24. Pricing Fixed Income Derivatives

through Black's Formula

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References for this Lecture:

John C. Hull, *Options, Futures & other Derivatives (Fourth Edition)*, Prentice Hall (2000)

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24a. Bond Options

A bond option is a contract in which the underlying asset is a bond, in consequence, a derivative or secondary financial instrument.

An examples can be the option to buy (or sell) a 30 US Treasury Bond at a determined strike and date[1].

Bond options are also included in callable bonds. A callable bond is a coupon bearing bond that includes a provision allowing the issuer of the bond to buy back the bond at a predermінated price and date (or dates) in the future.

When buying a callable bond we are:

- Buying a coupon bearing bond
- Selling an european bond option (to the issuer of the

[1] Options on US bonds of the American Type,i.e. they give the right to buy/sell the bond at any date up to maturity.

bond)

Similar situations arises in putable bonds, that include the provision for the holder to demand an early redemption of the bond at certain predermined price, and at a predetermined date(s).

When buying a putable bond, we are

- Buying a coupon bearing bond
- Buying a put option on the same bond.

This are called embeded bond options, as they form part of the bond buying contract.

24b. Black's Model for European Options

A standard procedure to price bond options is Black's Formula (1976)[2] that was initially proposed to price commodities options.

Assume that we want to price an option written on a financial instrument with value V, in a certain currency. Define

- T : Maturity of the option.
- F : Forward price of V.
- F_0 : Value of F at time $t = 0$.
- K : strike price of the option.
- $P(t, T)$: Price at time t of a zero cuopon bond paying 1 unity of the currency at time T
- $V(T)$: value of V at time T.

[2] Black, F. "The Pricing of Commodity Contracts" Journal of Financial Economics, 1976.

- σ : volatility of F.

The assumptions of Black's model are

- $V(T)$ has a lognormal distribution with standard deviation of $\log V(T)$ equal to $\sigma\sqrt{T}$.
- The expected value[3] of $V(T)$ is F_0.

Under this conditions, Black showed that the option price is

$$Call = P(0,T)\big[F_0\Phi(d_1) - K\Phi(d_2)\big],$$

where

$$d_1 = \frac{\log(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\log(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

[3] under a certain risk neutral probability measure

Similarly, the value of a put option is given by

$$Put = P(0,T)\big[K\Phi(-d_2) - F_0\Phi(-d_1)\big].$$

The similarity with the BS formula is clear, being the main differences:

- There is no assumption on the time dynamics of the price of the financial instrument, the assumption is on time T
- The risk-free interest rate does not appear, it is taken into account in the zero coupon bond.
- The price of the financial instrument is substituted by the its forward price, that includes the (risk neutral) expectatives about future behaviour of prices.

In this respects Black's formula is a generalization of Merton's time dependent Black-Scholes formula[4].

[4] Remember Lecture 16 "Time dependence in Black Scholes".

24c. Pricing Bond Options

The pricing computations under the Black Model are similar to the BS pricing, with some minor differences.

One main difference is that here the quoted price, or clean price, should be corrected in order to obtain the cash (or dirty) price. This correction applies both for the spot and the strike price.

Example Compute the Bond Call Option price under the following characteristics[5]. Thea 10-month European call option on a The underlying is a 9.75 year Bond with a face value of $1,000. Suppose that

- The option expires in 10 months.
- Current quoted (clean) bond price is $935

[5] Taken from John C. Hull, *Options, Futures & other Derivatives (Fourth Edition)*, Prentice Hall (2000)

- The (clean) strike price is \$1,000
- The ten month risk free interest rate is 10% p.a.
- The volatility of the forward bond price in 10 months is 9% p.a.
- The bond pays a semiannual coupon of 10% p.a. interest rate
- Coupon payments (\$50 each) are expected in three and nine months.
- Risk-free interet rates for three and nine months are 9.0% and 9.5% p.a. respectively.

The procedure to determine the necessary input in Black's formula follows.

