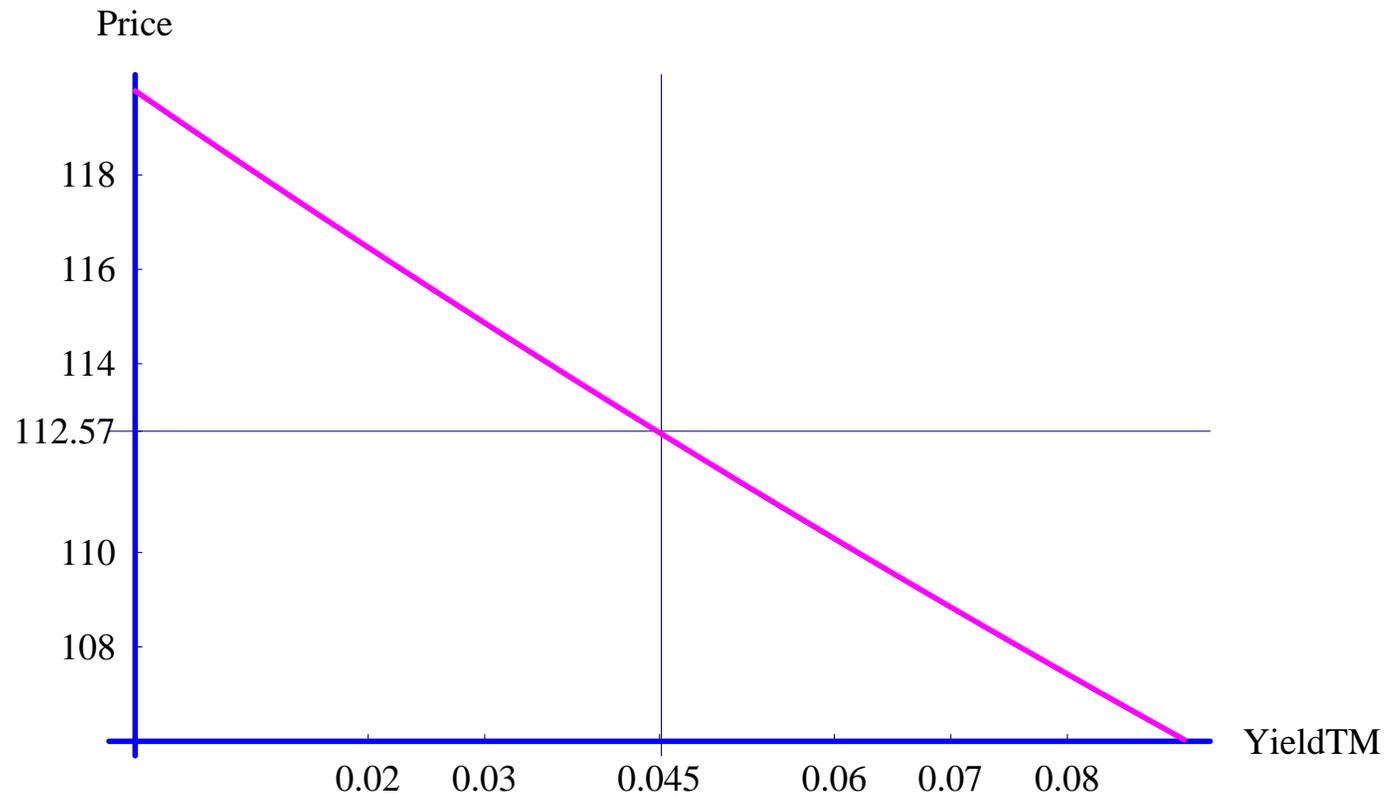
In conclusion, as

$$108.06 + 4.945 \times 0.912 = 112.57,$$

the YTM y is found through the equation

$$112.57 = \frac{104.945}{(1 + y)^{1.5 + \Delta}} + 4.945 \left[\frac{1}{(1 + y)^{\Delta}} + \frac{1}{(1 + y)^{0.5 + \Delta}} + \frac{1}{(1 + y)^{1 + \Delta}} \right]$$

The graphic shows the market price as a function of the yield:



In the graphic we see the yield ($0.045=4.5\%$) corresponding to the value 112.57.

23b. Bond Duration

A second relevant quantity related to a bond is its **duration**, that measures the variation of price as a function of the YTM.

