

25. Interest rates models

MA6622, Ernesto Mordecki, CityU, HK, 2006.

References for this Lecture:

John C. Hull, *Options, Futures & other Derivatives (Fourth Edition)*, Prentice Hall (2000)

Plan of Lecture 25

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- (25e) Arbitrage Free pricing Models
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25a. Modelling the interest rates

Black's Formula to price interest rate derivatives makes the assumption of lognormality at the maturity time of the underlying.

In order to price time dependent derivatives (as american options) or simply, more general distributed financial instruments it is necessary to develop a model for the evolution of the whole term structure of interest rates.

Let us now define the quantities we observe in the market:

- The price of zero coupon bonds at t , paying 1 at maturity T , denoted by $P(t, T)$.
- Equivalently, we observe the continuously compounded yield of the zero coupon bonds, defined as

$$Y(t, T) = -\frac{\log P(t, T)}{T - t}.$$

based on the relation

$$P(t, T) = 1 \times \exp \left[- (T - t)Y(t, T) \right].$$

25b. Equilibrium spot rates models (Vasicek)

A first approach to model the zero coupon bond prices, based on economics equilibrium arguments, assumes the existence of a

- a stochastic instantaneous interest rate, denoted by $r(t)$, also called the spot rate or short rate, that (theoretically) is the interest corresponding to a short time interval of the form $t, t + h$.

that satisfies the diffusion equation:

$$dr(t) = a(b - r(t)) dt + \sigma dW(t),$$

where a , b and σ are positive constants, $\{W(t)\}$ is a Wiener Process, and the process departs from some $r(0) = r_0 > 0$.

This is the Vasicek model¹.

¹Vasicek, O. "An equilibrium characterization of the term structure", Journal of Financial Economics, 1977, pp. 177-188.

It can be proved (with the help of Itô's formula) that the interest rates can be computed through:

$$r(t) = b + e^{-at}(r_0 - b) + \sigma e^{-at} \int_0^t e^{as} dW(s)$$

what gives that $r(t)$ is a gaussian random variable, with

$$\begin{aligned} \mathbf{E} r(t) &= b + e^{-at}(r_0 - b), \\ \mathbf{var} r(t) &= \frac{\sigma^2}{2a} [1 - e^{-2at}]. \end{aligned}$$

In particular, for big values of t we obtain

$$\mathbf{E} r(t) \sim b, \quad \mathbf{var} r(t) \sim \frac{\sigma^2}{2a}.$$

Vasicek shows that the price of a zero coupon bond is

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

where r is the value of r at time t , and

$$B(t, T) = (1 - \exp(-a(T - t))) / a,$$

$$A(t, T) = \exp \left[\frac{(B(t, T) - T + t)(a^2 b - \sigma^2 / 2)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a} \right]$$

From this we obtain the equation for the yield curve, as

$$Y(t, T) = \frac{-1}{T - t} \log A(t, T) + \frac{1}{T - t} B(t, T) r(t).$$

In order to [calibrate](#) the model, we introduce $\tau = T - t$, and $r(t) = r_0$. Then (with a slightly abuse of notation), we have

