4. Markowitz Diversification Portfolio Investment*†

Question:

How to invest v HKD into assets $A_1, \dots, A_n$

Central idea: Diversification

“Don’t put all your eggs in one basket”

*MA6622, Ernesto Mordecki, CityU, HK, 2006.

†Reference for Lecture 4: Essentials of Stochastic Finance, Albert N. Shiryaev, World Scientific (1999)

4a. Formulation of the Problem.

Suppose that you have v HKD to invest in several assets $A_1, \dots, A_n$.

Prices today ($t = 0$) are $S_1(0), \dots, S_n(0)$ (known).

Prices tomorrow ($t = 1$) will be

$$S_k(1) = S_k(0)\Big(1 + X_k\Big), \qquad k = 1, \dots, n,$$

where X_k is the random interest rate of each asset.

The random interest rates X_k have positive expectation

$$r_k = \mathbf{E}\, X_k \geq 0, \qquad k = 1, \dots, n,$$

variances

$$\sigma_k^2 = \mathbf{var}\, X_k, \qquad k = 1, \dots, n,$$

covariances

$$\mathbf{cov}_{jk} = \mathbf{E}(X_j - r_j)(X_k - r_k), \qquad j \neq k = 1, \dots, n,$$

and correlations

$$\rho_{jk} = \frac{\mathbf{cov}_{jk}}{\sigma_j \sigma_k}, \qquad j \neq k = 1, \dots, n.$$

So you will buy a portfolio $\pi = (\alpha_1, \dots, \alpha_n)$ such that

$$v = \alpha_1 S_1(0) + \cdots + \alpha_n S_n(0).$$

At time $t = 1$ your capital will be

$$V_\pi(1) = \alpha_1 S_1(1) + \cdots + \alpha_n S_n(1).$$

Problem: choose best possible $\pi = (\alpha_1, \dots, \alpha_n)$, according to your risk-return preferences.

The main instruments of analysis proposed by Markowitz (1952) are

- $\mathbf{E}\, V_\pi(1)$ to measure the expected return,

- $\mathbf{var}\, V_\pi(1)$ to measure the risk.

Let us compute these numbers.

Consider the proportions

$$a_1 = \frac{\alpha_1 S_1(0)}{v}, \ldots, a_n = \frac{\alpha_n S_n(0)}{v}$$

such that $a_1 + \cdots + a_n = 1$. Now

$$\begin{aligned} V_\pi(1) &= \alpha_1 S_1(1) + \cdots + \alpha_n S_n(1) \\ &= \alpha_1 S_1(0)(1 + X_1) + \cdots + \alpha_n S_n(0)(1 + X_n) \\ &= v a_1 (1 + X_1) + \cdots + v a_n (1 + X_n) \\ &= v(1 + a_1 X_1 + \cdots + a_n X_n) \\ &= v(1 + X_\pi). \end{aligned}$$

where we define the return of portfolio π as

$$X_\pi = a_1 X_1 + \cdots + a_n X_n.$$

We have

$$\mathbf{E}\, V_\pi(1) = v(1+\mathbf{E}\, X_\pi), \qquad \mathbf{var}\, V_\pi(1) = v^2\, \mathbf{var}\, X_\pi$$

and we can write

$$\mathbf{E}\, X_\pi = a_1 X_1 + \cdots + a_n X_n,$$

$$\mathbf{var}\, X_\pi = \sum_{k=1}^{n} (a_k \sigma_k)^2 + 2 \sum_{1 \le j < k \le n} a_j a_k\, \mathbf{cov}_{jk}\, .$$

4b. Re-formulation of the Problem.

Given $r_k \geq 0$, $\sigma_k > 0$, $-1 \leq \rho_{jk} \leq 1$, find $(a_1, \ldots, a_n)$ such that $a_1 + \cdots + a_n = 1$:

- expected return: $a_1 r_1 + \cdots + a_n r_n$ as higher as possible,
- risk: $\sum_{k=1}^{n} (a_k \sigma_k)^2 + 2 \sum_{1 \leq j < k \leq N} a_i a_j \sigma_j \sigma_k \, \rho_{jk}$ as lower as possible,

This is the risk-return tradeoff.

Let us examine the case $n = 2$. Results are

$$\mathbf{E}\, X_\pi = a_1 r_1 + a_2 r_2,$$

and, as $\mathbf{cov}_{12} = \sigma_1 \sigma_2\, \rho_{12}$, we have

$$\begin{aligned}\mathbf{var}\, X_\pi &= (a_1\sigma_1)^2 + (a_2\sigma_2)^2 + 2a_1a_2\sigma_1\sigma_2\, \rho_{12} \\ &= (a_1\sigma_1 - a_2\sigma_2)^2 + 2a_1a_2\sigma_1\sigma_2(1 + \rho_{12})\end{aligned}$$

Example: Assume that $r_1 < r_2$, $\sigma = \sigma_1 = \sigma_2$ and $\sigma_{12} = 0$

- $a_1 = 0, a_2 = 1$ gives the maximum possible return r_2, with variance σ^2.

