20. Diffusion with Jumps Modelling

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References for this Lecture:

R. C. Merton. "Option Pricing When Underlying Stock Returns are Discontinuous", Journal of Financial Economics, 5, pp 125-144 (1976).

See also: R.C. Merton, *Continuous Time Finance*, Basil Blackwell (1990).

R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall, CRC Press (2003).

Plan of Lecture 20

(20a) Jumps in financial data

(20b) When jumps occur: the exponential distribution

(20c) Simulating moments of jumps

(20d) The Poisson Process

(20e) Merton Jump Diffusion model

(20f) Risk Neutral condition

20a. Jumps in financial data

An alternative to BS models is to consider that, from time to time, the financial instrument being modelled has significative larger movements that the ones observed usually.

This larger movements, that can be produced by some unexpected information, are called jumps in the evolution of the asset.

The interest in modelling with jumps is based on the fact that

- Jumps are observed in financial time series
- The presence of jumps produces some of the stylized facts observed in time series as asymmetry (skewness) or heavy tails (kurtosis)
- But, mainly, because adding a few parameters the models can produce derivative pricing with smiles similar to the observed in option prices.

20b. When jumps occur: the exponential distribution

In order to model the occurence of jumps we introduce the exponential distribution.

The idea is that the first jump in our model will occur at an uncertain moment τ.

A random variable τ has exponential distribution with parameter λ if it has a density

$$p(t) = \lambda e^{-\lambda t}, \quad \text{for } t \geq 0.$$

(τ takes only positive values.) From this we can compute probabilities:

$$\mathbf{P}(\tau \leq t) = 1 - e^{-\lambda t}.$$

Sometimes it is easier to check, that

$$\mathbf{P}(\tau \geq t) = e^{-\lambda t}.$$

More important, we obtain that

$$\mathbf{P}(\tau \geq t + h \mid \tau \geq t) = \mathbf{P}(\tau \geq h).$$

This property says that, if we know that the jump has not occured at time t (i.e. conditional on $\tau \geq t$) the probability of τ arriving h unit of time later that t (i.e. at $\tau \geq t+h$) is the same as in the principle, i.e. the probability of $\tau \geq h$.

In other words, if we know that the first jump has not occured at time t, the situation is (in what respects the jump) the same as in the begining $t = 0$.

This property, the memoryless property of the exponential distribution means that, knowing that the jump has not arrived, we do not have any information to know when it will occur[1].

[1] It can be proved that the exponential distribution is the only one distribution with this property

As we want to model more jumps, we denote by τ_1 the first moment of jump.

In fact we assume that once the first jump has arrived, the situation starts anew, with a second (independent) exponential random variable τ_2, and so on.

In conclussion, given a sequence of independent random variables $\tau_1, \tau_2, \dots$, with exponential distribution with parameter λ, modelling with jumps consist in assuming that the asset will suffer a large movement at time $T_1 = \tau_1$, a second large movement at time $T_2 = \tau_1 + \tau_2$, and, in general, a n-th large movement at time $T_n = \tau_1 + \cdots + \tau_n$.

20c. Simulating moments of jumps

Assume that U is an uniform random variable on $[0, 1]$, and define

$$\tau = (-1/\lambda) \log U.$$

We have

$$\begin{aligned}\mathbf{P}(\tau \geq t) &= \mathbf{P}\left((-1/\lambda) \log U \geq t\right) \\ &= \mathbf{P}(U \leq e^{-\lambda t}) = e^{-\lambda t}.\end{aligned}$$

In conclusion, τ is an exponential random variable with parameter λ.

From this fact, we can simulate the jumps $T_1, \ldots, T_n$ of a jump diffusion on an interval $[0, T]$:

Step 1 Sample and uniform random variable U_1.

Step 2 Compute $T_1 = \tau_1 = (-1/\lambda) \log U_1$.