In formulas the duration D is computed as

$$D = \frac{1}{V} \left[(T - t) \times \frac{100 + C}{(1 + y)^{(T-t)}} + (t_1 - t) \times \frac{C}{(1 + y)^{t_1 - t}} + \dots + (t_{n-1} - t) \times \frac{C}{(1 + y)^{t_{n-1} - t}} \right]$$

The **duration** of a bond is the **mean** of the payments dates, ponderated by the respective discounted payments of the bond. It is also called the **average** or **McCauley** duration.

Remark The duration of a zero coupon bond is its maturity.

Example Let us compute the duration for the EFM 1081 of last example. We have a corrected current price of $V = 112.57$, so the duration is

$$D = \frac{1}{112.57} \left[\frac{104.945 \times (1.5 + \Delta)}{(1 + y)^{1.5 + \Delta}} + 4.945 \left[\frac{\Delta}{(1 + y)^\Delta} + \frac{0.5 + \Delta}{(1 + y)^{0.5 + \Delta}} + \frac{1 + \Delta}{(1 + y)^{1 + \Delta}} \right] \right].$$

With $\Delta = 0.0438$ and $y = 0.045$ we obtain

$$D = 1.3582 \text{ years.}$$

Remember that time to maturity was $1.5 + \Delta = 1.50438$, one year and a half.

Remark The duration is always smaller than the maturity.

In order to measure the sensitivity of the bond with respect to the yield, we compute the derivative. Consider the value as a function of y :

$$V(y) = \frac{P}{(1+y)^{T-t}} + C \times \sum_{k=1}^n \frac{1}{(1+y)^{t_k-t}}.$$

Remember that

$$\begin{aligned} \frac{\partial}{\partial y} \left[\frac{1}{(1+y)^r} \right] &= \frac{-r}{(1+y)^{r+1}} \\ &= \frac{-r}{(1+y)} \times \frac{1}{(1+y)^r}, \end{aligned}$$

This will help us to compute

$$\begin{aligned}
V'(y) &= \frac{\partial V(y)}{\partial y} \\
&= \frac{-P(T-t)}{(1+y)(1+y)^{T-t}} - C \times \sum_{k=1}^n \frac{t_k - t}{(1+y)(1+y)^{t_k-t}} \\
&= \frac{-1}{1+y} \left[\frac{P(T-t)}{(1+y)^{T-t}} + C \times \sum_{k=1}^n \frac{t_k - t}{(1+y)^{t_k-t}} \right] \\
&= \frac{-1}{1+y} [V(y)D],
\end{aligned}$$

where D is the duration. With this computations we obtain that

$$\frac{V'(y)}{V(y)} = -\frac{D}{(1+y)}.$$

This led us to consider the [modified duration](#), as

$$D_{mod} = \frac{D}{(1 + y)}$$

The applications of this concept is the following:

- The duration measures the [sensitivity](#) of the value with respect to the yield to maturity. Bonds with higher duration are more sensitive to variations in the yield, and in consequence, more risky.
- The duration is used to hedge, i.e. to lower the risk exposure of a bond.
- Similarly, one defines the [convexity](#) of a bond as the [second](#) derivative of the value with respect to the yield, also used to hedge.

23c. Yield Curve

The **yield curve** is the relation between the cost of borrowing, or yield, and the maturity of the debt for a given borrower in a given currency.

For example, the current U.S. dollar interest rates paid on U.S. Treasury securities for various constitute the US yield curve.

Let us compute the yield curve for the HKD.

We take the fixings of the HKMA release 4 July 2006, detailed in the following tables:

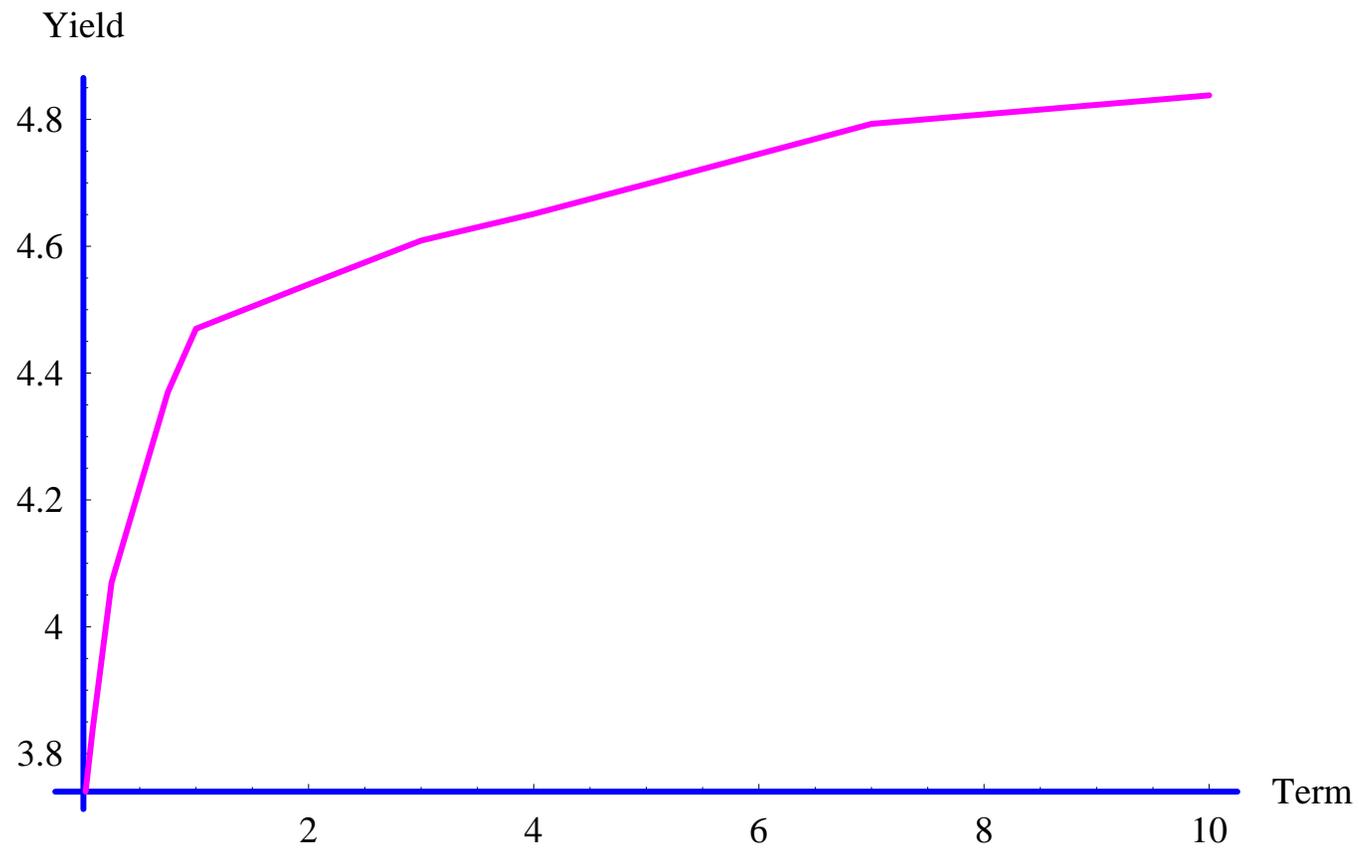
Term	Issue	Yield
1W	Q615	3.74
1M	Q619	3.84
3M	Q626	4.07
6M	H667	4.22
9M	Y688	4.37
12M	Y691	4.47

Term	Issue	Yield	Price
2YR	2805	4.540	99.66
3YR	5906	4.609	97.64
4YR	5006	4.651	95.47
5YR	5106	4.698	99.67
7YR	7305	4.793	100.01
10YR	1606	4.838	100.38

