10. Value at Risk (VaR)

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References for this Lecture:

The portable financial analyst (Second Edition). Mark P. Kritzman, Wiley (2003).

An Overview of Value at Risk. D. Duffie, J. Pan, Journal of Derivatives (1997)
[Available at: http://www.mit.edu/~junpan/]

Main Purposes of Lectures 10 and 11:

- Introduce the notion of Value at Risk (VaR)

- Notice the relevance of this risk measurement notion, in particular in relation to the Basel Second Accord (Basel II)

- Review the analytical and historical computation of VaR with emphasis on its tail behaviour dependence

- Discuss the difference between long and short positions in VaR (Lecture 11).

- Present the Monte Carlo approach to VaR (Lecture 11).

- Comment on VaR and derivatives (nonlienarity) (Lecture 11)

Plan of Lecture 10

(10a) Definition of VaR

(10b) VaR and Basel II

(10c) VaR with Normal returns

(10d) VaR with Lognormal returns

(10e) Normal versus Lognormal returns in VaR

(10f) Historical VaR

10a. Definition of VaR.

Value at Risk (VaR) is the minimum loss that could occur to a portfolio at a given confidence level α over a specified period T.

More in detail:

- The period, holding period, or time horizon T is the length of time over which we plan to hold the portfolio:

  - When market risk managment is measured through VaR, T is usually 1 or 10 days.

  - In credit risk managment and operational risk managment, T is usually one year.

- The confidence level at which we plan to make the estimate usually are $\alpha = 0.95$ and $\alpha = 0.99$. In our lectures usually we take $\alpha = 0.95$.

- The unit of the currency which will be used to denominate the VaR (HKD in our case).

- If the capital of our portfolio is $V(0)$, that (randomly) changes into $V(\text{T})$, we have, over time T A loss

$$L(\text{T}) = -[V(\text{T}) - V(0)],$$

(if $L(\text{T}) < 0$ we say we have a profit), and the VaR is defined through the equation

$$1 - \alpha = \mathbf{P}\left(L(\text{T}) \geq \text{VaR}\right) = \mathbf{P}\left(-[V(\text{T}) - V(0)] \geq \text{VaR}\right)$$
$$= \mathbf{P}\left(V(\text{T}) - V(0) \leq -\text{VaR}\right).$$

- In statistical terms, VaR is the $1 - \alpha$ quantile of the loss $L(\text{T})$, being $-\text{VaR}$ also the $1 - \alpha$ quantile of the profit or gain $V(\text{T}) - V(0)$.

10b. VaR and Basel II.

Basel II is an agreement between 13 countries (Belgium, Canada, France, Germany, Italy, Japan, Luxembourg Netherlands, Spain, Sweden, Switzerland, United Kingdom, and United States), related to best practices in banking supervision.

Value at Risk made its way in Basel II capital-adequacy framework as this agreement bases it analysis to promote stability in the financial system, in three pillars:

(1) minimum capital requirements;

(2) supervisory review;

(3) market discipline.

The first pillar provides improved risk sensitivity in the way that capital requirements are calculated for three major components of risk that a bank faces:

(a) credit risk (usually T equals one year),

(b) operational risk, (also usually T equals one year),

(c) market risk (usually T equals 1 or 10 days).

VaR is one of the possible techniques to quantify the aftermentioned risks, and prehaps the most popular technique.

10c. VaR with normal returns.

We now compute the VaR assuming that, during the holding period, our portfolio value has normal distribution with parameters μ and σ^2. That is, we assume that $V(\mathrm{T}) = V(0)(1 + X)$ where $X \sim \mathcal{N}(\mu, \sigma^2)$. So

$$V(\mathrm{T}) - V(0) = V(0)X.$$

Denoting

$$X = \mu + \sigma\mathcal{N}, \qquad \text{where} \qquad \mathcal{N} \sim \mathcal{N}(0, 1),$$

VaR must satisfy

$$0.05 = \mathbf{P}\left(V(0)X \leq -\mathrm{VaR}\right) = \mathbf{P}\left(V(0)[\mu + \sigma\mathcal{N}] \leq -\mathrm{VaR}\right)$$
$$= \mathbf{P}\left(\mathcal{N} \leq -[(\mathrm{VaR}/V(0)) - \mu]/\sigma\right)$$

As $t_{0.05} = -1.645$, we have $-1.645 = -\left[(\mathrm{VaR}/V(0)) - \mu\right]/\sigma$ that gives

$$\mathbf{VaR_{0.95}} = \mathbf{V(0)[1.645\sigma - \mu]}.$$

10d. VaR with lognormal returns.

For long intervals, compounding leads to a lognormal distribution rather than a normal distribution for the value of a portfolio.

Given a random variable $Z \sim \mathcal{N}(\delta, \gamma^2)$ the random variable $Y = \exp(Z)$ has a lognormal distribution with parameters (δ, γ^2). These are the distributions encountered in Black Scholes model for asset prices.