Step 3 If $T_1 > T$, then we have no jumps on $[0, T]$ and we have finished.

Step 4 Otherwise we sample a second uniform r.v. U_2 and compute $\tau_2 = (-1/\lambda) \log U_2$.

Step 5 If $T_2 = T_1 + \tau_2 > T$ we have one wump T_1.

Step 6 Otherwise we continue with $\tau_3 = (-1/\lambda) \log U_3$, compute $T_3 = T_2 + \tau_3$, and check $T_3 > T$. If this happens, we have jumps T_1, T_2.

Step 7 We continue until the first n such that $T_n > T$. In this case, the jumps in the interval are $T_1, \ldots, T_{n-1}$.

20d. The Poisson Process

An important (random) quantity is the number of jumps that occured up to time t, denoted by $N(t)$. Formally

$$N(t) = \max\{i\colon T_i < t\}.$$

It can be proved that

$$\mathbf{P}(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots,$$

and it is said that $N(t)$ has a Poisson distribution with parameter λt, and also that $\{N(t)\}$ is a Poisson process with intensity λ.

We can now compute the number of expected jumps over $[0,t]$, as

$$\mathbf{E}\,N(t) = \sum_{n=0}^{\infty} n\,\mathbf{P}(N(t)=n) = \sum_{n=0}^{\infty} n e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$
$$= (\lambda t)e^{-\lambda t}e^{\lambda t} = \lambda t.$$

We have obtained that the expected number of jumps is proportional to the length of the time interval, being λ, the intensity, the proportionality constant.

In practice the values of λ (measured in annualized terms) may range from 1 to 5,

$$\lambda \sim 1 \quad \text{to} \quad \lambda \sim 5.$$

In other words, in one year $t = 1$ we expect λ extraordinary price movements.

20e. Merton Jump Diffusion model

In order to specify our model with jumps we assume:

- Jumps are observed at times $T_1, T_2, \ldots$ constructed as above.
- The magnitudes of each log-stock jump is normally distributed, with paramters (ν, δ^2).
- Between jumps we assume a Black-Scholes dynamics
- The time of jumps, the magnitude of the jumps, and the Wiener process of the BS dynamics are independent.

In formulas, we denote by $Y_1, Y_2, \ldots$ a sequence of independent random variables with identical $\mathcal{N}(\nu, \delta^2)$.

The asset price is:

- Before the first jump, $t < T_1$:

$$\frac{S(t)}{S(0)} = \exp[(\alpha - \sigma^2/2)t + \sigma W(t)],$$

- At the first jump, $t = T_1$:

$$\frac{S(t)}{S(0)} = \exp[(\alpha - \sigma^2/2)T_1 + \sigma W(T_1) + Y_1],$$

- Between the first and second jumps, $T_1 \le t < T_2$:

$$\frac{S(t)}{S(0)} = \exp[(\alpha - \sigma^2/2)t + \sigma W(t) + Y_1],$$

- At the second jump, $t = T_2$:

$$\frac{S(t)}{S(0)} = \exp[(\alpha - \sigma^2/2)T_2 + \sigma W(T_2) + Y_1 + Y_2],$$

- ...

In the general case, up to time t we have accumulated $N(t)$ of jumps, and the asset price is

$$\frac{S(t)}{S(0)} = \exp\left[(\alpha - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i\right]$$

Remark The difference with the BS model is the the sum of accumulated jumps in the log asset evolution.

20f. Risk Neutral condition

In order to check the kisk neutral condition of the model[2] we use use the technique of conditioning.

The idea is to divide the probability space according to the occurence of $n = 0$ jumps, $n = 1$ jump, $n = 2$ jumps and so on.

In other terms, we consider the events

- $N(t) = 0$, i.e. no jump occured up to time t,
- $N(t) = 1$, i.e. one jump occured up to time t,
- $\cdots\cdots$ and so on.

Let us see how this principle works to determine the risk neutral condition the parameters should satisfy.