$$Y(\tau) = \frac{1}{\tau} \left[r_0 B(\tau) - \log A(\tau) \right],$$

where

$$B(\tau) = \frac{1 - e^{-a\tau}}{a},$$

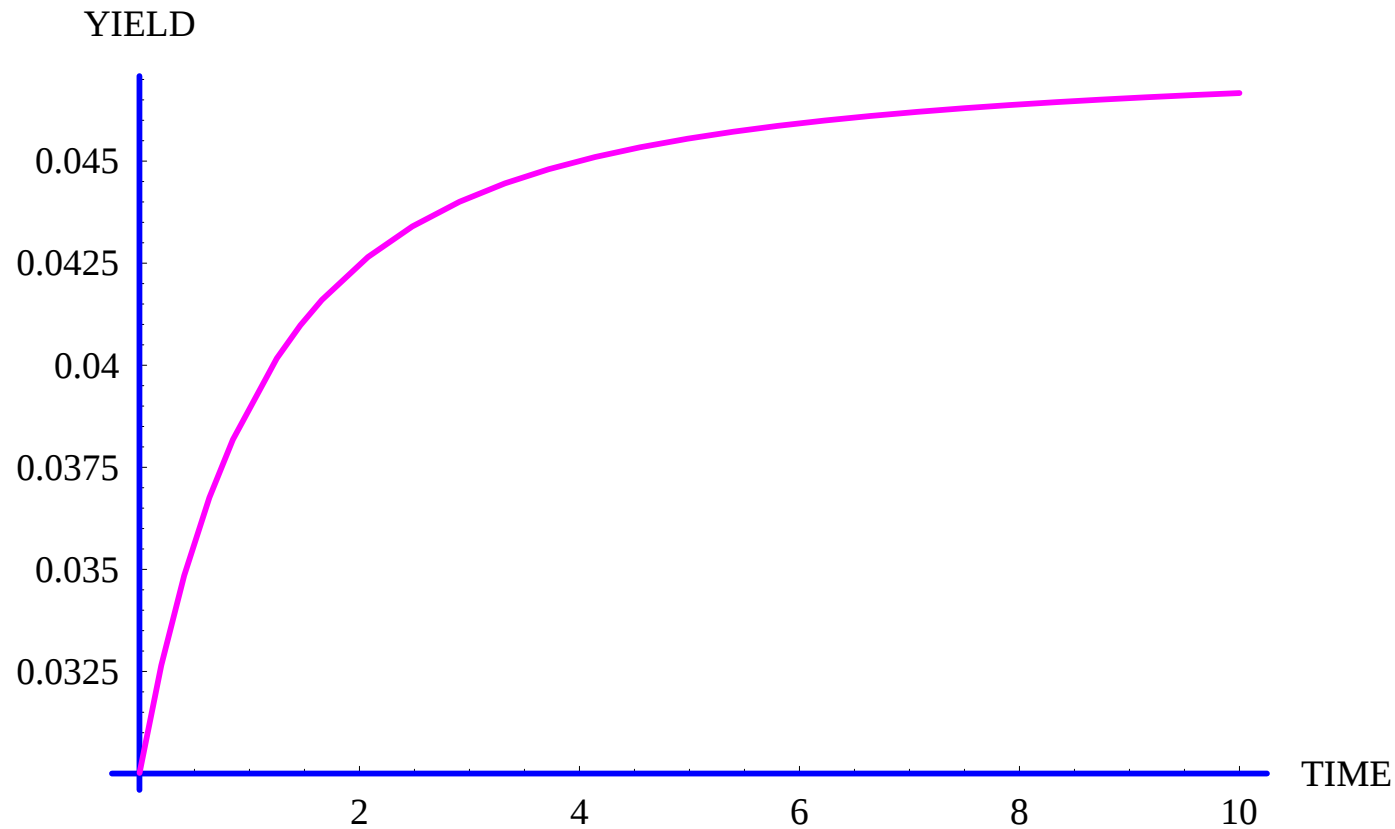
and

$$-\log A(\tau) = \left(b - \frac{\sigma^2}{2a^2}\right)(\tau - B(\tau)) + \frac{\sigma^2}{4a}B(\tau)^2.$$

We then observe:

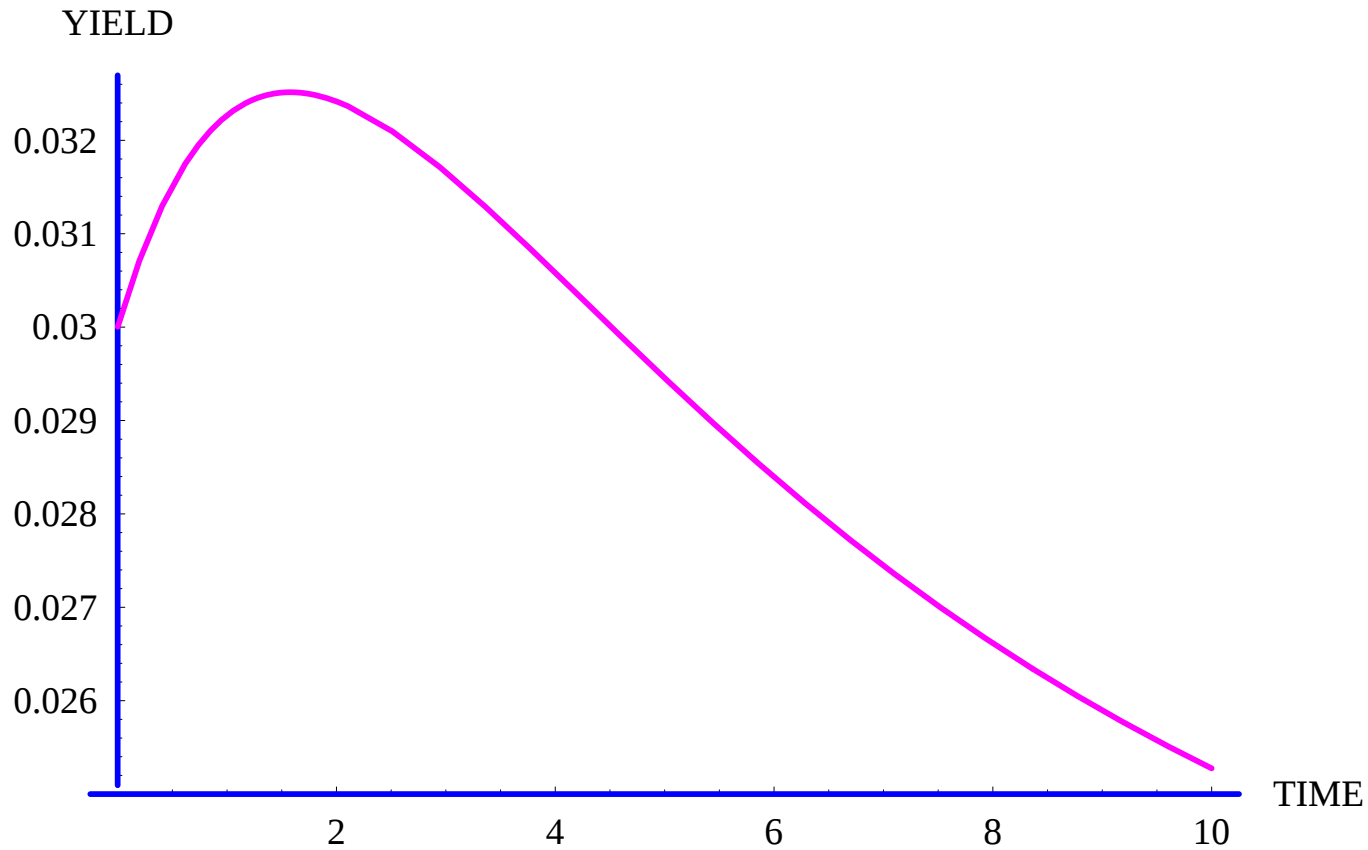
- $Y(0) = r_0$,
- $Y'(0) = a(b - r_0)/2$, so in order to have positive slope at $\tau = 0$ we require $b > r_0$.
- $\lim_{\tau \rightarrow \infty} Y(\tau) = b - \frac{\sigma^2}{2a^2}$ and this defines the form of the yield curve.

Let us see the three typical forms the yield curve has in Vasicek model:



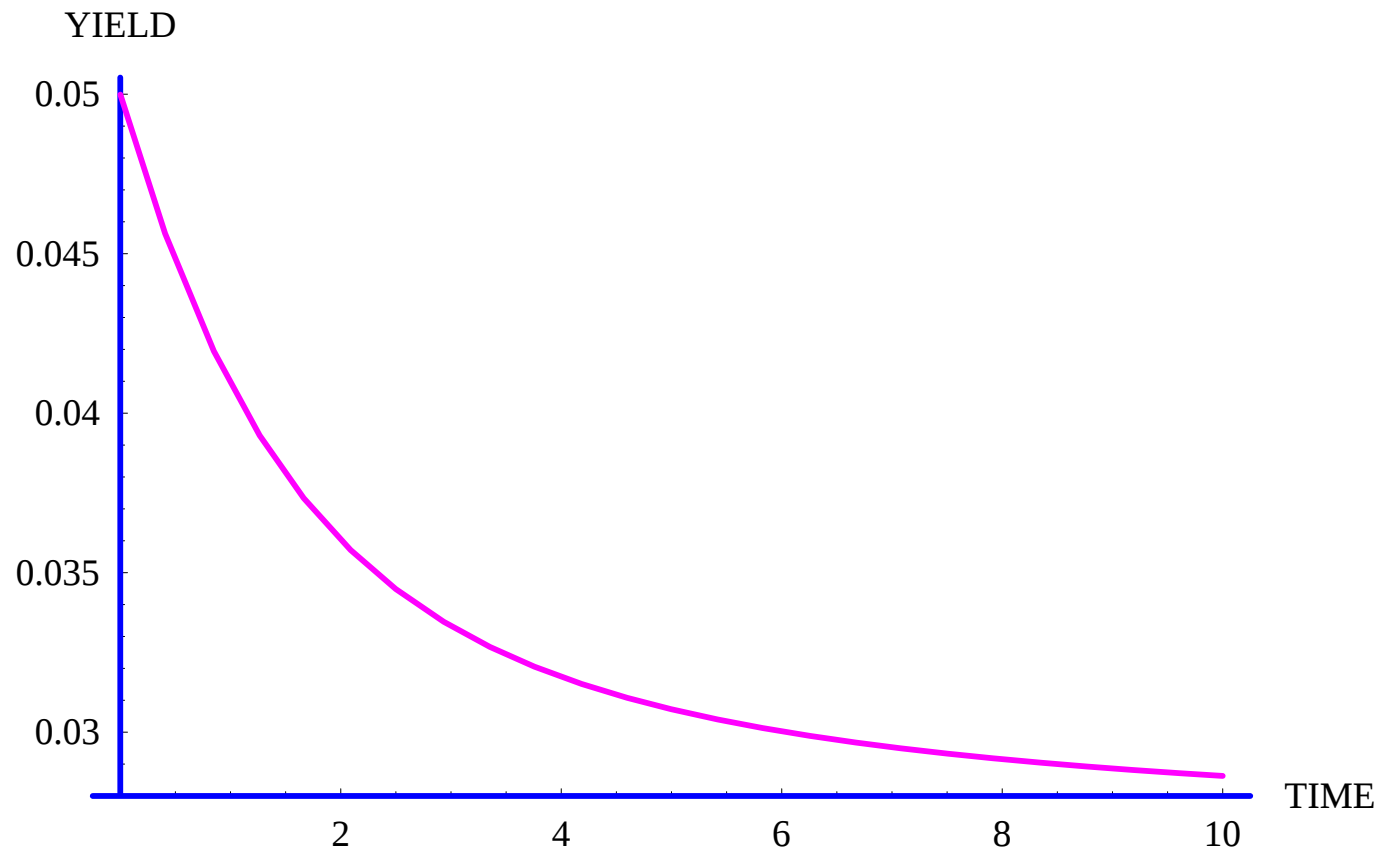
The value of the parameters are $a = 1.5$, $r_0 = 0.03$, $b = 0.05$, $\sigma = 0.10$. The characteristic values are $Y(0) = r_0 = 0.03$, $Y'(0) = 0.015$, $Y(\infty) = 0.0477$,

This is a **normal** yield curve.



The value of the parameters are $a = 0.3$, $r_0 = 0.03$, $b = 0.05$, $\sigma = 0.10$, The characteristic values are $Y(0) = r_0 = 0.03$, $Y'(0) = 0.03$, $Y(\infty) = 0$,

This is a **humped** yield curve.



The value of the parameters are $a = 1.2$, $r_0 = 0.05$, $b = 0.03$, $\sigma = 0.10$, The characteristic values are $Y(0) = r_0 = 0.05$, $Y'(\infty) = -0.012$, $Y(\infty) = 0.03$,

This is an **inverted** yield curve.

25c. Other equilibrium models

It must be observed that, as the interest rates in Vasicek model are gaussian random variables, the interest rates can take negative values².