Step 1. The maturity is $T = 10/12$ is in the contract.

Step 2. We compute the cash bond price. The accrued interest is $25, as the coupon is $50 to be paid in three months (i.e. $f = 1/2$).

In consequence the bond cash price is

$$935 + 25 = 960.$$

Step 3. Now we compute the forward price of the bond, begining by discounting the coupons the holder will recieve.

The present value of the two coupons to be paid are

$$50 \exp(-0.09 \times 0.25) + 50 \exp(-0.095 \times 0.75) = 95.45.$$

Then, the futures price is

$$F_0 = (960 - 95.45) \exp(0.1 \times 10/12) = 939.68$$

Step 4. We compute the cash strike price to use in the formula, taking into account that at month 10, we have 1 month since the last coupon payment date. The cash strike price is

$$1,000 + 50 \times (1/6) = 1008.33$$

Step 5. We have all the input for Black's formula:

- $P(0,T) = \exp(-0.1 \times 10/12) = 0.92$
- $F_0 = 939.68$
- $K = 1008.33$
- $\sigma = 0.09$
- $T = 10/12 = 0.833$

We obtain

$$d_1 = \frac{\log(939.68/1008.33) + (0.09)^2(0.833)/2}{0.09\sqrt{0.833}} = -0.681,$$

$$d_2 = d_1 - 0.09\sqrt{0.833} = -0.749.$$

So the call price is

$$0.92\big[939.68\Phi(-0.681) - 1008.33\Phi(-0.749)\big] = 7.968.$$

24d. Yield volatilities

Undoubtely the most difficult parameter to determine in the previous computation is the volatility. The usual procedure is to infer some implied volatility to be used in the formula.

If instead of the bond volatility, we get quotes of yield volatilities, we obtain the approximate bond volatilities by the following procedure.

We know the form of dependence of the value of a bond and its yield $V = V(y)$, and we have obtained that

$$\frac{V'(y)}{V(y)} = -D_{mod},$$

where $D_{mod} = D/(1+y)$ is the modified duration[6]

[6] If one uses the continuous compounded yield, the formulas are slightly different, in particular we have only one duration.

If we approximate $V'(y) = \Delta V/\Delta y$, we have

$$\frac{\Delta V}{V} = -(D_{mod} y)\Big(\frac{\Delta y}{y}\Big),$$

taking the variance in both sides

$$\begin{aligned} \sigma^2_{Bond} &= \mathbf{var}\left(\frac{\Delta V}{V}\right) = D^2_{mod} y^2\, \mathbf{var}\left(\frac{\Delta y}{y}\right) \\ &= D^2_{mod} y^2 \sigma^2_y, \end{aligned}$$

that gives

$$\sigma_{Bond} = (D_{mod} y)\, \sigma_y.$$

Remark Here we see that the larger the duration, for the same yield, the higher the Bond's volatility.

24e. Interest rate options

Instead of writting an option on a bond, it is possible and usual to write and option on a floating interest rate, tipically the Libor.

This options are produced in order to protect the buyer against large up or down movments of interest rates, and are called respectively [7] caps, to protect against high floating interest rates, and floors to protect against low floating interest rates.

Consider a series of dates $t_1 < \cdots < t_n$, with $\Delta = t_{k+1} - t_k$ fixed (the tenor) and denote by $r_1 < \cdots < r_n$ the a floating interest rate, applied for the period between dates t_k and t_{k+1}.

[7] You can think that you have a short position on a floating interest rate coupon Bond, and want to protect yourself against interest rates over a certain level.

Consider a principal P and a fixed interest rate r_c, the cap rate.

The holder of a cap will recieve the difference of the interest generated in the period t_k, t_{k+1} computed with the floating rate r_k and the fixed cap rate r_c, if this difference is positive, i.e. when $r_k > r_c$.