[2] In fact, we are assuming that the price process follows the Merton model under $\mathbf{Q}$, the risk neutral probability.

The risk neutral condition reduces to

$$\mathbf{E}_{\mathbf{Q}}\, S(t) = S(0)e^{rt}$$

where r is the risk free interest rate. According to Merton's model

$$S(t) = S(0)\exp\left[(\alpha - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i\right]$$

$$= S(0)e^{(\alpha-\sigma^2/2)t} \times\ e^{\sigma W(t)} \times\ e^{\sum_{i=1}^{N(t)} Y_i}.$$

As we have independence of the three factors

$$\mathbf{E}_{\mathbf{Q}}\, S(t) = S(0)\,\mathbf{E}_{\mathbf{Q}}\, e^{(\alpha-\sigma^2/2)t}\ \mathbf{E}_{\mathbf{Q}}\, e^{\sigma W(t)}\ \mathbf{E}_{\mathbf{Q}}\, e^{\sum_{i=1}^{N(t)} Y_i}$$

The first factor is deterministic:

$$\mathbf{E}_{\mathbf{Q}}\, e^{(\alpha-\sigma^2/2)t} = e^{(\alpha-\sigma^2/2)t}.$$

The second factor is the expectation of a log-normal random variable, as in BS model:

$$\mathbf{E}_{\mathbf{Q}}\, e^{\sigma W(t)} = e^{(\sigma^2/2)t}.$$

In the third term we use conditioning:

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \exp\left[\sum_{i=1}^{N(t)} Y_i\right] &= \sum_{n=0}^{\infty} \mathbf{E}_{\mathbf{Q}}\left[\mathbf{1}_{\{N(t)=n\}}\, e^{\sum_{i=1}^{n} Y_i}\right] \\ &= \sum_{n=0}^{\infty} \mathbf{Q}(N(t)=n)\ \mathbf{E}_{\mathbf{Q}}\left[e^{\sum_{i=1}^{n} Y_i}\right] \end{aligned} \tag{1}$$

Now we compute each summand separately:

$$\mathbf{E}_{\mathbf{Q}}\, e^{(Y_1+\cdots+Y_n)} = \mathbf{E}_{\mathbf{Q}}\left(e^{Y_1}\cdots e^{Y_n}\right) = \mathbf{E}_{\mathbf{Q}}\, e^{Y_1}\cdots\ \mathbf{E}_{\mathbf{Q}}\, e^{Y_n}.$$

But $Y_1, \ldots, Y_n$ are normal random variables with parameters (ν, δ^2), so $e^{Y_1}, \ldots, e^{Y_n}$ are lognormal with the same parameters, and

$$\mathbf{E}_{\mathbf{Q}}\, e^{Y_1} = e^{\nu+\delta^2/2}.$$

This gives

$$\mathbf{E}_{\mathbf{Q}}\, e^{\sum_{i=1}^{n} Y_i} = e^{(\nu+\delta^2/2)n},$$

and returning to our initial computation (1), taking into account that

$$\mathbf{Q}(N(t) = n) = e^{-\lambda t}(\lambda t)^n/n!,$$

we conclude that

$$\mathbf{E}_{\mathbf{Q}}\, e^{\sum_{i=1}^{N(t)} Y_i} = \sum_{n=0}^{\infty} e^{(\nu+\delta^2/2)n} e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \Big[e^{(\nu+\delta^2/2)} \frac{(\lambda t)}{n!} \Big]^n$$

$$= \exp\big[t\big(-\lambda + \lambda e^{(\nu+\delta^2/2)}\big)\big] = \exp\big[t\big(\lambda(e^{\nu+\delta^2/2} - 1)\big)\big].$$

With all this computations, the risk neutral condition is

$$\exp(rt) = \exp\left[t\big(\alpha - \sigma^2/2) + \sigma^2/t + \lambda(e^{\nu+\delta^2/2} - 1)\big)\right]$$

that, after taking logarithms, dividing by t, and denoting $\kappa = e^{\nu+\delta^2/2} - 1$, gives

$$\alpha = r - \lambda\kappa.$$

Observe, that if $\lambda = 0$, meaning that we have no jumps, we recover the BS risk neutral condition $r = \alpha$.

