3. Predictability of asset returns (second part)*

Martingale and Time Series approach

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Question:

Can we predict future values of an asset?

More precisely:

How to model the time evolution of prices of financial instruments, as stocks, indexes, commodities, etc.

3a. Martingale Hypothesis

A stronger form of an efficient market is the martingale hypothesis, that we describe now.

Consider

- a probability space $(\Omega, \mathcal{F}, \mathbf{P})$,

- a discrete time scheme $\mathbb{N} = \{0, 1, \ldots, n, \ldots\}$

With each time n we associate a σ-algebra $\mathcal{F}_n$, that represents the information up to (including) time n.

As information increases with time, we assume that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$$

The increasing family introduced is a filtration, denoted $\mathbb{F}$, i.e.

$$\mathbb{F} = \{\mathcal{F}_n\}$$

The quadruple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is a filtered probability space.

A (square integrable) martingale in $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is a sequence of random variables $\{X_n\}$:

(0) $\mathcal{F}_n$ contains the information of $X_1, \ldots, X_n$, write $X_n \in \mathcal{F}_n$

(1) $\mathbf{E}\,|X_n|$ is finite ($\mathbf{E}(X_n)^2$ is finite)

(CE) The conditional expectation

$$E\Big(X_{n+1} \mid \mathcal{F}_n\Big) = X_n$$

Condition (CE) mathematically expresses of no predictability of $\{X_n\}$: we can prove that

$Y = E\Big(X_{n+1} \mid \mathcal{F}_n\Big)$ minimizes $\mathbf{E}(X_{n+1} - Y)^2$ when $Y \in \mathcal{F}_n$: the prediction of X_{n+1} based on current information $\mathcal{F}_n$ is today's value X_n.

Example of a martingale: If $\{h_n\}$ are independent random variables such that $\mathbf{E}\, h_n = 0$. Then

$$H_n = h_1 + \cdots + h_n \text{ is a martingale:}$$

$$\begin{aligned}
\mathbf{E}\left(H_{n+1} \mid \mathcal{F}_n\right) &= \mathbf{E}\left(H_n + h_{n+1} \mid \mathcal{F}_n\right) \\
&= \mathbf{E}\left(H_n \mid \mathcal{F}_n\right) + \mathbf{E}\left(h_{n+1} \mid \mathcal{F}_n\right) \\
&= H_n + 0 = H_n
\end{aligned}$$

Assume a market model with a deterministic savings account

$$B_n = (1 + r)^n$$

and an asset with stochastic prices $\{S_n\}$.

We say that two probabilities are equivalent when

$$\mathbf{P}(A) = 0 \Leftrightarrow \mathbf{Q}(A) = 0,$$

i.e. they have the same nule sets.

The martingale property says:

There exists a probability measure $\mathbf{Q}$, equivalent to $\mathbf{P}$ such that

the process $\{B_n^{-1} S_n\}$ is a martingale

with respect to the filtration $\mathbb{F}$. In other terms

$$\mathbf{E}_{\mathbf{Q}}\left(\frac{S_{n+1}}{B_{n+1}} \mid \mathcal{F}_N\right) = \frac{S_n}{B_n},$$

for all $n = 1, 2, \ldots$. $\mathbf{Q}$ is a risk neutral martingale measure.

Consequence: If

$$X_n = S_n/(1+r)^n$$

is a martingale, we can not predict the values of S_{n+1} with the information of today $\mathcal{F}_n$. Our best prediction is

$$\widehat{S}_{n+1} = (1+r)S_n.$$

because the predictor (under $\mathbf{Q}$) of

$S_{n+1}/(1+r)^{n+1}$ is $S_n/(1+r)^n$.

3b. Predictability of asset returns:

If one or more hypothesis of the efficient market assumptions does not hold, we have the possibility of predicting the movementes of asset returns.

Remember that the random walk hypothesis assumed that

$$S_n = \exp(h_1 + \cdots + h_n), \qquad n \geq 1,$$

where $\{h_n\}$ are independent random variables.

Suppose that $\mathbf{P}$ is a risk neutral martingale measure and $B_n \equiv 1$. The idea is to look for a reasonable substitute hypothesis for the independence.

A simple assumption on the probabilistic nature of $\{h_n\}$ to substitute independence was proposed by Engle (1982) when analyzing the variance of UK inflation. He proposed to consider

$$h_n = \sigma_n \varepsilon_n,$$

where $\{\varepsilon_n\}$ is a sequence of independent standard normal random variables, and the volatility σ_n is itself a random variable that depends on the past, is predictable, i.e. σ_n is $\mathcal{F}_{n-1}$ measurable.

Engle proposed to consider

$$\sigma_n^2 = \alpha_0 + \alpha_1 h_{n-1}^2 + \cdots + \alpha_q h_{n-q}^2,$$

i.e. the volatility is a function of the q past observed prices, and this is called the ARCH (Auto Regressive Conditional Heteroskedasticity) model. In order to calibrate the model we must estimate the parameter q, and furthermore $\alpha_0 \ldots, \alpha_q$.

Observe that this generalizes the random walk hypothesis, that correspond to the case

$$\alpha_1 = \cdots = \alpha_q = 0.$$

A further developement in this direction was the introduction of the GARCH (Generalized - ARCH), where

$$\sigma_n^2 = \alpha_0 + \alpha_1 h_{n-1}^2 + \cdots + \alpha_q h_{n-q}^2 \\ + \beta_1 \sigma_1^2 + \cdots + \beta_p \sigma_{n-p}^2,$$

by Bollerslev (1986), that depend on the parameters p, q and

$$\alpha_0, \ldots, \alpha_q, \beta_1, \ldots, \beta_p.$$

In this time series scheme it is possible to predict, with certain confidence interval, the future prices of an asset.