

17. The volatility Smile and its Implied Tree

MA6622, Ernesto Mordecki, CityU, HK, 2006.

Reference for this Lecture:

E. Derman and I. Kani (Course web site),

The Volatility Smile and Its Implied Tree.

Also as: *Riding on a Smile*, Risk Journal, 1994, 7, 32.

Plan of Lecture 17

(17a) The implied tree.

(17b) Calibration of the first node.

(17c) Calibration of the second node.

(17d) Calibration of the third node.

17a. The Implied tree

In order to construct a model consistent with the volatility smile at each maturity, i.e. consistent with the volatility matrix, Derman and Kani (1994) proposed to calibrate a time-space dependent binomial tree, taking into account the smile at each step.

The binomial tree is implied by the option prices at each time step and at each node.

In order to examine the proposal, we construct a two step implied tree for the HSI, with dates June 8, June 29, and July 28.

Jn 8	Jn 29	Pr.	Jl 28	Pr.
			S_C	$p_1 p_2$
	S_A	p_1		
15816.5			S_0	$p_1(1 - p_2) + (1 - p_1)p_3$
	S_B	$1 - p_1$		
			S_D	$(1 - p_1)(1 - p_3)$

The idea is to determine:

- S_A, S_B and p_1 from the forward price of the HSI, and from an at the money option price, plus a centering condition (first node)
- S_C and p_2 from a forward price and an at the money call option price (second node).
- S_D and p_3 from a forward price and an at the money put option price (third node)

17b. Calibration of the first node

Time period: June 8 - June 29.

From S_0 the model assumes either:

- An upward movement to a value S_A , with probability p_1 ,
- A downward movement to a value S_B with probability $1 - p$,
- S_A, S_B, p_1 to be determined.

We need three conditions in order to find the parameters S_A, S_B, p .

- First: **Risk Neutral** condition: The expected value of the stock must be equal to the forward price.

$$p_1 S_A + (1 - p_1) S_B = S_0 \exp(rT) = F_1,$$

Observe that this condition can be transformed into

$$p_1 = \frac{F_1 - S_B}{S_A - S_B}.$$

- Second: **Centering condition**:

$$S_A \times S_B = (S_0)^2.$$

- Third: Calibration with a call option price at the money.

Now the numerical computations:

- For the forward price we use the risk-free interest rate of US Bonds (remember the observations of Lecture 16).

For the time period considered:

$$r(\text{June 8, June 29}) = 0.047.$$

As we have 21 calendar days in 360, we obtain

$$F_1 = 15816.5 \exp \left(0.047 \times (21/360) \right) = 15859.9$$

- Combining the [centering](#) condition we eliminate S_B :

$$p_1 = \frac{F_1 - (S_0^2/S_A)}{S_A - (S_0^2/S_A)} = \frac{F_1 S_A - S_0^2}{(S_A - S_0)(S_A + S_0)}.$$

In particular

$$p_1(S_A - S_0) = \frac{F_1 S_A - S_0^2}{S_A + S_0} \quad (1)$$

- We compute the price of an at the money option.

We interpolate in the smile:

Month	Strike	Price	$r(t, T)$ %	$\sigma(t, T)$ %
June, 29	15800	335	4.70	19.54
June, 29	15816.5	326.35	4.70	19.534
June, 29	16000	240	4.70	19.47

We use 15 trading days in 247. We interpolate to find the volatility for our at the money option:

$$\sigma = 19.54 + 16.5 \left[\frac{19.47 - 19.54}{200} \right] = 19.52$$

The option price in the Binomial model is

$$Call = e^{-rT} p(S_A - S_0),$$

and using (1), we obtain

$$\frac{FS_A - S_0^2}{S_A + S_0} = e^{rT} Call,$$

(that only depends on S_A).