21. Option Pricing for Diffusion with Jumps

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References for this Lecture:

R. C. Merton. "Option Pricing When Underlying Stock Returns are Discontinuous", Journal of Financial Economics, 5, pp 125-144 (1976).

See also: R.C. Merton, *Continuous Time Finance*, Basil Blackwell (1990).

R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall, CRC Press (2003).

Plan of Lecture 21

(21a) Option Prices for MJD model

(21b) Numerical Experiments with BS

(21c) Calibration of MJD for the S&P 500

(21d) Calibration for different maturities

21a. Option Prices for MJD model

Merton Jump Diffusion model for option prices, under the risk neutral measure, is

$$S(t) = S(0)\exp\left[(r - \lambda\kappa - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{N(t)} Y_i\right]$$

- r is the risk free interest rate,
- λ is the mean number of jumps in one year,
- σ is the standard deviation of the gaussian component
- $W(t)$ is a Wiener process
- $N(t)$ is a Poisson process with parameter λ,
- Y_i is the jump of the log-stock price, assumed to be lognormal with parameters (ν, δ),
- $\kappa = e^{\nu+\delta^2/2} - 1$.

With the technique of conditioning, Merton obtained the price of a call option under the jump diffusion model.

Suppose that we want to price an European Call Option with expiry T and strike K. Remember the BS price of such an option by

$$BS(S, K, T, r, \sigma) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}dt$ is the standard normal distribution, and

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

We consider the stock according to the number of jumps in $[0, T]$

If we have n jump in $[0,T]$, under the risk neutral condition (i.e. with $\alpha = r - \lambda\kappa$) the stock is

$$\frac{S_n(T)}{S(0)} = \exp\left[(r-\lambda\kappa-\sigma^2/2)T+\sigma W(T)+Y_1+\cdots+Y_n\right]$$

As $W(T)$ and the jumps Y_i are independent, we have

$$\sigma W(T) + Y_1 + \cdots + Y_n \sim \mathcal{N}(n\nu, \sigma^2 T + n\delta^2).$$

So, computing the option price as the expected reward under $\mathbf{Q}$, the price of the option is

$$\begin{aligned} CJ &= e^{-rT}\,\mathbf{E}_{\mathbf{Q}}\left(S(T) - K\right)^+ \\ &= \sum_{n=0}^{\infty} e^{-(\lambda+r)T}\frac{(\lambda T)^n}{n!}\,\mathbf{E}_{\mathbf{Q}}(S_n(T) - K)^+. \end{aligned}$$

Merton's proposal is the relate this price to the BS model, so he defines

$$\sigma_n^2 = \sigma^2 + \frac{n}{T}\delta^2,$$

$$r_n = r - \lambda\kappa + \frac{n}{T}\Big(\nu + \frac{\delta^2}{2}\Big)$$

and, after some transformations, obtains that

$$CJ = \sum_{n=0}^{\infty} e^{-\lambda' T}\frac{(\lambda' T)^n}{n!} BS(S, K, T, r_n, \sigma_n)$$

where r_n and σ_n are given above, and

$$\lambda' = \lambda(1+\kappa) = \lambda e^{\nu+\delta^2}.$$

Remarks

- The price of a call option in the jump diffusion model is a mixture of BS prices of call options, each with different rate and volatility, and the n-th mixture coefficient is related to the probability of having n jumps.
- Althogh formally we have an infinite series, in practice it is enough to sum the first 5 to 10 terms[3].

[3] More recent developements on option pricing with Jumps suggest the use of the Fourier Transform to compute the call option price.