- $a_1 = a_2 = 1/2$ gives the minimum variance $\sigma^2/4$, with medium return $(r_1 + r_2)/2$.

and these two portfolios are very different.

So:

- high expected return rises with an increase in risk,

- Low levels risk are associated with low expected returns

In other words: high returns requires risky investments (read Financial Times article4.pdf)

Negative correlated assets

If $\rho_{12} = -1$, i.e. A_1 and A_2 are negatively correlated,

$$\mathbf{var}\, X_\pi = (a_1\sigma_1 - a_2\sigma_2)^2$$

we can achieve $\mathbf{var}\, V_\pi(1) = 0$ by choosing $a_1\sigma_1 = a_2\sigma_2$, that gives

$$a = \frac{\sigma_2}{\sigma_1 + \sigma_2}, \quad b = \frac{\sigma_1}{\sigma_1 + \sigma_2}$$

In this situation we have no risk, and our return is

$$V_\pi(1) = v\Big(1 + \frac{r_1\sigma_2 + r_2\sigma_1}{\sigma_1 + \sigma_2}\Big) > v.$$

Phenomenon of negative correlation: When making an investment portfolio, negatively correlated assets reduces the risk.

Diversification

Assume that the assets $A_1, \ldots, A_n$ are uncorrelated:

$$\rho_{jk} = 0, \quad \text{for all } j \neq k,$$

with bounded variances:

$$\sigma_1^2 \leq C, \ldots, \sigma_n^2 \leq C.$$

Choose $a_1 = \cdots = a_n = 1/n$.

$$\mathbf{var}\, X_\pi = a_1^2 \sigma_1^2 + \cdots + a_n^2 \sigma_n^2 =$$

$$\frac{1}{n^2}(\sigma_1^2 + \cdots + \sigma_n^2) \leq \frac{C}{n} \to 0 \quad (n \to \infty).$$

Phenomenon of absence of correlation: In absence of correlation the number n of assets should be possible larger to reduce the risk (variance) of the investment.

4c. Systematic and Unsystematic Risk

If we have n correlated assets, the variance of a portfolio with proportions $a_1, \ldots, a_n$ and return

$$X_\pi = a_1 X_1 + \cdots + a_n X_n$$

is

$$\mathbf{var}\, X_\pi = \sum_{k=1}^{n} (a_k \sigma_k)^2 + 2 \sum_{1 \le j < k \le n} a_i a_j\, \mathbf{cov}_{jk}\,.$$

If we take $a_1 = \cdots = a_n = 1/n$, we obtain

$$\sum_{k=1}^{n} (a_k \sigma_k)^2 = \frac{1}{n} \bar{\sigma}_n,$$

defining the mean variance as

$$\bar{\sigma}_n = \frac{1}{n} \sum_{k=1}^{n} \mathbf{var}\, X_k.$$

The mean covariance is

$$\bar{c}_n = \frac{2}{n^2 - n} \sum_{1 \le i < j \le n} \mathbf{cov}(i, j),$$

and, we conclude

$$\mathbf{var}\, X_\pi = \frac{1}{n}\bar{\sigma}_n + \left(1 - \frac{1}{n}\right)\bar{c}_n$$

Assume that $\bar{\sigma}_n \le C$ and $\bar{c}_n \to \bar{c}$. Then

$$\mathbf{var}\, X_\pi \to \bar{c} \qquad (n \to \infty)$$

In conclusion:

- The first (variance) term can be reduced by diversification, it is the unsystematic risk

- The second (covariance) term can not be reduced, it is the systematic called also the market risk.

4d. Efficient Portfolio

In order to compare two portfolios, rational behaviour is:

- With equal expected returns, we prefer lower variance,

- With equal variance, we prefer higher expectated returns.

We now plot, for $a_k \geq 0$ with $a_1 + \cdots + a_n = 1$ all the possible values of

$$\left(\sqrt{\mathbf{var}\, V_\pi(1)}, \mathbf{E}\, V_\pi(1)\right).$$

It is not difficult to see, that

- Given a constant $\sigma_0^2 = \mathbf{var}\, V_\pi(1)$ we have an interval of values for $\mathbf{E}\, V_\pi(1)$,

- Given a constant $m_0 = \mathbf{E}\, V_\pi(1)$ we have an interval of values of $\mathbf{var}\, V_\pi(1)$

Conclussion: Given σ_0^2 we have one portfolio with this variance and maximum expectation, in the efficient curve.

Similarly, given m_0 there is only one portfolio with this expected return and minimum variance, in the same curve.

This portfolios are called efficient.

These results are known as the mean-variance analysis of Markowitz.