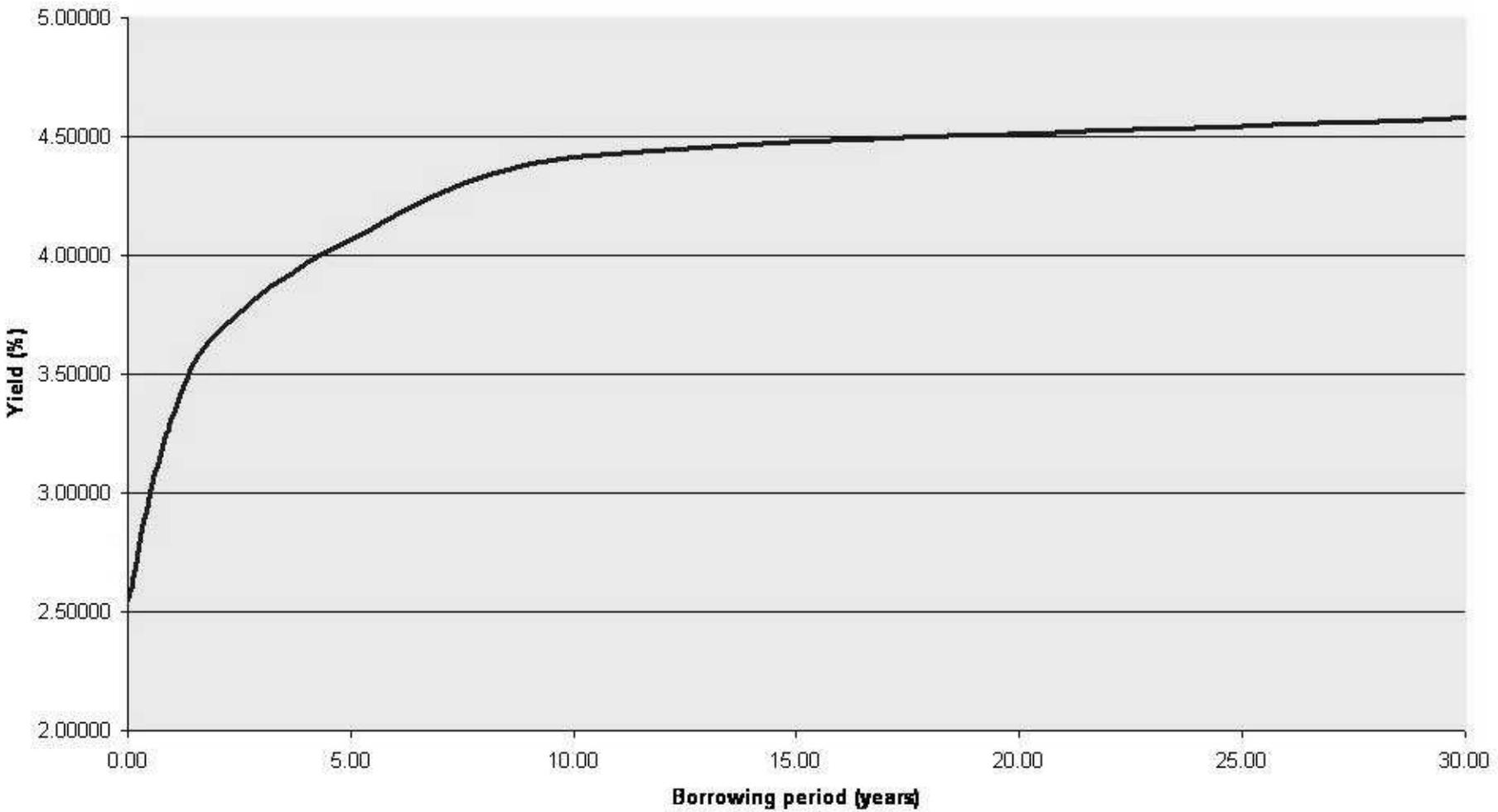
Exchange Fund Bills

Exchange Fund Notes

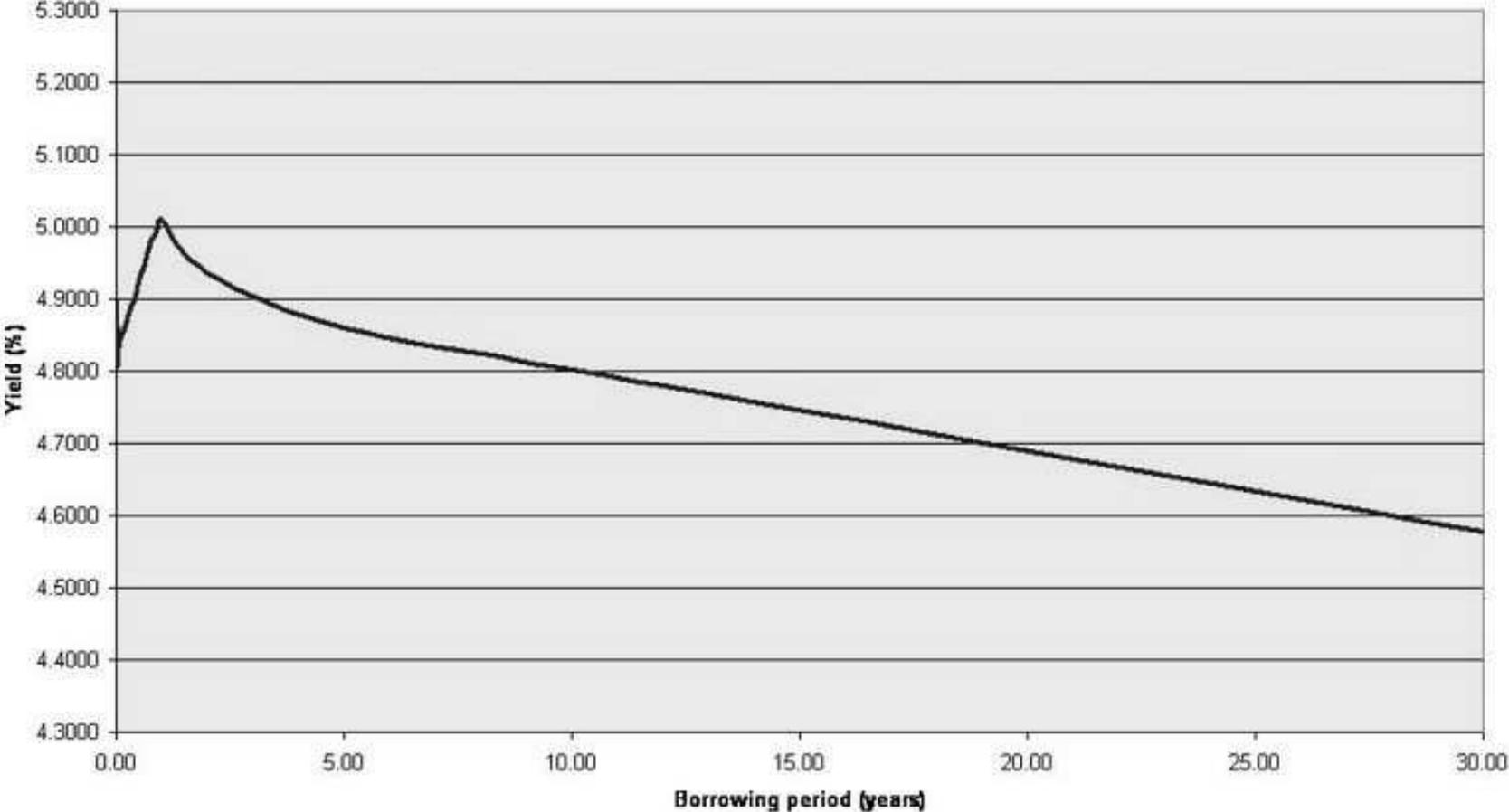
With this information we plot the Yield Curve of the HK Dollar



Yield curve as at 9th February 2005 for USD



Yield curve as at 9th February 2005 for GBP



23d. Cash Flows

In order to describe other fixed income financial instruments we introduce the notion of **cash flow**, i.e. a series of payments P_1, \dots, P_n scheduled at future dates t_1, \dots, t_n .

In a schematic way we have

Date	t_1	\dots	t_n
Amount	P_1	\dots	P_n

For example, a two year maturity Bond, issued today with principal P and coupons C has a cash flow with four dates:

Date	0.5	1	1.5	2
Amount	C	C	C	$P + C$

By the way, this coupon bearing bond is equivalent to **four** zero coupon bonds:

- The first with maturity in half a year and principal C ,
- The second with maturity in a year and principal C ,
- The third with maturity in one and a half year and principal C
- The last one with maturity in two years and principal $P + C$.

This is relevant as, if we have a method to **price** zero coupon bonds with different maturities, we can use this method to price the coupon bearing bond.

23e. Other Fixed Income securities

A forward rate agreement (FRA) is a contract in which one party pays a fixed interest rate, and receives a floating interest rate equal to a reference rate (the underlying rate).

The payments are calculated over a notional amount over a certain period.

An interest swap is a contractual agreement between two parties, in which they agree to make periodic payments to each other according to two different indices.

In a plain vanilla swap, one party A makes fixed interest rate payments to receive from B floating rate payments.

In other words, they agree to interchange a fixed rate cash flow for a floating rate cash flow. The agreement also includes a notional amount or face value of the swap, the

payment frequency (quarterly, etc.) the tenor or maturity of the agreement, and, of course, the fixed and floating rates. For example a swap can consist in

Dates	0.5	1	1.5	2
<i>A to B</i>	5%	5%	5%	5%
<i>B to A</i>	6M USD Lb	6M USD Lb	6M USD Lb	6M USD Lb

for a certain nominal.

More sophisticated contracts include [American](#) features:

- A [callable](#) bond is a coupon bearing bond with the additional feature that the issuer has the right to [call back](#) (i.e. to buy from the holder) the bond, at specified dates, and at specific price.
- A swap can also include the possibility of one party to interrupt the agreement.

More general, there are [derivatives](#) on fixed income instruments, the most typical being the [bond option](#), identical to a stock option, with the difference that the underlying is a bond