21b. Numerical Experiments with BS

We perform the calibration procedure under the BS assumption, with:

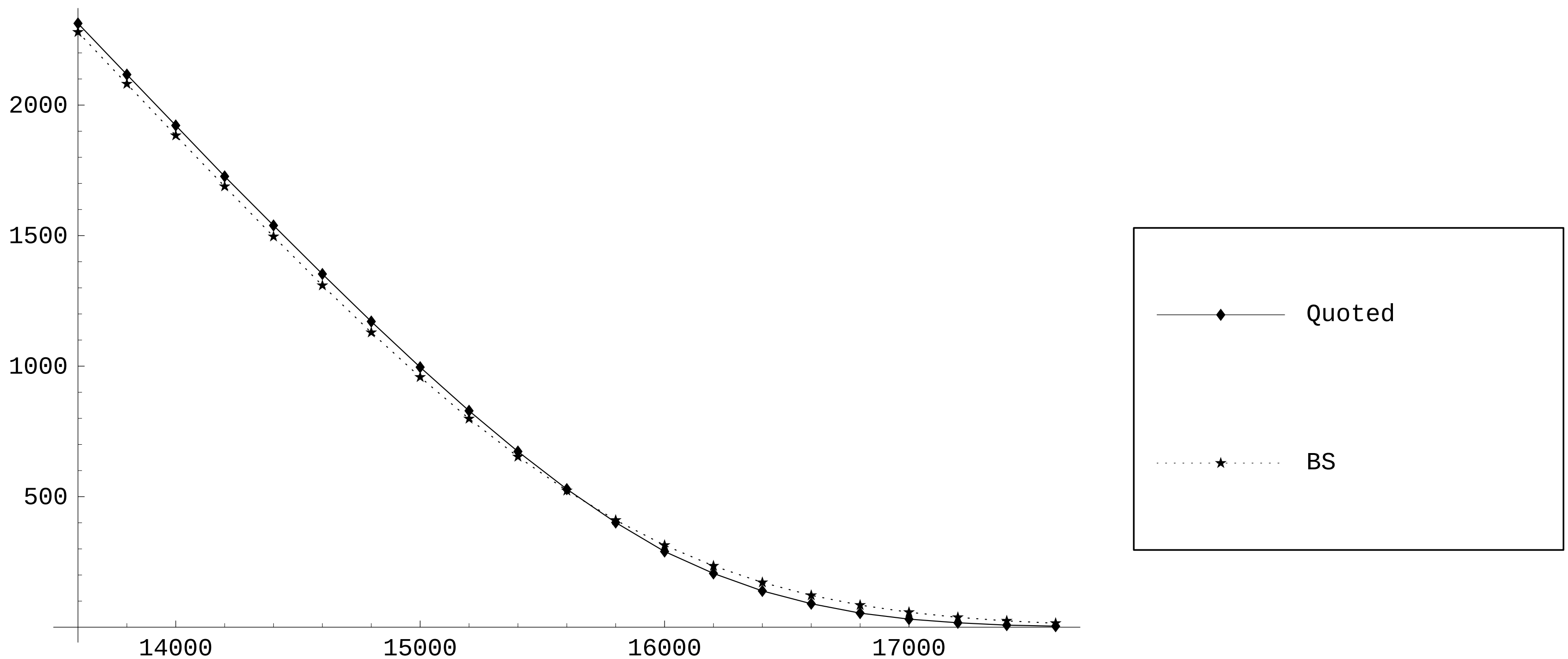
- Quoted prices of July options (see SCMP, June 30)
- Computed with BS

The known parameters are:

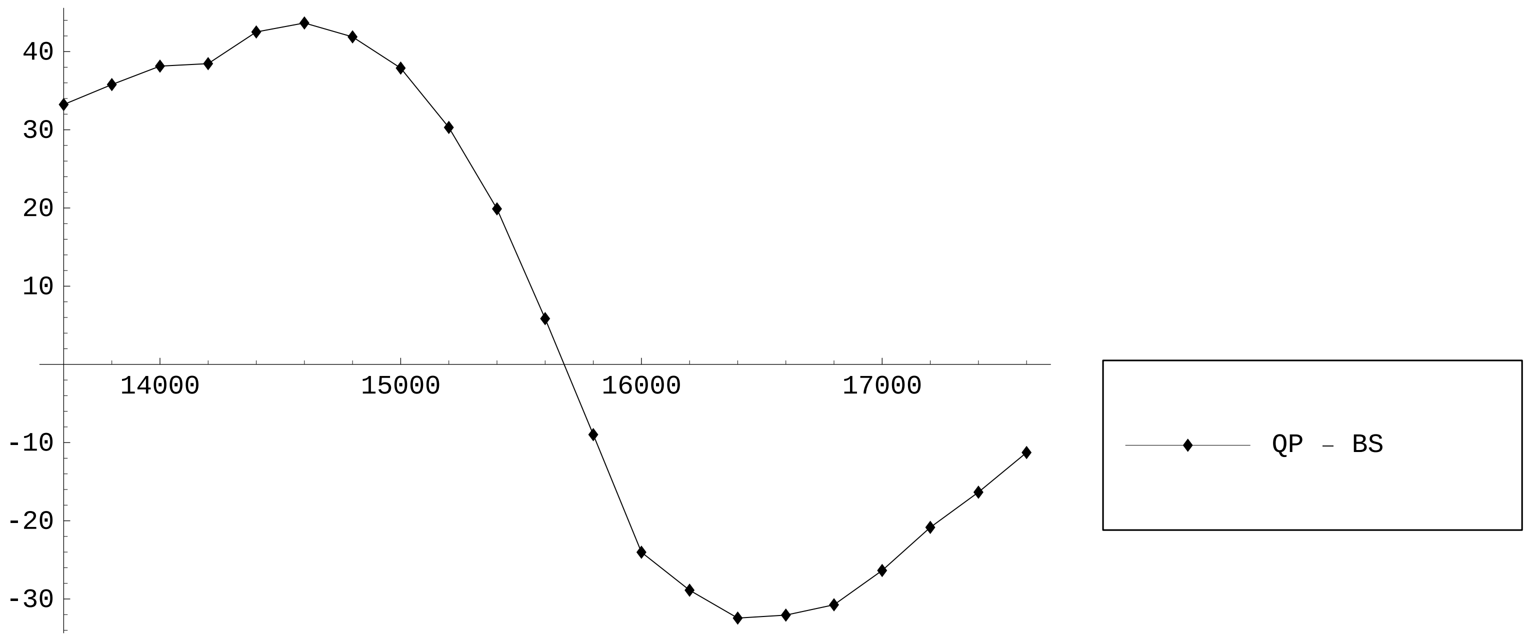
- Spot Price $S(0) = 15865.22$ (closing price of the HSI on June 29).
- $T = 21/247$, as we have 21 pricing days in the interval June 30, July 28 (the maturity of July options), and 247 trading days in 2006.
- Risk free interest rate (the futures quotation $F(T) = 15881$):

$$r = \frac{1}{T} \log \left[\frac{F(T)}{S(0)} \right] = \frac{247}{21} \log \left[\frac{15881}{15865.22} \right] = 0.012.$$

Let us first compare the prices given by BS, with a mean volatilty of 0.20, with the quoted prices:



In order to see the difference of prices, we plot it:



In the plot we see maximum difference between Quoted Prices and BS prices of about 40 points.

The point where the prices coincies, a strike of about 15750, is such that the implied volatility is exactly $\mathbf{v} = 0.20$, the volatility used in BS formula.