Numerical results follow:

- $S_A = \frac{S_0 e^{rT} + Call}{1 - (Call/S_0)} = 16438.7,$
- $S_B = \frac{S_0^2}{S_A} = 15217.8,$
- $p_1 = \frac{F - S_B}{S_A - S_B} = 0.526.$

The status of our implied tree is:

Jn 8	Jn 29	Pr.	Jl 28	Pr.
			S_C	$p_1 p_2$
	16438.7	0.526		
15816.5			S_0	$p_1(1 - p_2) + (1 - p_1)p_3$
	15217.8	0.474		
			S_D	$(1 - p_1)(1 - p_3)$

17c. Calibration of the second node

Time period June 30 - July 28. The term rate for this period is

$$r(\text{June 30, July 28}) = 0.04724,$$

as computed in Lecture 16.

Observe that the model assumes that the downward value is S_0 (centering condition).

We determine S_C and p_2 , based on

- The price F_2 of a forward during June 30, July 28.
- The price of an option during June 9, July 28, struck at $S_A = 16438.7$.

- We compute the forward (using calendar days):

$$r(\text{Jn30, Jl28}) = \frac{7 \times 0.0470 + 22 \times 0.0476}{7 + 22} = 0.04746$$

and

$$F_2 = S_A \exp(0.0476(29/360)) = 16501.9$$

From the risk neutral equation:

$$p_2 S_C + (1 - p_2) S_0 = F_2$$

we obtain

$$p_2 = \frac{F_2 - S_0}{S_C - S_0}.$$

- We compute the option price:

Month	Strike	Price	$r(t, T)$ %	$\sigma(t, T)$ %
July, 28	16400	246	4.724	18.09
July, 28	16438.7	231.17	4.724	17.97
July, 28	16600	185	4.724	17.49

We interpolate to find the implied volatility

$$\sigma = 18.09 + \left(\frac{38.7}{200}\right)(17.49 - 18.09) = 17.97$$

- The price of the option in the Binomial model is

$$Call = e^{-r(\text{Jn8, Jl28})T} p_1 p_2 (S_C - S_A).$$

Numerical values follow:

$$- S_C = 17569.9.$$

$$- p_2 = \frac{F_2 - S_0}{S_C - S_0} = 0.39107$$

The status of our implied tree is:

Jn 8	Jn 29	Pr.	Jl 28	Pr.
			17569.9	0.206
	16438.7	0.526		
15816.5			15816.5	$0.320 + (1 - p_1)p_3$
	15217.8	0.474		
			S_D	$(1 - p_1)(1 - p_3)$

17c. Calibration of the third node

We must determine S_D and p_3 , using

- The forward price (the interest rate is the same as in the previous node) for S_B ,
- A put option with dates June 8, July 28, struck at S_B .

The calculations are:

- The forward price is

$$F_3 = 15217.8 \exp(0.0476(29/360)) = 15276.3.$$

giving the risk neutral equation for the probability

$$1 - p_3 = \frac{S_0 - F_3}{S_0 - S_D}$$

We compute the price of the Put through put call parity. The implied volatility for a strike of 15200 is 0.195252, that

gives a Put price of¹

$$Put = 191.027$$

The price of this put under the binomial model is

$$\begin{aligned} Put &= e^{-r(\text{Jn8, Jl28})T} (1 - p_1)(1 - p_3)(S_B - S_D) \\ &= e^{-r(\text{Jn8, Jl28})T} (1 - p_1) \left(\frac{S_0 - F_3}{S_0 - S_D} \right) (S_B - S_D). \end{aligned}$$

Here the only unknown value is S_D , obtaining

$$S_D = 13412.5$$

The probability is

$$p_3 = 0.775288.$$

¹If we take Put quoted prices we can not calibrate the model, giving rise to [arbitrage](#) opportunities.

The final implied tree is:

Jn 8	Jn 29	Pr.	Jl 28	Pr.
			17569.9	0.206
	16438.7	0.526		
15816.5			15816.5	0.688
	15217.8	0.474		
			13412.5	0.107

Application: Based on the information on this implied tree, we can price a Bermudean option, i.e. a call option, written on June 8, that can be executed on June 29, or on July 28.

The procedure is the same as in the usual american options, we compare the expected reward of holding the option on June 29, with the reward of executing the option.