The first accrued interest is

$$P \times \Delta \times r_k,$$

while the second is

$$P \times \Delta \times r_c.$$

So the holder of a cap will recieve

$$\max(P\Delta\, r_k - P\Delta\, r_c, 0) = P\Delta \max(r_k - r_c)^+,$$

In consequence, a cap will provide the holder with the following cash-flow:

Dates	t_2	t_3	$\ldots$	t_n
Amount	$P\Delta(r_1 - r_c)^+$	$P\Delta(r_2 - r_c)^+$	$\ldots$	$P\Delta(r_{n-1} - r_c)^+$

Similarly, a floor with principal P and interest rate r_f, on dates $t_1, \ldots, t_n$ will provide to the holde the following payoff:

Dates	t_2	t_3	$\ldots$	t_n
Amount	$P\Delta(r_f - r_1)^+$	$P\Delta(r_f - r_2)^+$	$\ldots$	$P\Delta(r_f - r_{n-1})^+$

Remarks

- Each individual payment of a cap is called a caplet, and is in fact a call option written on the floating interest rate observed at t_k to be payed at t_{k+1}.
- There is a put-call paritiy relating prices of caps and floors. If both contracts have the same scheduled dates, on a same principal, and if $r_c = r_f$, then

$$\text{cap price}=\text{floor price} + \text{value of swap}$$

  as is is equivalent to hold a floor and a swap than a cap.
- A collar is a contract that pays when the interest rate is above a prefixed rate r_c or below another prefixed rate r_f, and is a combination of a long position in a cap and a short position in a floor.

24f. Pricing Interest rate options

The pricing of interest rate options can also be, in principle, priced with Black's formula.

Consider an individual caplet, that provides, at time t_{k+1}, a payoff

$$P\Delta(r_k - r_c)^+,$$

where r_k is the (floating) interest rate observed at time t_k.

In order to price the caplet, assuming that the rate r_k is lognormal with volatility σ_k, as we have to diferent time dates involved:

- Date t_k when the rate is observed,
- Date t_{k+1} when the payment is effective,

we use the following time modified Black's formula, discounting up to the moment of payment:

$$Call = P\Delta\, P(0, t_{k+1})\big[F_k\Phi(d_1) - r_c\Phi(d_2)\big]$$

where

$$d_1 = \frac{\log(F_k/r_c) + \sigma_k^2 t_k/2}{\sigma_k\sqrt{t_k}}, \quad d_2 = d_1 - \sigma_k\sqrt{t_k}.$$

Example Consider a contract that caps the interest rate on a $P = \$10,000$ loan at $r_c = 8\%$ p.a., with quarterly compounding (all interest rates will be with quarterly compounding), for three months, starting in one year. Assume that we know:

- The forward interest rate for 3 months, starting in one year is 7%,
- The current 15 months interest rate is 6.5%,
- The volatility of the forward three month rate underlying

the caplet is 20%,

The dates of the contract are:

$t = 0$	$t_k = 1$ (year)	$t_{k+1} = 1 + \Delta$, $\Delta = 1/4$ (3 mths)
Pay the caplet	Observe r_k	Recive $P\Delta(r_k - r_c)^+$

With this information:

- The forward intest rate is $F_k = 0.07$.
- The principal is $P = 10,000$.
- The cap interest is 0.08
- $\Delta = 3/12 = 0.25$
- The discount is $P(0, t_{k+1}) = \exp(-1.25 \times 0.065) = 0.922$,
- $\sigma_k = 0.20$, and $t_k = 1$.

We compute

$$d_1 = -0.5677, \quad d_2 = -0.7677$$

to obtain the price of the caplet as

$$0.25 \times 10,000 \times 0.922$$
$$\times \big[0.07\Phi(-0.5677) - 0.08\Phi(-0.7677) = 5.19$$

This sholud be compared to

$$P \times \Delta \times (0.09 - r_c) = 10,000 \times 0.25 \times 0.01 = 25$$

the difference of interest to be paid if $r_k = 0.09$.