In order to prove our calibration method, let us find σ that minimizes

$$\sum_{i=0}^{20} \Big(C(13600 + 200i) - BS(\sigma, 13600 + 200i)\Big)^2.$$

Here $BS(\sigma, K)$ is the BS price with volatility σ and strike K, where we do not include the other parameters that are fixed in the optimization.

After some numerical calculations we arrive to a minimun at the point

$$\sigma = 0.197$$

(very near to our initial $\sigma = 0.20$).

21c. Calibration of MJD for the S&P 500

The general procedure to calibrate parametric models, i.e. models where we have a fixed number of parameters is done similarly.

For instance, in MJD model we have to calibrate four parameters:

$$\theta = (\sigma, \lambda, \nu, \delta).$$

Here

- σ is the standard deviation of the BS part of the model,
- λ is the rate of jumps, i.e. the mean number of jumps in a year,
- ν is the mean value of the jumps,
- δ is the standard deviation of the jumps.

The parameter α is determined by the risk neutral condition.

The direct calibration numerical procedure has problems. The search methods like the Newton-Raphson gives solutions that depende too much on the initial values (it is and ill-posed problems).

In order to circumvent this difficulty, one adds a regularizing term, minimizing

$$\sum_{k,j} \big(C(\theta, K, T) - QP(K, T)\big)^2 + R(\theta).$$

We now review some calibration results from the literature[4].

[4] L. Andersen and J. Andreasen, "Jump-Diffusion Processes: Volatility Smile Fitting and Numerican Methods for OPtion Pricing", Review of Derivatives Research,4, (2000). It must be said that nowadays the models choosen to calibrate stock option prices are more sophisticated than the ones we examined, for instance, the combinate a one factor model with jumps, i.e.

The authors take prices from S&P 500 index on April 1999. They cover a wide range of strikes, and combine maturities from $T = 0.08$ (one month), $T = 0.25$ (quarter),... up to $T = 10$ (ten years).

The obtained result is:

σ	λ	ν	δ
0.176	0.089	-0.88	0.45

Some comments are in order

- Usually σ in MJD is smaller that the implied volatility in BS for the same option prices. This is due to the fact that in MJD we have two sources of risk:
  - diffusion risk
  - jumps risk,

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assume that the volatility is a function $\sigma(t, S(t))$ of the time and the spot price, where the evolution of the one factor model is interrupted from time to time (with exponential intervals) by a jump.

and the idea is that the superposition of both should equate the only source of risk in BS, the volatility.

- As we said, the intensity of jumps λ is in the order from one to five per year. A larger λ can be interpreted as model missespecification
- The value of ν is usually negative. This is related to the fact that the smile is not symmetric, giving larger values to out of the money options. I.e. in the risk-neutral world, jumps have negative expectation.

21d. Calibration for different maturities

We now review more recent results of calibration for the MJD, also for the S&P 500, with the additional property of different maturities[5].

Data consists of daily call option prices on the S&P500 futures from March 24, 2004 to March 16, 2005, with a total of 248 trading days.

The results are presented in the following table:

[5] K. Matsuda. Parametric Regularized Calibration of Merton Jump-Diffusion Model with Relative Entropy: What Difference Does It Make?, http://www.maxmatsuda.com/

Risk neutral parameters for the S&P500 index in MJD.

$T = \text{days}/248$	σ	λ	ν	δ
10/248	0.086	1.001	-0.054	0.043
40/248	0.092	1.296	-0.070	0.056
80/248	0.094	0.663	-0.107	0.081
120/248	0.095	0.732	-0.144	0.0082
180/248	0.094	0.486	-0.198	0.112
245/248	0.109	0.307	-0.267	0.140

- Typical values of σ are 10%, half of the typical implied volatility in Black Scholes of 20%.
- The mean number of jumps λ is larger for shorter maturities. For longer maturities the effect of jumps "passes" to the standard deviation of the BS part.
- As we said, expected values of jumps are negative.
- Jumps are more appropiate for short maturities.