An alternative similar model was proposed by Cox, Ingersoll and Ross³. It assumes that the interest rate process satisfies the diffusion

$$dr(t) = a(b - r(t)) dt + \sigma \sqrt{r(t)} dW(t),$$

where a , b and σ are positive constants, $\{W(t)\}$ is a Wiener Process, and the process departs from some $r(0) = r_0 > 0$.

In this model, called the CIR model, Bond prices have the

²If the model is correctly calibrated this will happen with very small probability

³J.C. Cox, J.E. Ingersoll Jr, S.A. Ross, A Theory of the Term Structure of Interest Rates
Econometrica, Vol. 53, No. 2 (March, 1985)

same form as in Vasicek model

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

but with different formulas for $A(t, T)$ and $B(t, T)$.

Some other characteristics of the CIR model are:

- The model produces the same type of yield curves as in Vasicek model.
- The authors provide closed formulas for European call and put options on zero coupon bonds.
- In order to calibrate it, it is necessary to fit parameters a, b, σ .
- The interest rates always remain positive.

25d. Pricing Bond options under Vasicek model

First consider zero coupon bonds. Jamshidian⁴ obtained the formula for a call option on a zero coupon bond with:

- a, σ the parameters in Vasicek model,
- The principal Pr .
- Bond maturity T ,
- Option maturity $t < T$,
- Strike K ,

as

$$Call = Pr. \times P(0, T)\Phi(h) - K \times P(0, t)\Phi(h - \sigma_P),$$

where the constants h and σ_P are:

⁴Jamshidian, Farshid “An Exact Bond Option Formula” The Journal of Finance (1989)

$$h = \frac{1}{\sigma_P} \log \left[\frac{Pr. \times P(0, T)}{K \times P(0, t)} \right] + \frac{\sigma_P}{2},$$

$$\sigma_P = \frac{\sigma}{a} \left[1 - e^{-a(s-T)} \right] \sqrt{\frac{1 - e^{-2aT}}{2a}}.$$

Decomposing a coupon bearing bond, as a series of zero coupon bonds, corresponding to Jamshidian obtains the price of coupon bearing bonds.

25e. Arbitrage Free pricing Models

The tale of bond pricing only begins with Vasicek (1976) model.

The paradigm of risk-neutral martingale measures enters into the fixed-income securities to produce arbitrage free pricing models.

In particular, this models fit exactly (theoretically) the term structure of interest rates.

In equilibrium models today's term structures of interest rates is an output of the model, whereas in arbitrage free interest rates it is an input.

This is possible at the cost of assuming that the drift in the equation governing the interest rate is time dependent.

An example of a risk-free interest rate is the Ho and Lee model⁵

$$dr(t) = \theta(t)dt + \sigma dW(t),$$

where σ is constant, and $\theta(t)$ is a function chosen such that the initial term structure.

In this simple model the function θ can be calculated analytically, as

$$\theta(t) = \frac{\partial}{\partial t}F(0, t) + \sigma^2 t.$$

where $F(0, t)$ is the instantaneous forward rate for maturity t .

⁵T. Ho; S. Lee “Term Structure Movements and Pricing Interest Rate Contingent Claims”
The Journal of Finance, (1986).

Characteristics of the Ho and Lee model are:

- It gives arbitrage free prices for bond options
- Option prices can be computed also as

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

but with corresponding analytic formulas for $A(t, T)$ and $B(t, T)$.

25f. Comments on more advanced models

- Two factor models for interest rates have also been proposed in the literature.
- A tree construction, similar to the Derman Kani tree can also be used to price fixed income securities.

An relevant step in the modelling of fixed income securities has been proposed by Heath Jarrow and Morton.

The main innovation in the HJM approach is that they model the zero coupon bonds directly, instead of modelling the interest rate.

In consequence, the HJM models is called a **forward rate model**, in contrast with the previous models, known as **short rate models**.

Consider the formula

$$P(t, T) = \exp \left[- \int_t^T f(t, s) ds \right]$$

where $f(t, s)$ is the forward interest rate observed today of a loan with duration $s, s + h$.

In particular, when $s = t$ we obtain as a particular case

$$f(t, t) = r(t),$$

the spot interest rate.