Assume then that the value of the portfolio today is $V(0)$, and, at time T will be
$$V(\mathrm{T}) = V(0) \exp(Z).$$

We have
$$V(\mathrm{T}) - V(0) = V(0)[\exp(Z) - 1],$$

so, representing $Z = \delta + \gamma\mathcal{N}$, the VaR satisfies

$$0.05 = \mathbf{P}\left(V(0)[\exp(Z) - 1] \leq -\text{VaR}\right)$$
$$= \mathbf{P}\left(\exp(Z) \leq 1 - (\text{VaR}/V(0))\right)$$
$$= \mathbf{P}\left(\delta + \gamma\mathcal{N} \leq \log[1 - (\text{VaR}/V(0))]\right)$$
$$= \mathbf{P}\left(\mathcal{N} \leq [\log(1 - \text{VaR}/V(0)) - \delta]/\gamma\right).$$

As $\mathcal{N}$ is a standard normal random variable:

$$-1.645 = [\log(1 - \text{VaR}/V(0)) - \delta]/\gamma,$$

and from this, we obtain

$$\mathbf{VaR_{0.95}} = \mathbf{V(0)}\left(\mathbf{1 - e^{\delta - 1.645\gamma}}\right).$$

Remark   If $\alpha = 0.99$ we put 2.326 (the 0.99 quantile) instead of 1.645.

Remark   Compare the two formulae

$$\text{VaR}_{0.95} = V(0)[1.645\sigma - \mu] \qquad \text{under normal returns}$$

$$\text{VaR}_{0.95} = V(0)[1 - e^{\delta - 1.645\gamma}] \qquad \text{under lognormal returns}$$

Note the fact that, for small values of x we have

$$1 - \exp(x) \sim -x$$

Then if $\delta - 1.645\gamma$ is small, we have

$$1 - e^{\delta - 1.645\gamma} \sim -(\delta - 1.645\gamma),$$

that substituted in the lognormal formua gives the normal one.

This indicates that, if expected returns and standard deviations are small (as expected to be for short holding periods), both approaches give similar VaR. (Observe also that, for short times $\mu \sim \delta$ and $\sigma \sim \gamma$.)

10e. Normal versus Lognormal returns in VaR.

Let us compare through an example the previous two formulas.

Assume that the 31 december 1996 we hold a long position of $100,000$ HKD in the HSI. We want to compute the VaR for 1 year.

We begin with the estimation of the parameters (δ, γ^2) in the lognormal case

- As we know[1] that the annualized volatiliy computed from daily data during 1986-1996 is $\gamma = 26.7\%$

- From the quotations 1st January 1986 ($S(0) = 2,568$) and 31st December 1996 quotation ($S(11) = 13,451$) we estimate the

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[1]An Overview of Value at Risk. D. Duffie, J. Pan, Journal of Derivatives (1997)

mean continuous return as

$$\delta = \frac{1}{10} \sum_{k=1}^{10} \log \frac{S(k)}{S(k-1)} = \frac{1}{10} \log \frac{S(n)}{S(0)} = 0.166$$

We now compute (μ, σ^2) from this values, under the condition

$$\mathbf{E}(1+X) = \mathbf{E}\exp(Z), \qquad \mathbf{var}(1+X) = \mathbf{var}\exp(Z).$$

Using the moment generating function

$$\mathbf{E}\exp(tZ) = \exp(t\delta + t^2\gamma^2/2),$$

we obtain

$$1 + \mu = \exp(\delta + \gamma^2/2) \qquad \sigma = (1+\mu)\sqrt{\exp(\gamma^2) - 1}.$$

and from this, and using the corresponding formulae for the VaR, we have the following table:

	Normal model	Lognormal model
mean	0.2234	0.1666
standard deviation	0.333	0.267
$\text{VaR}_{0.95}$	32,438	23,907
$\text{VaR}_{0.99}$	55,249	36,625

10f. Historical VaR

In the previous example we have seen that VaR is sensible to the distribution, more precisely to the tail of the distribution.

The historical VaR is $1 - \alpha$ empirical quantile of the profit and looses historical records of the portfolion in question.

In order to have a reliable estimation we should

- Dispose of historical series with aboundant data (say, at least, 100 daily values for a period of 1 day)

- Assume that the distribution is stationary, i.e. one can expect future prices to move as past ones.

- In order to compute 1 day VaR it is necessary to have daily records, for annual VaR one needs yearly data.

In practice given the historical daily records of the portfolio values

$$S(0), S(1), \ldots, S(n)$$

one computes the historical profit/loss daily results as

$$PL(1) = S(1) - S(0), PL(2) = S(2) - S(1), \ldots$$
$$\ldots, PL(n) = S(n) - S(n-1),$$

and finds the number $-$VaR such that

- $PL(k) < -$VaR for the 0.05% of the $PL(k)$ values

In general, if one has an interval of values for $-$VaR one can take also mean value of the two extremes of this interval.

For example, if your last 250 daily PL records are:

$$1.8, 0.59, 0.74, 1.4, 0.80, -0.017, 2.4, 2.4, \ldots$$

we order the data

$$-2.3, -1.9, -1.6, -1.4, -1.3, -1.3, -1.3, -1.2, -1.2, -1.2,$$
$$-1.1, -1.1, -1.0, -0.97, -0.96, -0.94, -0.93, \ldots$$

The 0.05 quantile is between the 12th and the 13th record, so

$$\text{VaR} = 1.05.$$

From a statistical point of view, the historical approach consists in the estimation of the (unknown) density function of the PL random variable through a histogram.